Equilibrium Pricing in Incomplete Markets under Translation Invariant Preferences*

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Abstract

We provide results on the existence and uniqueness of equilibrium in dynamically incomplete financial markets in discrete time. Our framework allows for heterogeneous agents, unspanned random endowments and convex trading constraints. In the special case where all agents have preferences of the same type and all random endowments are replicable by trading in the financial market we show that a one-fund theorem holds and give an explicit expression for the equilibrium pricing kernel. If the underlying noise is generated by finitely many Bernoulli random walks, the equilibrium dynamics can be described by a system of coupled backward stochastic difference equations, which in the continuous-time limit becomes a multi-dimensional backward stochastic differential equation. If the market is complete in equilibrium, the system of equations decouples, but if not, one needs to keep track of the prices and continuation values of all agents to solve it. As an example we simulate option prices in the presence of stochastic volatility, demand pressure and short-selling constraints.

Keywords: Competitive equilibrium, incomplete markets, heterogenous agents, trading constraints, backward stochastic difference equations.

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1 Introduction

We consider an equilibrium framework to price financial securities in dynamically incomplete markets in discrete time. Our main interest is in equilibrium prices of derivatives or structured products with maturities T that are short compared to the time horizon of a typical life-time consumption-investment problem. So the risk of fluctuating interest rates does not play a big role, and we assume them to be exogenously given. We suppose our agents invest in the financial market with the goal of optimizing the utility of their wealth at time T. Our setup is flexible enough to accommodate heterogeneous agents, unspanned random endowments and convex trading constraints. In dynamic models with general preferences, several consumption goods and incomplete security markets an equilibrium does not always exist, and if there is one, it is typically not unique. For an overview of equilibria with incomplete markets we refer to the review articles by Geanakoplos (1990) and Magill and Shafer (1991) or the textbook by Magill and Quinzii (1996).

In this paper money is the only consumption good and all agents are assumed to have translation invariant preferences that are updated in a time-consistent way as new information is becoming available. This allows us to prove existence and uniqueness of an equilibrium under general assumptions by backward induction. Typical examples of translation invariant preferences are those induced by expected exponential utility, the monotone mean-variance preferences of Maccheroni et al. (2009), mean-risk type preferences where risk is measured with a convex risk measure, optimized certainty equivalents à la Ben-Tal and Teboulle (1986, 1987) or the divergence utilities of Cherny and Kupper (2009). The assumption of translation invariant preferences is appropriate if, for instance, agents are understood as banks or insurance companies which evaluate investments in terms of expected values and risk capital, that is, buffer capital that needs to be held to make an investment acceptable from a risk management point of view. Recently, Gârleanu et al. (2009) have modelled option dealers as expected exponential utility maximizers to describe the effects of demand pressure on options prices, and in Carmona et al. (2010) equilibrium prices for emission certificates have been studied under linear preferences. We assume there exist two kinds of assets. The first type of assets are liquidly traded in large volumes and their prices are not affected by the actions of our agents. Their dynamics will be exogenously given. Assets of the second kind entitle their holders to an uncertain payoff at time T. We think of them as derivatives or structured products which can also depend on non-financial risk such as temperature, rain or political events. They exist in fixed supply and are only traded by our agents. The goal is to price them by matching demand and supply. The situation where there are no exogenous instruments and all assets are priced in equilibrium is a special case. If assets of the second kind are issued by our agents and not bought by anybody else, they exist in zero net supply. On the other hand, if they are originated outside of our group of traders, they are in positive net supply. An example would be CO_2-emission certificates designed and issued but not traded by the European Union. Similarly, if there is demand for them from outside of our group of traders, they will be in negative net supply. An example will be discussed in Subsection 5.2 below where end users are buying put options from a group of option dealers.

The standard way to price derivatives is to compute the expectation of their discounted payoffs under an equivalent martingale measure $\mathbb{Q}$, that is, $\mathbb{Q}$ has the same null sets as the reference measure $\mathbb{P}$ and the discounted price of the underlying is a martingale (or local martingale) with respect to $\mathbb{Q}$. Binomial tree models and the standard Black–Scholes model are complete, and there is exactly one equivalent martingale measure. But extensions such as trinomial tree, GARCH-type, stochastic volatility or jump-diffusion models are incomplete and admit infinitely many equivalent martingale measures. The question then is, which one should be used for pricing. In practice, models are often built directly under $\mathbb{Q}$, then calibrated to market prices of liquidly traded options and used to value more exotic ones; see for instance, Lipton (2002) for an overview of popular stochastic volatility models. Some pricing measures that have been discussed in the literature are the minimal martingale measure of Föllmer and Schweizer (1991), the Esscher transformed measure proposed by Gerber and Shiu (1994), the variance-optimal martingale measure studied in Schweizer (1995, 1996) and Delbaen and Schachermayer (1996) or the minimal entropy martingale measure of Frittelli (2000). Recently, several authors have applied utility indifference arguments to the valuation of complex financial products; see for instance, Henderson and Hobson (2009) for an overview. But utility indifference prices are personal, reflecting the risk preferences of a single agent. Our approach provides a way of deriving the pricing rule from equilibrium considerations. We will show that if at least one agent has differentiable preferences and open trading constraints, then our pricing method is given by a probability measure $\mathbb{Q}$ that is absolutely continuous with respect to $\mathbb{P}$. If in addition, the agent's preference functional is strictly monotone, $\mathbb{Q}$ will be equivalent to $\mathbb{P}$. On the other hand, if agents have closed trading constraints, equilibrium prices in our model are not necessarily given by a linear functional; see also Hugonnier (2010) and the references therein for equilibrium models with constraints and rational asset pricing bubbles or Avellaneda and Lipkin (2009) for a dynamic model of stock and option prices under short-selling constraints.

Our method to prove existence of an equilibrium is to recursively construct one-time-step representative agents with preferences over the space of financial gains realizable by investing in the financial market. In every step we take a Negishi approach similar to Borch (1984) and Filipović and Kupper (2008), where (constrained) Pareto optimal allocations and their relation to equlibrium prices are studied in static models with uncertainty. Since we work with translation invariant preferences, our one-time-step representative agents can be constructed as simple sup-convolutions of the preference functionals of the single agents. But due to market incompleteness and trading constraints, optimal allocations have to be found in suitably restricted subsets. The consumption sets in our framework are unbounded from below. To guarantee the existence of an equilibrium we assume that each agent either is sensitive to large losses or has conditionally compact trading constraints. Sensitivity to large losses means that a position which will be negative in some states of the world is becoming unacceptable if it is multiplied with a sufficiently large constant, irrespective of its upside potential (a precise definition is given in Subsection 2.3 below). Our argument is based on the fact that for an agent which is sensitive to large losses, it is sufficient to search for optimal one-time-step strategies in conditionally compact sets. For different conditions and concepts of compactness in equilibrium models with consumption sets that are unbounded from below, we refer to Werner (1987), Cheng (1991), Brown and Werner (1995), Dana et al. (1997, 1999) and the references therein. Duffie (1987) has shown the existence of an equilibrium in in a model with complete spot markets and an incomplete market of purely financial securities. The proof is based on a fixed point argument and in general, his equilibrium is not unique. In Cuoco and He (2001) a static representative agent is constructed in an economy with incomplete securities markets. But in that paper an equilibrium does not always exist and the construction of the representative agent involves a sup-convolution of the single agents with stochastic weights. Anthropelos and Žitković (2010) show existence and uniqueness in a setup similar to ours. There are agents with translation invariant preferences who negotiate the price of a bundle of contingent claims while they can invest in an incomplete market of exogenously given financial securities. But in contrast to our model, they only consider static investments in the contingent claims. Jofre et al. (2010) provide results on the existence of equilibrium in general models with incomplete market and money.

If uncertainty is generated by a finite event tree, our arguments can be carried out with standard finite-dimensional convex duality arguments. In the case of a general probability space we are confronted with infinite-dimensional spaces and need conditional analysis results from Cheridito et al. (2011). If an equilibrium exists and in addition, at least one agent has differentiable preferences and open constraints, we show that equilibrium prices are unique. To show uniqueness of the agents' optimal trading gains one needs strict convexity assumptions on the preferences. In the special case where all agents have preferences of the same type, for instance, expected exponential utility with different risk aversions, and at the same time, all random endowments are spanned by attainable trading gains, we show that a one-fund theorem holds. Under an additional differentiability assumption on the preferences, the equilibrium pricing kernel can be given in explicit form. If there are exogenous assets, the pricing kernel contains optimal trading gains from investing in them. Otherwise, similar to the standard CAPM, it just consists of the gradient of the base preference functional at the point corresponding to the sum of aggregate endowment and total supply of the financial assets. As an example we study the effects of stochastic volatility, demand pressure and short-selling constraints on prices of options on single stocks and indexes.

If the underlying noise is generated by finitely many Bernoulli random walks we show how equilibrium prices and optimal strategies can be obtained by solving a system of coupled BSΔEs (backward stochastic difference equations). This part of our work is related to Dumas and Lyasoff (2009), where under the assumption that an incomplete-market equilibrium exists, a method is developed to recursively compute it from first order conditions. In the continuous-time limit our system of BSΔEs becomes a multi-dimensional BSDE (backward stochastic differential equation). If the market turns out to be complete in equilibrium, both our systems of BSΔEs and BSDEs decouple. Conditions that guarantee market completeness in equilibrium have been studied in various frameworks; see for instance, Magill and Shafer (1990), Horst and Müller (2007), Anderson and Raimondo (2009) or Horst et al. (2010). However, if the market is incomplete in equilibrium, our equations do not decouple, and to solve it one has to keep track of the prices and the continuation values of all agents.

The remainder of the paper is organized as follows. In Section 2 we introduce the ingredients for our model together with the notation. In Section 3 we give a convex dual characterization of equilibrium and use it to show existence. In Section 4 we prove uniqueness of equilibrium prices if preferences are differentiable and uniqueness of optimal wealth dynamics if they satisfy a strict convexity property. Section 5 provides a one-fund theorem for the special case where agents have preferences of the same type and random endowments are replicable by trading in the financial market. As an application we discuss the effects of stochastic volatility, demand pressure and short-selling constraints on option prices. In Section 6 we assume that the noise is generated by finitely many Bernoulli random walks and characterize the equilibrium dynamics in terms of a coupled system of BSΔEs. All proofs are given in the appendix.

2 Notation and setup

We consider a finite group of agents $\mathbb{A}$ who trade in a financial market. Time is discrete and runs through the set $\{0, 1, ..., T\}$. Uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The flow of information is described by a filtration $(\mathcal{F}_t)_{t=0}^T$. We assume that at time t, all agents have access to the information represented by $\mathcal{F}_t$ and all events in $\mathcal{F}_0$ have probability 0 or 1. $\mathbb{P}$ is a reference probability measures that does not necessarily reflect the beliefs of our agents. But we assume they all agree that an event $A \in \mathcal{F}$ is impossible if $\mathbb{P}[A] = 0$. $L^0(\mathcal{F}_t)$ denotes the set of all $\mathcal{F}_t$-measurable random variables and $L^\infty(\mathcal{F}_t)$ the set of essentially bounded random variables, where random variables are identified if they are equal $\mathbb{P}$-almost surely. Accordingly, all equalities and inequalities between random variables will be understood in the $\mathbb{P}$-almost sure sense. Expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. Notation for expectations with respect to other probability measures will be introduced where it is needed. Each agent $a \in \mathbb{A}$ is initially endowed with an uncertain payoff $H^a \in L^0(\mathcal{F}_T)$ that is bounded from below. In the special case where the sample space Ω is finite, all random variables are bounded and the filtration $(\mathcal{F}_t)$ can be thought of as an event tree.

2.1 The financial market

All agents can lend funds to and borrow from a money market account at the same exogenously given interest rate and invest in a financial market consisting of $J + K$ assets. We use the money market as numeraire, that is, all prices will be expressed in terms of the value of one dollar invested in the money market at time 0. The prices of the first J assets are exogenously given by a J-dimensional bounded adapted process $(R_t)_{t=0}^T$. Our agents can buy and sell arbitrary quantities of them without influencing their prices. The prices of the other K assets will be determined endogenously by supply and demand. The k-th of them exists in net supply $n^k \in \mathbb{R}$ and yields a payoff of $S^k \in L^\infty(\mathcal{F}_T)$ per share at the final time T. Our goal is to find equilibrium price processes $(S_t^k)_{t=0}^T$ satisfying the terminal conditions $S_T^k = S^k$ together with optimal investment strategies for all agents $a \in \mathbb{A}$. By $n \in \mathbb{R}^K$ we denote the vector with components n^k and (S_t) is the K-dimensional process with components (S_t^k). In the special case $J = 0$, the prices of all assets are determined by supply and demand. The possibility to include exogenously given assets in the model is helpful for the study of derivatives and structured products. For instance a weather derivative might only be traded by an insurance company and a few end users. They can also invest in large cap stocks. But while their demands will determine the price of the weather derivative, they are too small to influence the stock prices.

A *trading strategy* for agent $a \in \mathbb{A}$ is given by an $\mathbb{R}^{J+K}$-valued predictable stochastic process $(\vartheta_t^a)_{t=1}^T$, that is, ϑ_t^a is measurable with respect to $\mathcal{F}_{t-1}$. By $\vartheta_t^{a,R}$ we denote the first J components of ϑ_t^a. They describe how many shares of the assets $R^1, \dots, R^J$ agent a is holding from time $t-1$ to t. $\vartheta_t^{a,S}$ are the remaining K components of ϑ_t^a and model agent a's investments in the assets $S^1, \dots, S^K$. The part of agent a's time $t-1$ wealth not invested in the financial assets is kept in the money market account. Since all prices are expressed in discounted terms, investments in the money market do not change their value, and investor a's investment gains from time $t-1$ to t are given by

$$\vartheta_t^{a,R} \cdot \Delta R_t + \vartheta_t^{a,S} \cdot \Delta S_t := \sum_{j=1}^J \vartheta_t^{a,R,j} \Delta R_t^j + \sum_{k=1}^K \vartheta_t^{a,S,k} \Delta S_t^k,$$

where we denote $\Delta R_t^j := R_t^j - R_{t-1}^j$ and $\Delta S_t^k := S_t^k - S_{t-1}^k$. We suppose there is no consumption or infusion of funds at intermediate times. So a strategy $(\vartheta_t^a)_{t=1}^T$ leads to a time T wealth of

$$H^a + \sum_{t=1}^T \vartheta_t^{a,R} \cdot \Delta R_t + \vartheta_t^{a,S} \cdot \Delta S_t.$$

We assume that the R-assets satisfy the following no-arbitrage condition:

(NA) No arbitrage in the R-assets: For every predictable trading strategy $(\vartheta_t)_{t=1}^T$ in the R-assets one has that

$$\sum_{t=1}^T \vartheta_t \cdot \Delta R_t \geq 0 \quad \text{implies} \quad \sum_{t=1}^T \vartheta_t \cdot \Delta R_t = 0.$$

By the Dalang–Morton–Willinger theorem (see Dalang et al., 1990) this is equivalent to the existence of a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $R_t^j = \mathbb{E}_{\mathbb{Q}}\left[R_T^j \mid \mathcal{F}_t\right]$ for all j and t.

2.2 Trading constraints

We suppose that our agents face trading constraints described by sets C_{t+1}^a of admissible one-step strategies $\vartheta_{t+1} \in L^0(\mathcal{F}_t)^{J+K}$ satisfying the following two conditions:

(C1) There exist strategies $\bar{\vartheta}_{t+1}^a \in C_{t+1}^a \cap L^\infty(\mathcal{F}_t)^{J+K}$ such that $\sum_{a \in \mathbb{A}} \bar{\vartheta}_{t+1}^{a,S} = n$
(C2) $\lambda \vartheta_{t+1} + (1-\lambda)\vartheta'_{t+1} \in C_{t+1}^a$ for all $\vartheta_{t+1}, \vartheta'_{t+1} \in C_{t+1}^a$ and $\lambda \in L^0(\mathcal{F}_t)$ such that $0 \leq \lambda \leq 1$.

Condition (C1) guarantees that there exists at least one admissible trading strategy for each agent such that aggregate demand is equal to supply. For example, condition (C1) is fulfilled if the S-assets exist in zero net supply and for all agents it is admissible to just keep their funds in the money market account. (C2) is a conditional convexity condition which will be needed in our proof that an equilibrium exists. In the case $C_{t+1}^a = L^0(\mathcal{F}_t)^{J+K}$, we say that agent a is unconstrained at time t.

For $x \in L^0(\mathcal{F}_t)^{J+K}$, we set

$$||x||_{\mathcal{F}_t} = \left(\sum_{i=1}^{J+K} (x^i)^2\right)^{1/2}$$

and say C_{t+1}^a is $\mathcal{F}_t$-bounded if there exists an $\mathcal{F}_t$-measurable random variable Y such that $||x||_{\mathcal{F}_t} \leq Y$ for all $x \in C_{t+1}^a$. Similarly, we call C_{t+1}^a $\mathcal{F}_t$-open if for every $x \in C_{t+1}^a$ there exists an $\mathcal{F}_t$-measurable random variable $\varepsilon > 0$ such that $x' \in C_{t+1}^a$ for all $x' \in L^0(\mathcal{F}_t)^{J+K}$ satisfying $||x' - x||_{\mathcal{F}_t} \leq \varepsilon$. We say C_{t+1}^a is sequentially closed if it contains every x that is an almost sure limit of a sequence of elements in C_{t+1}^a.[1]

[1]Note that sequentially closed sets are not complements of $\mathcal{F}_t$-open sets.

2.3 Translation-invariant preferences

Agent a's goal at time $t \in \{0, ..., T\}$ is to invest in the financial market so as to optimize a preference functional

$$U_t^a : L^0(\mathcal{F}_T) \to \underline{L}^0(\mathcal{F}_t),$$

where $\underline{L}^0(\mathcal{F}_t)$ denotes the set of all $\mathcal{F}_t$-measurable random variables with values in $\mathbb{R} \cup \{-\infty\}$. Usually, preference functionals take values in $\mathbb{R}$. But our agents update their preferences as they learn about information contained in $\mathcal{F}_t$. So their utilities at time t are $\mathcal{F}_t$-measurable, and allowing U_t^a to take values in $\underline{L}^0(\mathcal{F}_t)$ instead of $L^0(\mathcal{F}_t)$ allows for more interesting examples; see Examples 2.1 below. We will also need the larger sets $\bar{L}^0(\mathcal{F}_t)$ of $\mathcal{F}_t$-measurable random variables with values in $\mathbb{R} \cup \{\pm\infty\}$.

We assume that U_t^a has the following properties:

(N) Normalization: $U_t^a(0) = 0$

(M) Monotonicity: $U_t^a(X) \geq U_t^a(Y)$ for all $X, Y \in L^0(\mathcal{F}_T)$ such that $X \geq Y$

(C) $\mathcal{F}_t$-Concavity: $U_t^a(\lambda X + (1-\lambda)Y) \geq \lambda U_t^a(X) + (1-\lambda) U_t^a(Y)$ for all $X, Y \in L^0(\mathcal{F}_T)$ and $\lambda \in L^0(\mathcal{F}_t)$ such that $0 \leq \lambda \leq 1$, where $0(-\infty)$ is understood to be 0

(T) Translation property: $U_t^a(X + Y) = U_t^a(X) + Y$ for all $X \in L^0(\mathcal{F}_T)$ and $Y \in L^0(\mathcal{F}_t)$

Every preference functional $U_t : L^0(\mathcal{F}_T) \to \underline{L}^0(\mathcal{F}_t)$ satisfying $U_t(0) \in L^0(\mathcal{F}_t)$ can be normalized without changing the preference order by passing to $U_t(X) - U_t(0)$. So one can assume (N) without loss of generality as soon as $U_t^a(X) > -\infty$. The monotonicity assumption (M) is standard. It just means that more is preferred to less. Condition (C) is an extension of ordinary concavity to a situation where agents make decisions based on the information contained in $\mathcal{F}_t$.[2] Condition (T) means that our preference orders are invariant under a shift of random payoffs by certain amounts of cash. We need this assumption in our proof that an equilibrium exists. It is for instance satisfied by the certainty equivalent of expected exponential utility or mean-risk type preferences, and it covers the case of professional investors which maximize expectation under constraints on the amount of risk they are allowed to take; specific cases of preference functionals with the translation property (T) are discussed in Example 2.1 below.

Note that it follows from condition (C) that U_t^a has the following local property:

$$1_A U_t^a(X) = 1_A U_t^a(Y) \quad \text{for all } X, Y \in L^0(\mathcal{F}_T) \text{ and } A \in \mathcal{F}_t \text{ such that } 1_A X = 1_A Y. \tag{2.1}$$

Indeed, due to (C), one has $1_A U_t^a(X) = 1_A U_t^a(1_A Y + 1_{A^c} X) \geq 1_A U_t^a(Y)$ and by symmetry, $1_A U_t^a(X) \leq 1_A U_t^a(Y)$. That is, in the event A, the utility $U_t(X)$ only depends on values X can attain in states of the world contained in A.

In addition to (N), (M), (C) and (T) we also assume that the preferences are time-consistent in the following sense:

(TC) Time-consistency: For all $X, Y \in L^0(\mathcal{F}_T)$ and $t = 0, \ldots, T-1$,

$$U_{t+1}^a(X) \geq U_{t+1}^a(Y) \quad \text{implies} \quad U_t^a(X) \geq U_t^a(Y). \tag{2.2}$$

By (N) and (T) one has $U_{t+1}^a(U_{t+1}^a(X)) = U_{t+1}^a(X)$ for all random variables X belonging to the domain

$$\operatorname{dom} U_{t+1}^a := \left\{ X \in L^0(\mathcal{F}_T) : U_{t+1}^a(X) \in L^0(\mathcal{F}_{t+1}) \right\}.$$

Applying (2.2) to the random variable $Y = U_{t+1}^a(X)$ shows that time-consistency implies the following recursive structure of the preference functionals:

$$U_t^a(X) = U_t^a(U_{t+1}^a(X)) \quad \text{for all} \quad t = 0, ..., T-1 \text{ and } X \in \operatorname{dom} U_{t+1}^a. \tag{2.3}$$

For some of the results in this paper we will also need the preferences to satisfy one or more of the following conditions:

[2] Convex preferences correspond to quasi-concave preference functionals. However, quasi-concavity and the translation property (T) imply concavity; see Lemma 2.1 in Cheridito and Kupper (2009).

(SL) Sensitivity to large losses: $\lim_{\lambda\to\infty} U_0^a(\lambda X) = -\infty$ for all $X \in L^0(\mathcal{F}_T)$ with the property $\mathbb{P}[X < 0] > 0$.

(SM) Strict monotonicity: $U_0^a(X) > U_0^a(Y)$ for all $X, Y \in \operatorname{dom} U_0^a$ such that $X \geq Y$ and $\mathbb{P}[X > Y] > 0$.

(SC) Strict concavity modulo translation: $U_0^a(\lambda X + (1-\lambda)Y) > \lambda U_0^a(X) + (1-\lambda)U_0^a(Y)$ for all $\lambda \in \mathbb{R}$ with $0 < \lambda < 1$ and $X, Y \in \operatorname{dom} U_0^a$ such that $X - Y$ is not constant.

For example, we will prove that an equilibrium exists if all agents have sequentially closed trading constraints and either are sensitive to large losses or at every time t, the their constraints are $\mathcal{F}_t$-bounded. Furthermore, we will show that if the market is in equilibrium and at least one agent has $\mathcal{F}_t$-open constraints and strictly monotone preferences, then there exists an equilibrium pricing measure $\mathbb{Q}$ equivalent to $\mathbb{P}$. In Section 4 we show that equilibrium prices are unique if at least one agent has differentiable preferences and $\mathcal{F}_t$-open trading constraints. Moreover, we show that in equilibrium the optimal trading gains of all agents satisfying (SC) are unique. Note that since the functionals U_0^a have the translation property (T), they cannot be strictly concave under translations by constants. But condition (SC) will be sufficient for our purposes.

Examples 2.1

1. Entropic preference functionals

The standard example of preference functionals satisfying (N), (M), (C), (T), (TC) is given by the conditional certainty equivalents of expected exponential utility, also called entropic preference functionals:

$$U_t^a(X) = -\frac{1}{\gamma} \log \mathbb{E}\left[\exp(-\gamma X) \mid \mathcal{F}_t\right] \quad \text{for a constant } \gamma > 0. \tag{2.4}$$

They induce the same preferences as the conditional expected exponential utilities $\mathbb{E}\left[\exp(-\gamma X) \mid \mathcal{F}_t\right]$. But only in the form (2.4) do they have the translation property (T). U_0^a also satisfies (SL), (SM) and (SC). (SL) and (SM) are obvious. (SC) follows from Theorem 5.3 in Cheridito and Li (2009).

2. Pasting together one-step preference functionals

A general method of constructing time-consistent preference functionals in discrete time is by pasting together one-step preference functionals. Assume, for instance, that

$$v_t : L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t), \quad t = 0, \dots, T-1,$$

are mappings with the properties (N), (M), (C) and (T) such that the extensions

$$V_t(X) = \lim_{n\to\infty} \lim_{m\to-\infty} v_t((X \wedge n) \vee m) \text{ map } \underline{L}^0(\mathcal{F}_{t+1}) \text{ to } \underline{L}^0(\mathcal{F}_t). \tag{2.5}$$

(general conditions for this to be true are given in Cheridito et al., 2006). In Example 5 below we provide a wide class of functionals for which it can be shown directly that (2.5) holds. Then the compositions

$$U_t(X) = V_t \circ \cdots \circ V_{T-1}(X), \quad X \in L^0(\mathcal{F}_T) \tag{2.6}$$

inherit (N), (M), (C), (T) and are automatically time-consistent.

In the sequel we give some specific examples of one-step preference functionals $v_t : L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t)$.

3. Monotone mean-variance preferences

Standard conditional mean variance

$$\mathrm{MV}_t^\lambda(X) = \mathbb{E}\left[X \mid \mathcal{F}_t\right] - \frac{\lambda}{2}\mathrm{Var}(X \mid \mathcal{F}_t)$$

fulfills (N), (C), (T) but not the monotonicity property (M); see for instance, Maccheroni et al. (2009). This can be corrected by slightly modifying its dual representation. For $X \in L^\infty(\mathcal{F}_1)$, $\mathrm{MV}_0^\lambda(X)$ has a dual representation of the form

$$\mathrm{MV}_0^\lambda(X) = \inf_{\xi \in \mathcal{E}_1} \mathbb{E}\left[X\xi + G^\lambda(\xi) \mid \mathcal{F}_t\right],$$

where

$$\mathcal{E}_1 = \left\{\xi \in L^1(\mathcal{F}_1) : \mathbb{E}[\xi] = 1\right\} \quad \text{and} \qquad G^\lambda(x) = \frac{1}{2\lambda}(x-1)^2.$$

This extends to

$$\mathrm{MV}_t^\lambda(X) = \operatorname*{ess\,inf}_{\xi \in \mathcal{E}_{t+1}} \mathbb{E}\left[X\xi + G^\lambda(\xi) \mid \mathcal{F}_t\right], \quad X \in L^\infty(\mathcal{F}_{t+1}), \tag{2.7}$$

where

$$\mathcal{E}_{t+1} = \left\{\xi \in L^1(\mathcal{F}_{t+1}) : \mathbb{E}[\xi \mid \mathcal{F}_t] = 1\right\}$$

and ess inf denotes the largest lower bound of a family of random variables with respect to the $\mathbb{P}$-almost sure order; see for instance, Proposition VI.1.1 of Neveu (1975). If one modifies (2.7) to

$$v_t(X) = \operatorname*{ess\,inf}_{\xi \in \mathcal{D}_{t+1}} \mathbb{E}\left[X\xi + G^\lambda(\xi) \mid \mathcal{F}_t\right], \tag{2.8}$$

for

$$\mathcal{D}_{t+1} = \left\{\xi \in L^1(\mathcal{F}_{t+1}) : \xi \geq 0, \mathbb{E}[\xi \mid \mathcal{F}_t] = 1\right\},$$

one obtains one-step preference functionals satisfying (N), (M), (C), (T). They belong to the class of divergence utilities, which are shown to satisfy condition (2.5) in Example 5 below.

4. Mean-risk preferences

Instead of modifying mean-variance as in (2.8), one can also replace the variance term by a convex risk measure and set

$$v_t(X) = \lambda \mathbb{E}[X \mid \mathcal{F}_t] - (1-\lambda)\rho_t(X), \tag{2.9}$$

where λ is a number in $(0,1)$ and

$$\rho_t : L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t)$$

a normalized conditional convex risk measure, that is, $-\rho_t$ satisfies (N), (M), (C) and (T); see Föllmer and Schied (2004) for an introduction to convex risk measures in a static framework and Cheridito and Kupper (2009) for dynamic risk measures. Whether condition (2.5) holds depends on ρ.

5. Divergence utilities

The monotone mean-variance preference functional (2.8) can be generalized by replacing the function G^λ with a more general divergence function $G : \mathbb{R}_+ \to \mathbb{R}$ such that $\operatorname{ess\,inf}_{\xi \in \mathcal{D}_{t+1}} \mathbb{E}[G(\xi) \mid \mathcal{F}_t] = 0$. Then

$$v_t(X) = \operatorname*{ess\,inf}_{\xi \in \mathcal{D}_{t+1}} \mathbb{E}[X\xi + G(\xi) \mid \mathcal{F}_t], \tag{2.10}$$

has all the properties (N), (M), (C), (T). For suitable functions $H : \mathbb{R} \to \mathbb{R}$, optimized certainty equivalents

$$\operatorname*{ess\,sup}_{s \in \mathbb{R}} \{s - \mathbb{E}[H(s - X) \mid \mathcal{F}_t]\} \tag{2.11}$$

are of this form (ess sup denotes the least upper bound of a family of random variables in the $\mathbb{P}$-almost sure order). For instance, if H is increasing and convex such that $\max_{x \in \mathbb{R}}(x - H(x)) = 0$, then (2.11) is of the form (2.10) with

$$G(y) = H^*(y) = \sup_{x \in \mathbb{R}} \{xy - H(x)\}.$$

$\mathbb{E}[G(\xi)]$ is an f-divergence after Csiszar (1967). In the special case $G(x) = \frac{1}{\lambda} x \log(x)$ it is relative entropy and v_t becomes the entropic preference functional of Example 2.1.1. Unconditional functionals of the form (2.10) and (2.11) have, for instance, been studied in the papers Ben-Tal and Teboulle (1987), Cheridito and Li (2009) or Cherny and Kupper (2009). It is shown in the appendix that the extensions

$$V_t(X) = \lim_{n \to \infty} \lim_{m \to -\infty} v_t((X \wedge n) \vee m) \quad \text{satisfy (2.5)} \tag{2.12}$$

and that

$$U_0 = V_0 \circ \cdots \circ V_{T-1} \text{ is sensitive to large losses.} \tag{2.13}$$

2.4 Definition of equilibrium

At every time $t = 0, 1..., T-1$ the goal of each agent $a \in \mathbb{A}$ is to invest in such a way that the utility of final wealth becomes maximal. Having invested according to some trading strategy $\vartheta_1^a, ..., \vartheta_t^a$ up to time t the agent's optimization problem is given by

$$\operatorname*{ess\,sup}_{\vartheta_s^a \in C_s^a\,,\, s \geq t+1} U_t^a \left(H^a + \sum_{s=1}^{T} \vartheta_s^{a,R} \cdot \Delta R_s + \vartheta_s^{a,S} \cdot \Delta S_s \right). \tag{2.14}$$

But since U_t^a has the translation property (T), $U_t^a \left(H^a + \sum_{s=1}^{T} \vartheta_s^{a,R} \cdot \Delta R_s + \vartheta_s^{a,S} \cdot \Delta S_s \right)$ can be written as

$$U_t^a \left(H^a + \sum_{s=t+1}^{T} \vartheta_s^{a,R} \cdot \Delta R_s + \vartheta_s^{a,S} \cdot \Delta S_s \right) + \sum_{s=1}^{t} \vartheta_s^{a,R} \cdot \Delta R_s + \vartheta_s^{a,S} \cdot \Delta S_s.$$

So (2.14) is equivalent to

$$\operatorname*{ess\,sup}_{\vartheta_s^a \in C_s^a\,,\, s \geq t+1} U_t^a \left(H^a + \sum_{s=t+1}^{T} \vartheta_s^{a,R} \cdot \Delta R_s + \vartheta_s^{a,S} \cdot \Delta S_s \right). \tag{2.15}$$

We say the market is in equilibrium if the following holds:

Definition 2.2 *An equilibrium consists of a bounded, $\mathbb{R}^K$-valued, adapted process $(S_t)_{t=0}^T$ satisfying the terminal condition $S_T = S$ together with admissible trading strategies $(\hat{\vartheta}_t^a)_{t=1}^T$ for all agents $a \in \mathbb{A}$, such that the following two conditions hold:*

(i) Individual optimality

$$U_t^a \left(H^a + \sum_{s=t+1}^{T} \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s \right) \geq U_t^a \left(H^a + \sum_{s=t+1}^{T} \vartheta_s^{a,R} \cdot \Delta R_s + \vartheta_s^{a,S} \cdot \Delta S_s \right)$$

for all $t = 0, \ldots, T-1$ and admissible continuation strategies $(\vartheta_s^a)_{s=t+1}^T$.

(ii) Market clearing $\quad \sum_{a \in \mathbb{A}} \hat{\vartheta}_t^{a,S} = n \quad$ *for all* $\quad t = 1, \ldots, T.$

Remark 2.3 If U_0^a is strictly monotone, then individual optimality at all times t follows from the time 0 optimality condition

$$U_0^a \left(H^a + \sum_{s=1}^{T} \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s \right) \geq U_0^a \left(H^a + \sum_{s=1}^{T} \vartheta_s^{a,R} \cdot \Delta R_s + \vartheta_s^{a,S} \cdot \Delta S_s \right) \tag{2.16}$$

for all admissible strategies $(\vartheta^a_s)_{s=1}^T$. Indeed, let us assume to the contrary that (2.16) holds but there exist $t \geq 1$ and an admissible continuation strategy $(\vartheta^a_s)_{s=t+1}^T$ such that

$$U^a_t\left(H^a + \sum_{s=t+1}^T \hat{\vartheta}^{a,R}_s \cdot \Delta R_s + \hat{\vartheta}^{a,S}_s \cdot \Delta S_s\right) < U^a_t\left(H^a + \sum_{s=t+1}^T \vartheta^{a,R}_s \cdot \Delta R_s + \vartheta^{a,S}_s \cdot \Delta S_s\right)$$

on an $\mathcal{F}_t$-measurable set A with $\mathbb{P}[A] > 0$. Then by time-consistency, one has

$$\begin{aligned}
& U^a_0\left(H^a + \sum_{s=1}^T \hat{\vartheta}^{a,R}_s \cdot \Delta R_s + \hat{\vartheta}^{a,S}_s \cdot \Delta S_s\right) \\
= \;& U^a_0\left(\sum_{s=1}^t \hat{\vartheta}^{a,R}_s \cdot \Delta R_s + \hat{\vartheta}^{a,S}_s \cdot \Delta S_s + U^a_t\left(H^a + \sum_{s=t+1}^T \hat{\vartheta}^{a,R}_s \cdot \Delta R_s + \hat{\vartheta}^{a,S}_s \cdot \Delta S_s\right)\right) \\
< \;& U^a_0\left(\sum_{s=1}^t \hat{\vartheta}^{a,R}_s \cdot \Delta R_s + \hat{\vartheta}^{a,S}_s \cdot \Delta S_s + U^a_t\left(H^a + \sum_{s=t+1}^T \tilde{\vartheta}^{a,R}_s \cdot \Delta R_s + \tilde{\vartheta}^{a,S}_s \cdot \Delta S_s\right)\right) \\
= \;& U^a_0\left(H^a + \sum_{s=1}^t \hat{\vartheta}^{a,R}_s \cdot \Delta R_s + \hat{\vartheta}^{a,S}_s \cdot \Delta S_s + \sum_{s=t+1}^T \tilde{\vartheta}^{a,R}_s \cdot \Delta R_s + \tilde{\vartheta}^{a,S}_s \cdot \Delta S_s\right)
\end{aligned}$$

for the admissible strategies $\tilde{\vartheta}^a_s = 1_{A^c}\hat{\vartheta}^a_s + 1_A \vartheta^a_s$, $s = t+1, \ldots, T$. But this contradicts (2.16).

3 Dual characterization and existence of equilibrium

In complete markets, competitive equilibria are Pareto optimal and every Pareto optimal allocation can be supported as an equilibrium by constructing a suitable representative agent. A standard way of proving existence of an equilibrium in complete markets is therefore via a representative agent. The specific choice of the representative agent typically depends on the equilibrium to be supported and the proof involves complex fixed-point arguments. However, for translation invariant preferences the situation turns out to be simpler, and we will be able to take a representative agent approach to construct an equilibrium step by step going backwards in time even if markets are incomplete. Assume that equilibrium prices $S_{t+1}, ..., S_T$ and admissible trading strategies $\hat{\vartheta}^a_{t+2}, ..., \hat{\vartheta}^a_T$ for all agents $a \in \mathbb{A}$ have already been determined such that the components of $S_{t+1}, ..., S_T$ are bounded. We then define the continuation value of agent $a \in \mathbb{A}$ at time $t+1$ by

$$H^a_{t+1} = U^a_{t+1}\left(H^a + \sum_{s=t+2}^T \hat{\vartheta}^{a,R}_s \cdot \Delta R_s + \hat{\vartheta}^{a,S}_s \cdot \Delta S_s\right) \quad \text{and} \quad H^a_T = H^a. \tag{3.1}$$

Since we assumed H^a to be bounded from below and there exist bounded admissible one-step strategies $\bar{\vartheta}^a_s$, $s = t+2, \ldots, T$, there is a constant $c \in \mathbb{R}$ such that

$$H^a_{t+1} = U^a_{t+1}\left(H^a + \sum_{s=t+2}^T \hat{\vartheta}^{a,R}_s \cdot \Delta R_s + \hat{\vartheta}^{a,S}_s \cdot \Delta S_s\right) \geq U^a_{t+1}\left(H^a + \sum_{s=t+2}^T \bar{\vartheta}^{a,R}_s \cdot \Delta R_s + \bar{\vartheta}^{a,S}_s \cdot \Delta S_s\right) \geq c.$$

In particular, H^a_{t+1} belongs to $L^0(\mathcal{F}_t)$ and the following recursive relation holds:

$$\begin{aligned}
H^a_{t+1} &= U^a_{t+1}\left(U^a_{t+2}\left(H^a + \sum_{s=t+3}^T \hat{\vartheta}^{a,R}_s \cdot \Delta R_s + \hat{\vartheta}^{a,S}_s \cdot \Delta S_s\right) + \hat{\vartheta}^{a,R}_{t+2} \cdot \Delta R_{t+2} + \hat{\vartheta}^{a,S}_{t+2} \cdot \Delta S_{t+2}\right) \\
&= U^a_{t+1}\left(H^a_{t+2} + \hat{\vartheta}^{a,R}_{t+2} \cdot \Delta R_{t+2} + \hat{\vartheta}^{a,S}_{t+2} \cdot \Delta S_{t+2}\right).
\end{aligned}$$

The usual approach of defining a representative agent in a complete market framework would be to pool all available resources and redistribute them in a socially optimal manner. But in our model the agents cannot pool and redistribute resources arbitrarily. They can only exchange their risk exposures by trading the financial assets. In addition, they are subject to trading constraints. To account for that we construct a one-step representative agent at time t with preferences over one-step gains that can be realized by taking admissible positions in the financial market. This will allow us to construct equilibrium prices S_t, continuation values H^a_t and optimal strategies $\hat{\vartheta}^a_{t+1}$ recursively from S_{t+1} and H^a_{t+1}. Observe that agent a's time-t utility from investing according to an admissible one-step trading strategy $\vartheta^a_{t+1} \in C^a_{t+1}$ would be

$$U^a_t(H^a_{t+1} + \vartheta^{a,R}_{t+1} \cdot \Delta R_{t+1} + \vartheta^{a,S}_{t+1} \cdot \Delta S_{t+1}).$$

We want to extend this to any $\vartheta^a_{t+1} \in L^0(\mathcal{F}_t)^{J+K}$ by setting it equal to $-\infty$ for those $\omega \in \Omega$ where the trading constraints are violated. This can be done in an $\mathcal{F}_t$-measurable way by defining

$$\tilde{u}^a_t(\vartheta^a_{t+1}) := \begin{cases} U^a_t(H^a_{t+1} + \vartheta^{a,R}_{t+1} \cdot \Delta R_{t+1} + \vartheta^{a,S}_{t+1} \cdot \Delta S_{t+1}) & \text{on the set } \left\{c^a_t(\vartheta^a_{t+1}) = 1\right\} \\ -\infty & \text{on the set } \left\{c^a_t(\vartheta^a_{t+1}) = 0\right\} \end{cases}, \tag{3.2}$$

where $c^a_t(\vartheta^a_{t+1})$ is the $\mathcal{F}_t$-measurable function $c^a_t(\vartheta^a_{t+1}) = \operatorname{ess\,sup}_{\vartheta_{t+1} \in C^a_{t+1}} 1_{\left\{\vartheta_{t+1} = \vartheta^a_{t+1}\right\}}$. However, at this point, S_t is not known yet. So we replace the increment ΔS_{t+1} in (3.2) by S_{t+1} and define the mapping $u^a_t : L^0(\mathcal{F}_t)^{J+K} \to \underline{L}^0(\mathcal{F}_t)$ by

$$u^a_t(\vartheta^a_{t+1}) := \begin{cases} U^a_t(H^a_{t+1} + \vartheta^{a,R}_{t+1} \cdot \Delta R_{t+1} + \vartheta^{a,S}_{t+1} \cdot S_{t+1}) & \text{on the set } \left\{c^a_t(\vartheta^a_{t+1}) = 1\right\} \\ -\infty & \text{on the set } \left\{c^a_t(\vartheta^a_{t+1}) = 0\right\} \end{cases}. \tag{3.3}$$

The role of the one-step representative agent at time t will be played by the conditional sup-convolution

$$\hat{u}_t(x) = \operatorname*{ess\,sup}_{\substack{\vartheta^a_{t+1} \in L^0(\mathcal{F}_t)^{J+K} \\ \sum_{a \in \mathbb{A}} \vartheta^{a,S}_{t+1} = x}} \sum_{a \in \mathbb{A}} u^a_t\left(\vartheta^a_{t+1}\right), \quad x \in L^0(\mathcal{F}_t)^K. \tag{3.4}$$

Note that if time-t equilibrium prices S_t exist, the replacement of (3.2) by (3.3) just results in a shift of $\hat{u}_t(x)$ by $x \cdot S_t$, which changes the marginal utilities of the representative agent by S_t. If $\infty - \infty$ is understood as $-\infty$, the mapping $\hat{u}_t : L^0(\mathcal{F}_t)^K \to \bar{L}^0(\mathcal{F}_t)$ is $\mathcal{F}_t$-concave. Moreover, since the preference functionals of all agents have the local property (2.1), one has

$$1_A \hat{u}_t(x) = 1_A \hat{u}_t(y) \quad \text{for all } x, y \in L^0(\mathcal{F}_t)^K \text{ and } A \in \mathcal{F}_t \text{ such that } 1_A x = 1_A y.$$

We define the *conditional concave conjugate* $\hat{u}^*_t : L^0(\mathcal{F}_t)^K \to \underline{L}^0(\mathcal{F}_t)$ by

$$\hat{u}^*_t(y) = \operatorname*{ess\,inf}_{x \in L^0(\mathcal{F}_t)^K} \{x \cdot y - \hat{u}_t(x)\}$$

and call a random vector $y \in L^0(\mathcal{F}_t)^K$ a *conditional supergradient* of $\hat{u}_t$ at $x \in L^0(\mathcal{F}_t)^K$ if

$$\hat{u}_t(x) \in L^0(\mathcal{F}_t) \quad \text{and} \quad \hat{u}_t(x+z) - \hat{u}_t(x) \le z \cdot y \text{ for all } z \in L^0(\mathcal{F}_t)^K.$$

The *conditional superdifferential* $\partial \hat{u}_t(x)$ is the set of all conditional supergradients of $\hat{u}_t$ at x. As in standard convex analysis, one has

$$\hat{u}_t(x) + \hat{u}^*_t(y) \le x \cdot y \quad \text{for all } x, y \in L^0(\mathcal{F}_t)^K$$

with equality if and only if $y \in \partial \hat{u}_t(x)$. With this notation we now are ready to give a dual characterization of equilibrium:

Theorem 3.1 *A bounded $\mathbb{R}^K$-valued adapted process $(S_t)_{t=0}^T$ satisfying $S_T = S$ together with admissible trading strategies $(\hat{\vartheta}^a_t)_{t=1}^T$ for all agents $a \in \mathbb{A}$ form an equilibrium if and only if for all times $t = 0, \ldots, T-1$ the following three conditions hold:*

(i) $S_t \in \partial \hat{u}_t(n)$

(ii) $\sum_{a \in \mathbb{A}} U_t^a(H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot S_{t+1}) = \hat{u}_t(n)$

(iii) $\sum_{a \in \mathbb{A}} \hat{\vartheta}_{t+1}^{a,S} = n.$

In particular, if (i)–(iii) *hold and* $\partial \hat{u}_t(n) = \{S_t\}$ *for all* $t = 0, \dots, T-1$*, then* $(S_t)_{t=0}^T$ *is the unique equilibrium price process.*

The characterization of equilibrium in Theorem 3.1 is reminiscent of the complete market case: equilibrium prices correspond to marginal utilities of the representative agent. However, the representative agent $\hat{u}_t$ is only defined in directions spanned by attainable one-step trading gains, and optimal trading strategies are only constrained Pareto optimal.

The following proposition shows that there exists an equilibrium pricing measure if at least one of the agents has strictly monotone preferences and open trading constraints.

Proposition 3.2 *If the market is in equilibrium and there exists at least one agent* $a \in \mathbb{A}$ *such that* U_0^a *is strictly monotone and* C_{t+1}^a *is* $\mathcal{F}_t$*-open for all* $t \leq T-1$*, then there exists a probability measure* $\mathbb{Q}$ *on* $(\Omega, \mathcal{F})$ *equivalent to* $\mathbb{P}$ *such that*

$$R_t = \mathbb{E}_{\mathbb{Q}}\left[R_T \mid \mathcal{F}_t\right] \quad \textit{and} \quad S_t = \mathbb{E}_{\mathbb{Q}}\left[S_T \mid \mathcal{F}_t\right] \quad \textit{for all } t = 0, \dots, T. \tag{3.5}$$

To ensure existence of an equilibrium one needs assumptions on the preferences and trading constraints which guarantee that at every time t, the one-step representative agent's utility is finite and attained. To motivate these assumptions, we give a simple example where an equilibrium does not exist.

Example 3.3 Assume the probability space contains only finitely many elements $\{\omega_1, \dots, \omega_N\}$, the time horizon is 1 and the preferences of the agents are given by expectations $U_0^a(.) = \mathbb{E}^a[.]$ corresponding to probability measures $\mathbb{P}^a$, $a \in \mathbb{A}$. If there exist agents $a, b \in \mathbb{A}$ with no trading constraints and a payoff S^k such that $\mathbb{E}^a[S^k] \neq \mathbb{E}^b[S^k]$, an equilibrium price for this payoff cannot exist. Indeed, no matter how one chooses the initial price $S_0^k \in \mathbb{R}$, at least one of the expectations $\mathbb{E}^a[S^k]$, $\mathbb{E}^b[S^k]$ is different from S_0^k. If for instance, $\mathbb{E}^a[S^k] \neq S_0^k$, then

$$\sup_{\vartheta^a \in \mathbb{R}^{J+K}} U_0^a(H^a + \vartheta^{a,R} \cdot \Delta R_1 + \vartheta^{a,S} \cdot \Delta S_1) \geq \mathbb{E}^a[H^a] + \sup_{\vartheta \in \mathbb{R}} \mathbb{E}^a[\vartheta \Delta S_1^k] = \infty,$$

and there exists no optimal trading strategy for agent a.

Of course, if in Example 3.3, all agents have preferences given by $\mathbb{E}_{\mathbb{Q}}[.]$ for the same probability measure $\mathbb{Q}$ and $R_0^j = \mathbb{E}_{\mathbb{Q}}\left[R_1^j\right]$ for all $j = 1, \dots, J$, then $S_0^k = \mathbb{E}_{\mathbb{Q}}\left[S^k\right]$, $k = 1, \dots, K$, are equilibrium prices for the S-assets, and one has

$$\sup_{\vartheta^a \in \mathbb{R}^{J+K}} \mathbb{E}_{\mathbb{Q}}\left[H^a + \vartheta^{a,R} \cdot \Delta R_1 + \vartheta^{a,S} \cdot \Delta S_1\right] = \mathbb{E}_{\mathbb{Q}}\left[H^a + \bar{\vartheta}^{a,R} \cdot \Delta R_1 + \bar{\vartheta}^{a,S} \cdot \Delta S_1\right] = \mathbb{E}_{\mathbb{Q}}\left[H^a\right]$$

for all $a \in \mathbb{A}$. That is, every trading strategy leads to the same utility, and all of them are optimal.

Another extreme case is when the agents have general preferences but for every agent a there exists only one admissible trading strategy $(\bar{\vartheta}_t^a)_{t=1}^T$. Then any process $(S_t)_{t=0}^T$ with bounded components together with $(\hat{\vartheta}_t^a)_{t=1}^T = (\bar{\vartheta}_t^a)_{t=1}^T$, $a \in \mathbb{A}$, forms an equilibrium.

In the following theorem we give a general existence result. We say that the trading constraints C_{t+1}^a factorize if they are of the form

$$C_{t+1}^a = D_{t+1}^{a,1} \times \dots \times D^{a,J} \times E_{t+1}^{a,1} \times \dots \times E_{t+1}^{a,K}$$

for non-empty $\mathcal{F}_t$-convex subsets $D_{t+1}^{a,1}, \dots, D_{t+1}^{a,J}$, $E_{t+1}^{a,1}, \dots, E_{t+1}^{a,K}$ of $L^0(\mathcal{F}_t)$.

Theorem 3.4 *Assume that for all $a \in \mathbb{A}$ and $t \leq T-1$, C^a_{t+1} factorizes and is sequentially closed. If there exists a (possibly empty) subset $\mathbb{A}' \subset \mathbb{A}$ such that*
(i) *U^a_0 is sensitive to large losses for all $a \in \mathbb{A}'$ and*
(ii) *C^a_{t+1} is $\mathcal{F}_t$-bounded for all $a \in \mathbb{A} \setminus \mathbb{A}'$ and $t \leq T-1$,*
then an equilibrium exists.

Condition (ii) in Theorem 3.4 amounts to a conditional compactness assumption. Similarly, it can be deduced from condition (i) that it is enough for agents to optimize over conditionally compact sets in the strategy space. A full proof of the theorem is technical. We give it in the appendix using conditional analysis results from Cheridito et al. (2011). Here we shortly sketch the argument in a simple two-period model: Assume $T = 1$, $J = 0$, $K = 1$, and for all $a \in \mathbb{A}$, $H^a = 0$ and C^a is a non-empty closed interval in $\mathbb{R}$ such that $n \in \sum_{a \in \mathbb{A}} C^a$. The mapping $\hat{u}_0 : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is then given by

$$\hat{u}_0(x) = \sup_{\substack{\vartheta^a \in C^a \\ \sum_a \vartheta^a = x}} U^a_0(\vartheta^a S) \quad \text{for } x \in \sum_a C^a \tag{3.6}$$

and

$$\hat{u}_0(x) = -\infty \quad \text{for } x \notin \sum_a C^a.$$

If it can be shown that for all $x \in \sum_a C^a$, the supremum in (3.6) is attained, $\hat{u}_0$ is a concave function from $\mathbb{R}$ to $\mathbb{R} \cup \{-\infty\}$ and conditions (i)–(iii) of Theorem 3.1 hold. So an equilibrium exists. But since all U^a_0 have the translation property (T), the supremum in (3.6) is attained if and only if for all $x \in \sum_a C^a$, the supremum

$$\sup_{\substack{\vartheta^a \in C^a \\ \sum_a \vartheta^a = x}} U^a_0(\vartheta^a (S - \mathbb{E}[S])) \tag{3.7}$$

is attained. In the special case $S = \mathbb{E}[S]$, this holds trivially. Otherwise, both sets $\{S - \mathbb{E}[S] > 0\}$ and $\{S - \mathbb{E}[S] < 0\}$ have positive probability. So if for each $a \in \mathbb{A}$, either U^a_0 is sensitive to large losses or C^a is bounded, then (3.7) is equal to

$$\sup_{\substack{\vartheta^a \in I^a \\ \sum_a \vartheta^a = x}} U^a_0(\vartheta^a (S - \mathbb{E}[S])) \tag{3.8}$$

for suitable compact intervals $I^a \subset \mathbb{R}$. Since S is bounded, the mapping $\vartheta^a \mapsto U^a_0(\vartheta^a(S - \mathbb{E}[S]))$ is continuous for every $a \in \mathbb{A}$, and it follows that the supremum (3.8) is attained.

4 Differentiable preferences and uniqueness of equilibrium

In this section we introduce a differentiability condition on the preferences and give conditions that guarantee uniqueness of equilibrium prices and optimal wealth dynamics. Condition (D) in the following definition is a conditional version of Gâteaux-differentiability.

Definition 4.1 *We say that a preference functional $U_t : L^0(\mathcal{F}_T) \to \underline{L}^0(\mathcal{F}_t)$ satisfies the differentiability condition* (D) *if for all $X \in \operatorname{dom} U_t$ there exists a random variable $Z \in L^1(\mathcal{F}_T)$ such that*

$$\lim_{m \to \infty} \frac{U_t(X + Y/m) - U_t(X)}{1/m} = \mathbb{E}[YZ \mid \mathcal{F}_t] \quad \textit{for all} \quad Y \in L^\infty(\mathcal{F}_T). \tag{4.1}$$

If such a random variable Z exists, it has to be unique and we denote it by $\nabla U_t(X)$. If for $X \in \operatorname{dom} U_t \cap L^0(\mathcal{F}_{t+1})$, there exists a $Z \in L^1(\mathcal{F}_{t+1})$ such that (4.1) holds for all $Y \in L^\infty(\mathcal{F}_{t+1})$, it is also unique and we denote it by $\tilde{\nabla} U_t(X)$.

If for some random variable $X \in L^0(\mathcal{F}_{t+1})$ the gradient $\nabla U_t(X)$ exists, then so does $\tilde{\nabla} U_t(X)$, and it is equal to

$$\tilde{\nabla} U_t(X) = \mathbb{E}\left[\nabla U_t(X) \mid \mathcal{F}_{t+1}\right].$$

On the other hand, if $U_0, ..., U_{T-1}$ is a time-consistent family of preference functionals and there is an $X \in L^0(\mathcal{F}_T)$ such that $\tilde{\nabla} U_t(U_{t+1}(X))$ exists for all $t \leq T-1$, then $\nabla U_t(X)$ exists too and can be written as

$$\nabla U_t(X) = \prod_{s=t}^{T-1} \tilde{\nabla} U_s(U_{s+1}(X)). \tag{4.2}$$

Proposition 4.2 *Assume that for at least one agent $a \in \mathbb{A}$, the preference functionals U_t^a satisfy* (D) *and the sets C_{t+1}^a are $\mathcal{F}_t$-open for all $t \leq T-1$. Then there exists at most one equilibrium price process $(S_t)_{t=0}^T$. Moreover, if the market is in equilibrium,*

$$\frac{d\mathbb{Q}_t^a}{d\mathbb{P}} = \nabla U_t^a \left(H^a + \sum_{s=1}^T \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s \right) \tag{4.3}$$

defines a probability measure $\mathbb{Q}_t^a$ on $(\Omega, \mathcal{F})$ such that $\mathbb{Q}_t^a \mid_{\mathcal{F}_t} = \mathbb{P} \mid_{\mathcal{F}_t}$ and

$$R_t = \mathbb{E}_{\mathbb{Q}_t^a}\left[R_T \mid \mathcal{F}_t\right], \quad S_t = \mathbb{E}_{\mathbb{Q}_t^a}\left[S_T \mid \mathcal{F}_t\right]. \tag{4.4}$$

If in addition, U_0^a is strictly monotone, then $\mathbb{Q}^a := \mathbb{Q}_0^a$ is equivalent to $\mathbb{P}$ and one has,

$$R_t = \mathbb{E}_{\mathbb{Q}^a}\left[R_T \mid \mathcal{F}_t\right] \quad \textit{and} \quad S_t = \mathbb{E}_{\mathbb{Q}^a}\left[S_T \mid \mathcal{F}_t\right] \quad \textit{for all } t. \tag{4.5}$$

Remark 4.3 If under the assumptions of Proposition 4.2 an equilibrium exists but the preference functional U_0^a is not strictly monotone, one can still write

$$R_t = \mathbb{E}_{\mathbb{Q}^a}\left[R_T \mid \mathcal{F}_t\right] \quad \text{and} \quad S_t = \mathbb{E}_{\mathbb{Q}^a}\left[S_T \mid \mathcal{F}_t\right] \quad \mathbb{Q}^a\text{-almost surely}. \tag{4.6}$$

But $\mathbb{Q}^a$ might not be equivalent to $\mathbb{P}$ and it can happen that there exists an event $A \in \mathcal{F}_t$ such that $\mathbb{P}[A] > 0$ and $\mathbb{Q}[A] = 0$. (4.6) then does not give any information about R_t and S_t in the event A. So (4.6) is weaker than (4.4). On the other hand, if $\mathbb{Q}^a$ is equivalent to $\mathbb{P}$, then (4.4) and (4.6) are equivalent.

Since we made no assumptions on non-redundancy of the financial assets, we cannot say anything about the uniqueness of optimal trading strategies $(\hat{\vartheta}_t^a)$. If for instance, $R_t^1 = R_t^2$ for all t, then any investment in R^1 can arbitrarily be replaced by one in R^2. However, if equilibrium prices are unique and U_0^a is strictly concave modulo translation, it can be shown that the optimal one-step trading gains of the corresponding agent are unique.

Proposition 4.4 *If there exists a unique equilibrium price process $(S_t)_{t=0}^T$ and U_0^a is strictly concave modulo translation for some agent $a \in \mathbb{A}$, then the optimal one-step trading gains*

$$\hat{\vartheta}_t^{a,R} \cdot \Delta R_t + \hat{\vartheta}_t^{a,S} \cdot \Delta S_t, \quad t = 1, \ldots, T,$$

are unique.

5 Base preferences and attainable initial endowments

In this section we consider the case where all agents have preferences of the same type and all endowments can be attained by trading in the financial market. Then after hedging the endowment, every agent invests in the same portfolio. If preferences are differentiable, the equilibrium pricing kernel can be given in explicit form. In Subsection 5.1 we show our one-fund theorem. In Subsection 5.2 we discuss option prices under stochastic volatility, demand pressure and short-selling constraints.

5.1 One-fund theorem

Note that mean-variance preferences of the form

$$\mathrm{MV}^{\gamma}(\mathrm{X}) = \mathbb{E}[\mathrm{X}] - \gamma \mathrm{Var}(\mathrm{X})$$

for a parameter $\gamma > 0$ can be written as

$$\mathrm{MV}^{\gamma}(\mathrm{X}) = \frac{1}{\gamma}\mathrm{MV}(\gamma \mathrm{X})$$

for the functional $\mathrm{MV} = \mathrm{MV}^1$. If the preferences of our agents are related in the same way to a base preference functional and all initial endowments are attainable by trading, the following holds:

Theorem 5.1 (One fund theorem)
Assume there exists an equilibrium such that $(S_t)_{t=0}^T$ as well as all optimal one-step trading gains

$$\hat{\vartheta}_t^{a,R} \cdot \Delta R_t + \hat{\vartheta}_t^{a,S} \cdot \Delta S_t$$

are unique and the initial endowments are of the form

$$H^a = c^a + \sum_{t=1}^{T} \eta_t^{a,R} \cdot \Delta R_t + \eta_t^{a,S} \cdot \Delta S_t,$$

for constants $c^a \in \mathbb{R}$ and trading strategies $(\eta_t^a)_{t=1}^T$, $a \in \mathbb{A}$. Moreover, suppose there exist base preference functionals

$$U_t : L^0(\mathcal{F}_T) \to \underline{L}^0(\mathcal{F}_t), \quad \text{and non-empty subsets} \quad C_{t+1} \in L^0(\mathcal{F}_t)^{J+K}, \quad t = 0, \dots, T-1,$$

such that the preferences and trading constraints of agent $a \in \mathbb{A}$ are given by

$$U_t^a(X) = \frac{1}{\gamma^a} U_t(\gamma^a X) \quad \text{and} \quad C_{t+1}^a = \frac{1}{\gamma^a} C_{t+1} - \eta_{t+1}^a \tag{5.1}$$

for parameters $\gamma^a > 0$, $a \in \mathbb{A}$. Denote

$$\gamma = \left(\sum_{a \in \mathbb{A}} \frac{1}{\gamma^a} \right)^{-1} \quad \text{and} \quad \eta_t^S = \sum_{a \in \mathbb{A}} \eta_t^{a,S}.$$

Then there exists a J-dimensional trading strategy $(\hat{\vartheta}_t^R)_{t=1}^T$ which for all $t \leq T-1$, maximizes

$$U_t \left(\sum_{s=t+1}^{T} \vartheta_s^R \cdot \Delta R_s + \gamma(n + \eta_s^S) \cdot \Delta S_s \right) \tag{5.2}$$

over all $(\vartheta_s^R)_{s=t+1}^T$ satisfying $\left(\vartheta_s^R, \gamma\left(n + \eta_s^S\right)\right) \in C_s$, $s = t+1, \dots, T$, and agent a's optimal one-step trading gains are of the form

$$\left(\frac{1}{\gamma^a} \hat{\vartheta}_t^R - \eta_t^{a,R} \right) \cdot \Delta R_t + \left(\frac{\gamma}{\gamma^a}(n + \eta_t^S) - \eta_t^{a,S} \right) \cdot \Delta S_t, \quad t = 1, \dots, T. \tag{5.3}$$

If, in addition, U_t satisfies the differentiability condition (D) *and C_{t+1} is $\mathcal{F}_t$-open for all $t \leq T-1$, then*

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}} = \nabla U_t \left(\sum_{s=1}^{T} \hat{\vartheta}_s^R \cdot \Delta R_s + \gamma(n + \eta_s^S) \cdot \Delta S_s \right)$$

defines probability measures satisfying $\mathbb{Q}_t |_{\mathcal{F}_t} = \mathbb{P} |_{\mathcal{F}_t}$ *such that*

$$R_t = \mathbb{E}_{\mathbb{Q}_t}[R_T \mid \mathcal{F}_t] \quad \textit{and} \quad S_t = \mathbb{E}_{\mathbb{Q}_t}[S_T \mid \mathcal{F}_t] \quad \textit{for all } t \leq T-1.$$

If moreover, U_0 *is strictly monotone, then* $\mathbb{Q} := \mathbb{Q}_0$ *is equivalent to* $\mathbb{P}$*, and one has*

$$R_t = \mathbb{E}_{\mathbb{Q}}[R_T \mid \mathcal{F}_t] \quad \textit{as well as} \quad S_t = \mathbb{E}_{\mathbb{Q}}[S_T \mid \mathcal{F}_t] \quad \textit{for all } t.$$

Remark 5.2 If under the assumptions of Theorem 5.1, there are no R-assets ($J = 0$) and the endowments are of the form $H^a = c^a + \eta^{a,S} \cdot S$ for deterministic vectors $\eta^{a,S} \in \mathbb{R}^K$, one can write

$$H^a = c^a + \eta^{a,S} \cdot S_0 + \sum_{t=1}^{T} \eta^{a,S} \Delta S_t.$$

So it follows from Theorem 5.1 that agent a'a optimal one-step trading gains are of the form

$$\left(\frac{\gamma}{\gamma^a}(n + \eta^S) - \eta^{a,S}\right) \cdot \Delta S_t, \quad t = 1, \dots, T.$$

That is, after hedging the endowment, every agent, like in a one-time-step CAPM, takes a static position in the market portfolio. Moreover, if U_0 is strictly monotone and has the differentiability property (D), the equilibrium pricing kernel simplifies to

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \nabla U_0\left(\gamma(n + \eta^S) \cdot S\right), \quad \text{where } \eta^S = \sum_{a \in \mathbb{A}} \eta^{a,S}.$$

Thus, the equilibrium pricing measure only depends on aggregate endowment and supply and not on the distribution of wealth among the agents. Moreover, the introduction of new assets in zero-net supply does not change existing security prices. Of course, the situation is different when agents are truly heterogeneous or endowments are unspanned.

Example 5.3 If the agents have entropic utility functionals

$$U^a_t(X) = -\frac{1}{\gamma^a} \log \mathbb{E}[\exp(-\gamma^a X) \mid \mathcal{F}_t] \quad \text{for constants } \gamma^a > 0, \quad a \in \mathbb{A},$$

one can write $U^a_t(X) = U_t(\gamma^a X)/\gamma^a$ for the base preference functionals

$$U_t(X) = -\log \mathbb{E}[\exp(-X) \mid \mathcal{F}_t], \quad t = 0, \dots, T.$$

We know from Example 2.1 that they have the properties (M), (T), (C), (TC) and U_0 satisfies (SL), (SM), (SC). Moreover, U_t has the differentiability property (D) with

$$\nabla U_t(X) = \frac{\exp(-X)}{\mathbb{E}[\exp(-X) \mid \mathcal{F}_t]}.$$

So if all agents are unconstrained and have endowments of the form

$$H^a = c^a + \sum_{t=1}^{T} \eta^{a,R}_t \cdot \Delta R_t + \eta^{a,S}_t \cdot \Delta S_t,$$

one obtains from Theorem 3.4 that an equilibrium exists. By Propositions 4.2 and 4.4, the equilibrium prices and optimal one-step trading gains are unique, and it follows from Theorem 5.1 that for $\gamma =$

$(\sum_{a\in\mathbb{A}}(\gamma^a)^{-1})^{-1}$ and $\eta^S_t = \sum_{a\in\mathbb{A}}\eta^{a,S}_t$, there exists a J-dimensional trading strategy $(\hat{\vartheta}^R_t)^T_{t=1}$ such that (5.2)–(5.3) hold. Moreover,

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \nabla U_0\left(\sum_{s=1}^{T}\hat{\vartheta}^R_s\cdot\Delta R_s + \gamma(n+\eta^S_s)\cdot\Delta S_s\right) = \frac{\exp\left(-\sum_{s=1}^{T}\left\{\hat{\vartheta}^R_s\cdot\Delta R_s + \gamma(n+\eta^S_s)\cdot\Delta S_s\right\}\right)}{\mathbb{E}\left[\exp\left(-\sum_{s=1}^{T}\left\{\hat{\vartheta}^R_s\cdot\Delta R_s + \gamma(n+\eta^S_s)\cdot\Delta S_s\right\}\right)\right]} \tag{5.4}$$

defines a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ for which

$$R_t = \mathbb{E}_{\mathbb{Q}}\left[R_T \mid \mathcal{F}_t\right] \quad \text{and} \quad S_t = \mathbb{E}_{\mathbb{Q}}\left[S_T \mid \mathcal{F}_t\right] \quad \text{for all } t = 0,\ldots,T.$$

For the special case where there are no R-assets ($J = 0$) and endowments are of the form $H^a = c^a + \eta^{a,S}\cdot S$ for deterministic vectors $\eta^{a,S}\in\mathbb{R}^K$, the pricing kernel simplifies to

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\exp\left(-\gamma(n+\eta^S)\cdot S_T\right)}{\mathbb{E}\left[\exp\left(-\gamma(n+\eta^S)\cdot S_T\right)\right]} \qquad \text{for } \eta^S = \sum_{a\in\mathbb{A}}\eta^{a,S}.$$

5.2 Simulation of option prices in a discrete Heston model

As an application of Theorem 5.1 we calculate equilibrium prices of equity options and study the effects of stochastic volatility, demand pressure and short-selling constraints. It has been observed that implied volatility smiles of index options and options on single stocks look differently even though the underlying distributions are similar. Typically, index options appear to be more expensive and their smiles are steeper. A possible explanation for this difference is that there usually is positive aggregate demand for out-the-money put index options by end users. If option dealers sell these options to end users and cannot fully hedge themselves, they expose themselves to the risk of a decline of the index. To compensate for that they are asking higher prices; see Bakshi et al. (2003), Bollen and Whaley (2004), Gârleanu et al. (2009) and the references therein. We follow Gârleanu et al. (2009) and assume our agents $\mathbb{A}$ are option dealers with expected exponential utility preferences with absolute risk aversions $\gamma_a > 0$, $a\in\mathbb{A}$. They have no endowments and trade in the underlying and the options. On the other side there are end users such as for instance, pension funds who buy put options to insure their investment portfolios. We assume that end users demand a fixed portfolio of put options and pay the price at which it is offered by the dealers. For our simulations we suppose they demand $m \geq 0$ put options with discounted strike $K_0 = 92$ and maturity T. The net supply among the dealers is $n = -m \leq 0$. We assume the dealers do not influence the price of the underlying R but determine the option prices. Suppose the underlying moves according to a discretized Heston model

$$\begin{aligned} R_{t+h} &= |R_t + \mu R_t h + \sqrt{v_t}R_t\Delta b^1_{t+h}|, \quad R_0 = 100 \\ v_{t+h} &= |v_t + \alpha(m - v_t)h + \beta\sqrt{v_t}\Delta b^2_{t+h}|, \quad v_0 = 0.04. \end{aligned}$$

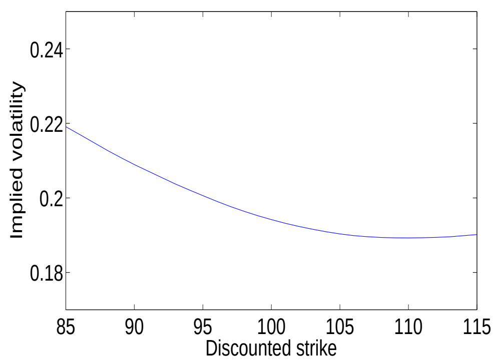
The absolute values are here to guarantee that R_t and v_t stay above zero. We choose maturity $T = 0.5$ years and make 100 steps of size $h = 0.005$. The other parameters are $\mu = 0.1$, $\alpha = 0.2$, $m = 0.04$, $\beta = 0.3$. $(b^1_{nh})^{100}_{n=0}$ and $(b^2_{nh})^{100}_{n=0}$ are two Bernoulli random walks with independent increments that have distribution $\mathbb{P}[\Delta b^i_t = \pm\sqrt{h}] = 1/2$ and correlation $\mathbb{E}\left[\Delta b^1_t \Delta b^2_t\right] = -0.3h$. We shall be interested in the prices of put options on R. The discounted time-T payoff corresponding to discounted strike K and maturity T is $S = (K - R_T)^+$. While for the simulation of (R_t) we make steps of size $h = 0.005$, we assume the trading dates to be a subset $\mathbb{T}$ of $\mathcal{T} = \{0, h, \ldots, T\}$ containing $\{0, T\}$. If $\mathbb{T}$ is coarse, option dealers can rebalance their portfolios less frequently, and the model becomes more incomplete. We think of situations where transaction costs are high or there are trading constraints. Denote by $\Theta^R_{\mathbb{T}}$ the set of all investment strategies in the underlying that are constant on the intervals $[t_{i-1}, t_i)$, where $\mathbb{T} = \{t_0 = 0, t_1, \ldots, T\}$. By formula (5.4), the equilibrium pricing kernel takes the form

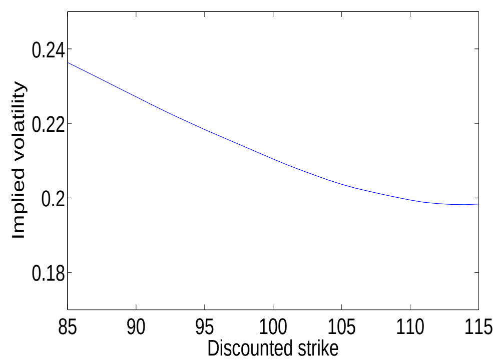
$$\frac{\exp\left(-\sum \hat{\vartheta}_t^R \Delta R_t - \gamma n P\right)}{\mathbb{E}\left[\exp\left(-\sum_{t\in\mathbb{T}} \hat{\vartheta}_t^R \Delta R_t - \gamma n P\right)\right]},$$

where $\gamma = (\sum_{a\in\mathbb{A}}(\gamma^a)^{-1})^{-1}$, $P = (K_0 - R_T)^+$ and $(\hat{\vartheta}_t^R) \in \Theta_{\mathbb{T}}^R$ is the maximizer of the expected utility

$$-\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\sum \vartheta_t^R \Delta R_t - \gamma n P\right)\right] \quad \text{over the set } \Theta_{\mathbb{T}}^R.$$

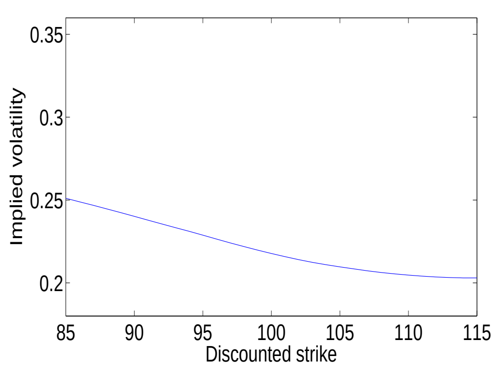
In the following we calculate implied volatilities of put options with discounted strikes between 85 and 115 for different choices of n and $\mathbb{T}$. We first assume $n = 0$ (no demand pressure) and think of R as the price of a single stock. The first of the two figures below shows implied volatilities for the case $n = 0$ and $\mathbb{T} = \mathcal{T}$ (option dealers rebalance their portfolios frequently). The second figure shows the situation for $n = 0$ and $\mathbb{T} = \{0, T\}$ (option dealers have to form their portfolios at time 0 and keep them constant until T).

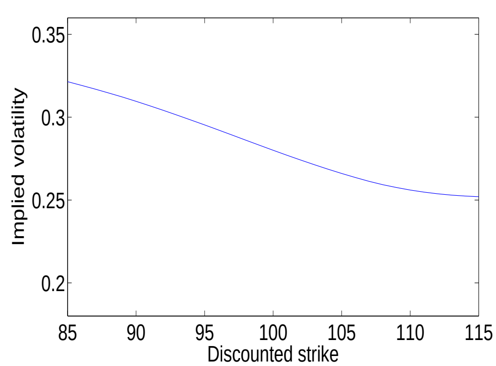




For $n = 0$, trading restrictions increase implied volatilities, and therefore option prices, only slightly because option dealers do not have to hedge the options. The only difference between frequent and less frequent trading is the quality of the dealers' investment strategy in the underlying R.

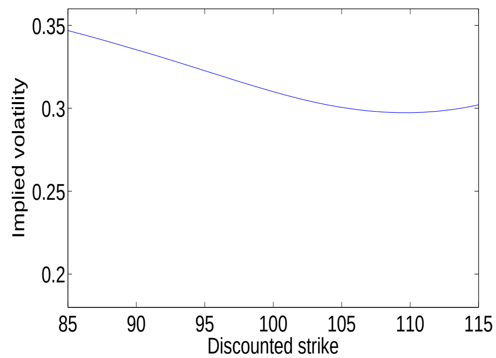
Now assume that net demand by end users for put options with discounted strike K_0 is positive and correspondingly, $n < 0$. This is typical for index options (see Gârleanu et al. (2009)). The first of the following two figures shows implied volatilities for the case $n < 0$ (positive demand) and $\mathbb{T} = \mathcal{T}$ (dealers rebalance frequently). The second one is for $n < 0$ (positive demand) and $\mathbb{T} = \{0, T\}$ (dealers have to invest statically).





It can be seen that net demand for put options with discounted strike $K_0 = 92$ increases prices of put options of all strikes, but especially those corresponding to low strikes. Also, trading restrictions have more of an influence on prices than in the case $n = 0$.

As a limit case, the next figure shows results for $n < 0$ and $\mathbb{T} = \emptyset$. That is, there is positive demand by end users for put options with discounted strike K_0. But option dealers are not allowed to trade the underlying. This can be interpreted as short-selling constraints. If dealers are short in put options, they would like to hedge by shorting the underlying. But under short-selling constraints, the best they can do is to have a zero position in the underlying. This increases prices of put options further compared to the case of demand pressure and few trading dates.



See also Avellaneda and Lipkin (2009) for a continuous-time model for hard-to-borrow stocks and the valuation of options on them.

6 Random walks and BSΔEs

We here consider the case where the noise is generated by d independent Bernoulli random walks but in contrast to Subsection 5.2, agents can be heterogenous and random endowments unspanned. Then equilibrium prices $(S_t)_{t=0}^T$ and continuation values $(H_t^a)_{t=0}^T$ can be obtained as solution to a system of coupled BSΔEs (backward stochastic difference equations). For notational convenience we restrict ourselves to the case $J = K = 1$, that is, R and S are both one-dimensional. The S-asset is in net supply $n \in \mathbb{R}$. We assume throughout this section that all agents are unconstrained, U_0^a is sensitive to large losses for all $a \in \mathbb{A}$, and U_t^a satisfies the differentiability condition (D) for all $a \in \mathbb{A}$ and $t \leq T - 1$. We let the time between two successive trading periods be given by some step size $h > 0$. The set of trading dates is $\mathbb{T} = \{0, h, \ldots, T\}$, where $T = Nh$ for some $N \in \mathbb{N}$. Let $b^i = (b_t^i)_{t \in \mathbb{T}}$, $i = 1, ..., d$, be random walks starting at 0 such that the increments $\Delta b_{t+h}^i = b_{t+h}^i - b_t^i$ are independent for different i and t with distribution

$$\mathbb{P}[\Delta b_{t+h}^i = \pm\sqrt{h}] = 1/2.$$

Let $(\mathcal{F}_t)_{t \in \mathbb{T}}$ be the filtration generated by (b_t^i), $i = 1, \ldots, d$. Note that $\mathcal{F}_t$ is generated by $2^{dt/h}$ atoms. In particular, $L^0(\mathcal{F}_t)$ just contains bounded random variables and can be identified with $\mathbb{R}^{dt/h}$.

6.1 The predictable representation property

It is well-known that for $d = 1$, the random walk b has the predictable representation property

$$\left\{x + z\Delta b_{t+h} : x, z \in L^0(\mathcal{F}_t)\right\} = L^0(\mathcal{F}_{t+h}) \quad \text{for all } t \leq T - h. \tag{6.1}$$

On the other hand,

$$\left\{x + \sum_{i=1}^d z^i \Delta b_{t+h}^i : x \in L^0(\mathcal{F}_t), z \in L^0(\mathcal{F}_t)^d\right\} \subsetneq L^0(\mathcal{F}_{t+h}) \quad \text{if } d \geq 2.$$

However, the following result shows how the predictable representation property can be obtained for $d \geq 2$ by adding enough orthogonal Bernoulli random walks.[3].

Lemma 6.1 *There exist $(\mathcal{F}_t)$-adapted random walks $b^{d+1}, \dots, b^D$ for $D = 2^d - 1$ such that*

$$b_0^i = 0 \quad \text{and} \quad \mathbb{P}[\Delta b_{t+h}^i = \pm\sqrt{h} \mid \mathcal{F}_t] = 1/2 \quad \text{for} \quad i = d+1, \dots, D,$$

$$\mathbb{E}\left[\Delta b_{t+h}^i \Delta b_{t+h}^l \mid \mathcal{F}_t\right] = 0 \quad \text{for all} \quad 1 \leq i \neq l \leq D$$

and

$$\left\{x + z \cdot \Delta b_{t+h} : x \in L^0(\mathcal{F}_t), z \in L^0(\mathcal{F}_t)^D\right\} = L^0(\mathcal{F}_{t+h}),$$

where b is the D-dimensional random walk with components b^i, $i = 1, \dots, D$.

It follows from Lemma 6.1 that every $X \in L^0(\mathcal{F}_{t+h})$ can be written as

$$X = \mathbb{E}[X \mid \mathcal{F}_t] + \pi_t(X) \cdot \Delta b_{t+h}$$

for

$$\pi_t^i(X) = \frac{1}{h}\mathbb{E}\left[X \Delta b_{t+h}^i \mid \mathcal{F}_t\right], \quad i = 1, \dots, D.$$

So one obtains from the translation property of the mappings U_t^a that

$$U_t^a(X) = U_t^a\left(\mathbb{E}\left[X|\mathcal{F}_t\right] + \pi_t(X) \cdot \Delta b_{t+h}\right) = \mathbb{E}\left[X \mid \mathcal{F}_t\right] - f_t^a(\pi_t(X))h, \tag{6.2}$$

where $f_t^a : L^0(\mathcal{F}_t)^D \to L^0(\mathcal{F}_t)$ is the $\mathcal{F}_t$-convex function given by

$$f_t^a(z) := -\frac{1}{h} U_t^a(z \cdot \Delta b_{t+h}).$$

Since U_t^a satisfies the differentiability condition (D), one has for all $z, z' \in L^0(\mathcal{F}_t)^D$,

$$\begin{aligned} \lim_{m \to \infty} \frac{f_t^a(z + z'/m) - f_t^a(z)}{1/m} &= -\frac{1}{h}\mathbb{E}\left[z' \cdot \Delta b_{t+h} \nabla U_t^a(z \cdot \Delta b_{t+h}) \mid \mathcal{F}_t\right] \\ &= z' \cdot \frac{1}{h}\mathbb{E}\left[-\Delta b_{t+h} \nabla U_t^a(z \cdot \Delta b_{t+h}) \mid \mathcal{F}_t\right]. \end{aligned}$$

That is,

$$\lim_{m \to \infty} \frac{f_t^a(z + z'/m) - f_t^a(z)}{1/m} = z' \cdot \nabla f_t^a(z)$$

for the random vector

$$\nabla f_t^a(z) = \mathbb{E}\left[-\Delta b_{t+h} \nabla U_t^a(z \cdot \Delta b_{t+h}) \mid \mathcal{F}_t\right] \in L^0(\mathcal{F}_t)^D.$$

6.2 First order conditions and equilibrium dynamics

We know from Theorem 3.4 that an equilibrium exists, and by Proposition 4.2, the equilibrium price process $(S_t)_{t \in \mathbb{T}}$ is unique. Introduce the random vectors

$$Z_{t+h}^R := \pi_t(R_{t+h}), \quad Z_{t+h}^S := \pi_t(S_{t+h}), \quad Z_{t+h}^a := \pi_t(H_{t+h}^a), \quad Z_{t+h} := \left(Z_{t+h}^R, Z_{t+h}^S, (Z_{t+h}^a)_{a \in \mathbb{A}}\right).$$

[3]Since the newly introduced random walks are adapted to the original filtration, they do not change the information structure of the economy

In terms of these quantities, H^a_t can be written as

$$\begin{aligned} H^a_t &= U^a_t\left(H^a_{t+h} + \hat{\vartheta}^{a,R}_{t+h}\Delta R_{t+h} + \hat{\vartheta}^{a,S}_{t+h}\Delta S_{t+h}\right) \\ &= \mathbb{E}\left[H^a_{t+h} + \hat{\vartheta}^{a,R}_{t+h}\Delta R_{t+h} + \hat{\vartheta}^{a,S}_{t+h}\Delta S_{t+h} \mid \mathcal{F}_t\right] - f^a_t\left(Z^a_{t+h} + \hat{\vartheta}^{a,R}_{t+h} Z^R_{t+h} + \hat{\vartheta}^{a,S}_{t+h} Z^S_{t+h}\right) h. \end{aligned}$$

So optimal trading strategies $\hat{\vartheta}^a_{t+h}$ have to satisfy the following first order conditions:

$$\mathbb{E}\left[\Delta R_{t+h} \mid \mathcal{F}_t\right] = Z^R_{t+h} \cdot \nabla f^a_t\left(Z^a_{t+h} + \hat{\vartheta}^{a,R}_{t+h} Z^R_{t+h} + \hat{\vartheta}^{a,S}_{t+h} Z^S_{t+h}\right) h \tag{6.3}$$

$$\mathbb{E}\left[\Delta S_{t+h} \mid \mathcal{F}_t\right] = Z^S_{t+h} \cdot \nabla f^a_t\left(Z^a_{t+h} + \hat{\vartheta}^{a,R}_{t+h} Z^R_{t+h} + \hat{\vartheta}^{a,S}_{t+h} Z^S_{t+h}\right) h, \tag{6.4}$$

and the mapping $\hat{u}_t : L^0(\mathcal{F}_t) \to L^0(\mathcal{F}_t)$ is of the form

$$\hat{u}_t(x) = \mathbb{E}\left[\sum_a H^a_{t+h} + x S_{t+h} \mid \mathcal{F}_t\right] - f_t(x Z^S_{t+h}, Z_{t+h}) h \tag{6.5}$$

for the function $f_t : L^0(\mathcal{F}_t)^{(3+|\mathbb{A}|)D} \to L^0(\mathcal{F}_t)$ given by the convolution

$$f_t(v, Z_{t+h}) = \min_{\substack{\vartheta^a \in L^0(\mathcal{F}_t)^2 \\ \sum_{a\in\mathbb{A}} \vartheta^{a,2} = 0}} \sum_{a\in\mathbb{A}} f^a_t\left(\frac{v}{|\mathbb{A}|} + Z^a_{t+h} + \vartheta^{a,1} Z^R_{t+h} + \vartheta^{a,2} Z^S_{t+h}\right) - \vartheta^{a,1}\frac{\mathbb{E}\left[\Delta R_{t+h} \mid \mathcal{F}_t\right]}{h}. \tag{6.6}$$

Furthermore, one has the following result:

Proposition 6.2 *f_t is conditionally differentiable in v at $v = nZ^S_{t+h}$ with*

$$\nabla^v f_t(nZ^S_{t+h}, Z_{t+h}) = \frac{1}{|\mathbb{A}|}\sum_{a\in\mathbb{A}} \nabla f^a_t\left(Z^a_{t+h} + \hat{\vartheta}^{a,R}_{t+h} Z^R_{t+h} + \hat{\vartheta}^{a,S}_{t+h} Z^S_{t+h}\right). \tag{6.7}$$

As a consequence one obtains from (6.3)–(6.4) the two equations

$$\mathbb{E}\left[\Delta R_{t+h} \mid \mathcal{F}_t\right] = Z^R_{t+h} \cdot \nabla^v f_t\left(nZ^S_{t+h}, Z_{t+h}\right) h \tag{6.8}$$

$$\mathbb{E}\left[\Delta S_{t+h} \mid \mathcal{F}_t\right] = Z^S_{t+h} \cdot \nabla^v f_t\left(nZ^S_{t+h}, Z_{t+h}\right) h. \tag{6.9}$$

Since $\nabla^v f_t$ also depends on Z^a_{t+h}, $a \in \mathbb{A}$, one needs the dynamics of the processes (H^a_t), $a \in \mathbb{A}$, to obtain a full characterization of equilibrium:

$$\begin{aligned} H^a_t &= U^a_t\left(H^a_{t+h} + \hat{\vartheta}^{a,R}_{t+h}\Delta R_{t+h} + \hat{\vartheta}^{a,S}_{t+h}\Delta S_{t+h}\right) \\ &= \mathbb{E}\left[H^a_{t+h} \mid \mathcal{F}_t\right] + \hat{\vartheta}^{a,R}_{t+h}\mathbb{E}\left[\Delta R_{t+h} \mid \mathcal{F}_t\right] + \hat{\vartheta}^{a,S}_{t+h}\mathbb{E}\left[\Delta S_{t+h} \mid \mathcal{F}_t\right] \\ &\quad - f^a_t\left(Z^a_{t+h} + \hat{\vartheta}^{a,R}_{t+h} Z^R_{t+h} + \hat{\vartheta}^{a,S}_{t+h} Z^S_{t+h}\right) h \\ &= \mathbb{E}\left[H^a_{t+h} \mid \mathcal{F}_t\right] + \hat{\vartheta}^{a,R}_{t+h}\mathbb{E}\left[\Delta R_{t+h} \mid \mathcal{F}_t\right] + \hat{\vartheta}^{a,S}_{t+h} Z^S_{t+h} \cdot \nabla^v f_t(nZ^S_{t+h}, Z_{t+h}) h \\ &\quad - f^a_t\left(Z^a_{t+h} + \hat{\vartheta}^{a,R}_{t+h} Z^R_{t+h} + \hat{\vartheta}^{a,S}_{t+h} Z^S_{t+h}\right) h \\ &= \mathbb{E}\left[H^a_{t+h} \mid \mathcal{F}_t\right] - g^a_t(Z_{t+h}) h, \end{aligned} \tag{6.10}$$

where

$$g^a_t(Z_{t+h}) = \min_{\vartheta \in L^0(\mathcal{F}_t)^2} f^a_t\left(Z^a_{t+h} + \vartheta^1 Z^R_{t+h} + \vartheta^2 Z^S_{t+h}\right) - \vartheta^1\frac{\mathbb{E}\left[\Delta R_{t+h} \mid \mathcal{F}_t\right]}{h} - \vartheta^2 Z^S_{t+h} \cdot \nabla^v f_t(nZ^S_{t+h}, Z_{t+h}).$$

Since

$$S_{t+h} = \mathbb{E}\left[S_{t+h} \mid \mathcal{F}_t\right] + Z^S_{t+h} \cdot \Delta b_{t+h} \quad \text{and} \quad H^a_{t+h} = \mathbb{E}\left[H^a_{t+h} \mid \mathcal{F}_t\right] + Z^a_{t+h} \cdot \Delta b_{t+h},$$

equations (6.9) and (6.10) yield the following result:

Theorem 6.3 *The processes (S_t) and (H^a_t) satisfy the following coupled system of BSΔEs:*

$$\begin{aligned}\Delta S_{t+h} &= Z^S_{t+h} \cdot \left\{ \nabla^v f_t(nZ^S_{t+h}, Z_{t+h})h + \Delta b_{t+h} \right\}, \quad S_T = S & (6.11)\\ \Delta H^a_{t+h} &= g^a_t\left(Z_{t+h}\right)h + Z^a_{t+h} \cdot \Delta b_{t+h}, \quad H^a_T = H^a. & (6.12)\end{aligned}$$

Remark 6.4 If in equilibrium the market becomes complete, the system of BSΔEs (6.11)–(6.12) decouples. Indeed, if the market is complete, there exist one-step trading strategies (η^a_{t+h}) such that

$$H^a_{t+h} - \mathbb{E}\left[H^a_{t+h} \mid \mathcal{F}_t\right] = \eta^{a,R}_{t+h}(R_{t+h} - \mathbb{E}\left[R_{t+h} \mid \mathcal{F}_t\right]) + \eta^{a,S}_{t+h}(S_{t+h} - \mathbb{E}\left[S_{t+h} \mid \mathcal{F}_t\right]),$$

and it follows that f_t can be written as

$$f_t(v, Z_{t+h}) = \min_{\substack{\vartheta^a \in L^0(\mathcal{F}_t)^2 \\ \sum_{a\in\mathbb{A}} \vartheta^{a,2} = 0}} \sum_{a\in\mathbb{A}} f^a_t\left(\frac{v}{|\mathbb{A}|} + (\eta^{a,R}_{t+h} + \vartheta^{a,1})Z^R_{t+h} + (\eta^{a,S}_{t+h} + \vartheta^{a,2})Z^S_{t+h}\right) - \vartheta^{a,1}\frac{\mathbb{E}\left[\Delta R_{t+h} \mid \mathcal{F}_t\right]}{h}.$$

In particular, $\nabla^v f_t(nZ^S_{t+h}, Z_{t+h})$ does not depend on Z^a_{t+h}, $a \in \mathbb{A}$, and equation (6.11) decouples from (6.12).

We refer to Horst and Müller (2007) and Horst et al. (2010) for sufficient conditions for market completeness in continuous-time and characterization of equilibrium by BSDEs (backward stochastic differential equations). Results on market completeness in more general equilibrium models can be found in Magill and Shafer (1990) and Anderson and Raimondo (2009).

6.3 Example with entropic preferences

We now study a concrete example where the agents have entropic preferences. For small time steps h, one can approximate f^a_t with quadratic drivers, and (6.11)–(6.12) becomes a coupled system of BSΔEs with drivers of quadratic growth. We first derive explicit expressions for the approximate equilibrium dynamics in discrete time. Then we give a formal discussion of the continuous-time limit $h \downarrow 0$.

6.3.1 Approximate dynamics in discrete time

Assume (R_t) evolves according to

$$\Delta R_{t+h} = R_t(\mu h + \sigma \Delta b^1_{t+h}), \quad R_0 \in \mathbb{R}_+, \tag{6.13}$$

where b^1 is the first component of the random walk b and μ, σ are constants such that $-\sigma\sqrt{h} < \mu h < \sigma\sqrt{h}$. Then (R_t) satisfies the no-arbitrage condition (NA) from above. The random vector Z^R_t is given by $Z^R_t = (\sigma R_t, 0, \ldots, 0)$ and cannot vanish. Suppose the agents have entropic preferences of the form

$$U^a_t(X) = -\frac{1}{\gamma^a}\log \mathbb{E}\left[\exp(-\gamma^a X) \mid \mathcal{F}_t\right] \quad \text{for constants} \quad \gamma^a > 0, \quad a \in \mathbb{A}.$$

The functions f^a_t are then given by

$$f^a_t(z) = \frac{1}{h\gamma^a}\log \mathbb{E}\left[\exp(-\gamma^a z \cdot \Delta b_{t+h})\right].$$

This expression is not very convenient for calculations. But for small time steps h, it can be approximated by a quadratic function. The idea is to ignore the orthogonal random walks $b^{d+1}, ..., b^D$ and approximate f^a_t

by a polynomial in z. To do that, let us write $z \in L^0(\mathcal{F}_t)^D$ as $z = (\bar{z}, \tilde{z})$ for $\bar{z} \in L^0(\mathcal{F}_t)^d$ and $\tilde{z} \in L^0(\mathcal{F}_t)^{D-d}$. For $X \in L^0(\mathcal{F}_T)$ and small $h > 0$,

$$X = \mathbb{E}[X] + \sum_{t<T} \pi_t(X) \cdot \Delta b_{t+h} \approx \mathbb{E}[X] + \sum_{t<T} \bar{\pi}_t(X) \cdot \Delta \bar{b}_{t+h}$$

is a good approximation which in the limit $h \downarrow 0$ becomes $X = \mathbb{E}[X] + \int_0^T Z_t \cdot dB_t$ for a d-dimensional Brownian motion $(B_t)_{0 \le t \le T}$ and a predictable process $(Z_t)_{0 \le t \le T}$. We therefore neglect the components $b^{d+1}, \dots, b^D$ and approximate $f^a_t(z)$ as follows:

$$f^a_t(z) \approx \frac{1}{h\gamma^a} \log \mathbb{E}\left[\exp\left(-\gamma^a \bar{z} \cdot \Delta \bar{b}_{t+h}\right)\right] = \frac{1}{h\gamma^a} \sum_{i=1}^d \log\cosh\left(\sqrt{h}\gamma^a z^i\right) \approx \frac{\gamma^a}{2} ||\bar{z}||_2^2. \tag{6.14}$$

Denote

$$z^R_{t+h} = \bar{Z}^R_{t+h}, \quad z^S_{t+h} = \bar{Z}^S_{t+h}, \quad z^a_{t+h} = \bar{Z}^a_{t+h}, \quad z_{t+h} = (z^R_{t+h}, z^S_{t+h}, (z^a_{t+h})_{a \in \mathbb{A}}) \quad \text{and} \quad z^A_{t+h} = \sum_a z^a_{t+h}.$$

To identify the equilibrium drift

$$g^S_t = \frac{\mathbb{E}[\Delta S_{t+h} \mid \mathcal{F}_t]}{h},$$

note that using approximation (6.14), the first order conditions (6.3)–(6.4) become

$$\begin{aligned} \mathbb{E}[\Delta R_{t+h} \mid \mathcal{F}_t] &= \gamma^a z^R_{t+h} \cdot \left(z^a_{t+h} + \hat{\vartheta}^{a,R}_{t+h} z^R_{t+h} + \hat{\vartheta}^{a,S}_{t+h} z^S_{t+h}\right) h & (6.15) \\ \mathbb{E}[\Delta S_{t+h} \mid \mathcal{F}_t] &= \gamma^a z^S_{t+h} \cdot \left(z^a_{t+h} + \hat{\vartheta}^{a,R}_{t+h} z^R_{t+h} + \hat{\vartheta}^{a,S}_{t+h} z^S_{t+h}\right) h. & (6.16) \end{aligned}$$

Substituting $\mu R_t h$ for $\mathbb{E}[\Delta R_{t+h} \mid \mathcal{F}_t]$ and summing over a gives

$$\begin{aligned} \mu R_t &= \gamma z^R_{t+h} \cdot \left(z^A_{t+h} + \hat{\vartheta}^{A,R}_{t+h} z^R_{t+h} + n z^S_{t+h}\right) & (6.17) \\ g^S_t(z_{t+h}) &= \gamma z^S_{t+h} \cdot \left(z^A_{t+h} + \hat{\vartheta}^{A,R}_{t+h} z^R_{t+h} + n z^S_{t+h}\right) & (6.18) \end{aligned}$$

for $\hat{\vartheta}^{A,R}_{t+h} = \sum_a \hat{\vartheta}^{a,R}_{t+h}$. It follows from (6.17) that

$$\hat{\vartheta}^{A,R}_{t+h} \sigma R_t = \frac{\mu}{\gamma\sigma} - (z^{A,1}_{t+h} + n z^{S,1}_{t+h}).$$

So one obtains from (6.18)

$$g^S_t(z_{t+h}) = z^S_{t+h} \cdot \begin{pmatrix} \mu/\sigma \\ \gamma(z^{A,l}_{t+h} + n z^{S,l}_{t+h}) \end{pmatrix},$$

where $z^{A,l}_{t+h}$ and $z^{S,l}_{t+h}$ are the vectors consisting of the last $d-1$ components of z^A_{t+h} and z^S_{t+h}, respectively.

With the approximation (6.14) the drivers g^a_t take the form

$$g^a_t(z_{t+h}) = \min_{\vartheta \in L^0(\mathcal{F}_t)^2} \frac{\gamma^a}{2} \|z^a_{t+h} + \vartheta^1 z^R_{t+h} + \vartheta^2 z^S_{t+h}\|_2^2 - \vartheta^1 \mu R_t - \vartheta^2 g^S_t(z_{t+h}), \tag{6.19}$$

and the approximate equilibrium prices and continuation values can be obtained from the coupled system of BSΔEs

$$\begin{aligned} \Delta S_{t+h} &= g^S_t(z_{t+h}) h + z^S_{t+h} \cdot \Delta \bar{b}_{t+h}, \quad S_T = S & (6.20) \\ \Delta H^a_t &= g^a_t(z_{t+h}) h + z^a_{t+h} \cdot \Delta \bar{b}_{t+h}, \quad H^a_T = H^a. & (6.21) \end{aligned}$$

In particular, they can be computed recursively by solving the following equations:

$$S_t = \mathbb{E}[S_{t+h} \mid \mathcal{F}_t] - g_t^S(z_{t+h})h, \quad S_T = S \tag{6.22}$$
$$H_t^a = \mathbb{E}[H_{t+h}^a \mid \mathcal{F}_t] - g_t^a(z_{t+h})h, \quad H_T^a = H^a. \tag{6.23}$$

Optimal strategies have to minimize (6.19). The corresponding first order conditions are

$$\begin{pmatrix} c_{t+h}^{RR} & c_{t+h}^{RS} \\ c_{t+h}^{RS} & c_{t+h}^{SS} \end{pmatrix} \begin{pmatrix} \hat{\vartheta}_{t+h}^{a,R} \\ \hat{\vartheta}_{t+h}^{a,S} \end{pmatrix} = \frac{1}{\gamma^a} \begin{pmatrix} \mu R_t - \gamma^a c_{t+h}^{Ra} \\ g_t^S(z_{t+h}) - \gamma^a c_{t+h}^{Sa} \end{pmatrix}, \tag{6.24}$$

where

$$c_{t+h}^{RR} := z_{t+h}^R \cdot z_{t+h}^R, \quad c_{t+h}^{RS} := z_{t+h}^R \cdot z_{t+h}^S, \quad c_{t+h}^{Sa} := z_{t+h}^S \cdot z_{t+h}^a, \quad \text{etc.}$$

There are two possible cases:

Case 1: $c_{t+h}^{RR} c_{t+h}^{SS} - c_{t+h}^{RS} c_{t+h}^{RS} > 0$. This condition means that from t to $t+h$ the two assets are non-redundant. So $\hat{\vartheta}_{t+h}^{R,a}$ and $\hat{\vartheta}_{t+h}^{S,a}$ are uniquely given by (6.24):

$$\begin{pmatrix} \hat{\vartheta}_{t+h}^{a,R} \\ \hat{\vartheta}_{t+h}^{a,S} \end{pmatrix} = \frac{1}{\gamma^a(c_{t+h}^{SS} c_{t+h}^{RR} - c_{t+h}^{RS} c_{t+h}^{RS})} \begin{pmatrix} c_{t+h}^{SS} \mu R_t - c_{t+h}^{RS} g_t^S(z_{t+h}) + \gamma^a(c_{t+h}^{RS} c_{t+h}^{Sa} - c_{t+h}^{SS} c_{t+h}^{Ra}) \\ c_{t+h}^{RR} g_t^S(z_{t+h}) - c_{t+h}^{RS} \mu R_t + \gamma^a(c_{t+h}^{RS} c_{t+h}^{Ra} - c_{t+h}^{RR} c_{t+h}^{Sa}) \end{pmatrix}$$

Case 2: $c_{t+h}^{RR} c_{t+h}^{SS} - c_{t+h}^{RS} c_{t+h}^{RS} = 0$. In this case one of the two assets is redundant in equilibrium, the optimal trading strategies are not unique and one can, for instance, choose

$$\hat{\vartheta}_{t+h}^{a,R} = \frac{1}{\gamma^a c_{t+h}^{RR}} \left(\mu R_t - \gamma^a c_{t+h}^{Ra} - n\gamma c_{t+h}^{RS} \right) \quad \text{and} \quad \hat{\vartheta}_{t+h}^{a,S} = \frac{n\gamma}{\gamma_a}.$$

To realistically simulate the system (6.22)–(6.23) is a non-trivial numerical task and left for future research.

6.3.2 Continuous-time limit

Let $(B_t)_{0 \le t \le T}$ be a d-dimensional Brownian motion and denote by $(\mathcal{F}_t)$ the augmented filtration generated by (B_t). Let (R_t) be a financial asset whose price is exogenously given by

$$dR_t = R_t(\mu dt + \sigma dB_t^1), \quad R_0 \in \mathbb{R}_+,$$

where μ, σ, r_0 are positive constants and B^1 is the first component of the Brownian motion B. Additionally, there exists an instrument with final payoff $S \in L^\infty(\mathcal{F}_T)$ that is traded by agents $a \in \mathbb{A}$ with initial endowments $H^a \in L^\infty(\mathcal{F}_T)$ and preferences of the form

$$U_t^a(X) = -\frac{1}{\gamma^a} \log \mathbb{E}\left[\exp(-\gamma^a X) \mid \mathcal{F}_t\right], \quad \gamma^a > 0.$$

The continuous-time analog of the discrete-time equations (6.20)–(6.21) is the following coupled system of BSDEs:

$$dS_t = g_t^S(Z_t)dt + Z_t^S \cdot dB_t, \quad S_T = S \tag{6.25}$$
$$dH_t^a = g_t^a(Z_t)dt + Z_t^a \cdot dB_t, \quad H_T^a = H^a \tag{6.26}$$

for $Z_t^R = (\sigma R_t, 0, \ldots, 0)$, $Z_t^A = \sum_a Z_t^a$, $Z_t = (Z_t^R, Z_t^S, (Z_t^a)_{a \in \mathbb{A}})$,

$$g_t^S(Z_t) = Z_t^S \cdot \begin{pmatrix} \mu/\sigma \\ \gamma(Z_t^{A,l} + nZ_t^{S,l}) \end{pmatrix},$$

where $Z_t^{A,l}$ and $Z_t^{S,l}$ are the vectors consisting of the last $d-1$ components of Z_t^A and Z_t^S, respectively, and

$$g_t^a(Z_t) = \min_{\vartheta \in L^0(\mathcal{F}_t)^2} \frac{\gamma^a}{2} \|Z_t^a + \vartheta^1 Z_t^R + \vartheta^2 Z_t^S\|_2^2 - \vartheta^1 \mu R_t - \vartheta^2 g_t^S(Z_t). \tag{6.27}$$

As in discrete time, optimal strategies $(\hat{\vartheta}_t^{a,R}, \hat{\vartheta}_t^{a,S})$ must minimize (6.27) for all t.

At this time there exists no general result which guarantees existence or uniqueness of solutions to the coupled system of BSDEs (6.25)–(6.26). So it is currently a conjecture that there exist processes (S_t), (H_t^a), (Z_t) solving (6.25)–(6.26) and that (S_t) is an equilibrium price.

A Proofs of Section 2

Proof of (2.12)
Let $X \in \underline{L}^0(\mathcal{F}_{t+1})$ and introduce the $\mathcal{F}_t$-measurable sets

$$A_0 = \{\mathbb{P}[X \leq 0 \mid \mathcal{F}_t] > 0\}, \quad A_l = \{\mathbb{P}[X \leq l \mid \mathcal{F}_t] > 0 \text{ and } \mathbb{P}[X \leq l-1 \mid \mathcal{F}_t] = 0\} \quad \text{for } l \geq 1.$$

Since Ω is the disjoint union of the sets $A_0, A_1, \dots$, the random variable

$$\xi = \sum_{l \in \mathbb{N}} 1_{A_l} \frac{1_{\{X \leq l\}}}{\mathbb{P}[X \leq l \mid \mathcal{F}_t]}$$

is in $\mathcal{D}_{t+1}$, and one has

$$\mathbb{E}\left[([X \wedge n] \vee m)\xi + G(\xi) \mid \mathcal{F}_t\right] \leq \sum_{l \in \mathbb{N}} 1_{A_l} \left(l + \mathbb{E}\left[G(\xi) \mid \mathcal{F}_t\right]\right) < \infty$$

for all $n \in \mathbb{N}$ and $m \in -\mathbb{N}$. This shows (2.12). □

Proof of (2.13)
Let $X \in \underline{L}^0(\mathcal{F}_{t+1})$ such that $\mathbb{P}[X < 0] > 0$. We first show that for every $n \in \mathbb{N}$, there exists a constant $\lambda_t \geq 1$ such that

$$\mathbb{P}[V_t(\lambda_t X) \leq -n] > 0. \tag{A.1}$$

To do that, we introduce the $\mathcal{F}_t$-measurable set $A = \{\mathbb{P}[X < 0 \mid \mathcal{F}_t] > 0\}$ and the conditional density

$$\xi = 1_A \frac{1_{\{X<0\}}}{\mathbb{P}[X < 0 \mid \mathcal{F}_t]} + 1_{A^c} \in \mathcal{D}_{t+1}.$$

The claim (A.1) now follows from the fact that

$$1_A V_t(\lambda_t X) \leq 1_A \left(\mathbb{E}\left[\lambda_t X \xi + G(\xi) \mid \mathcal{F}_t\right]\right) \to -\infty 1_A \quad \text{as} \quad \lambda_t \to \infty.$$

So one obtains that for every $n \in \mathbb{N}$ there exist constants $\lambda_{t-1}, \lambda_t \geq 1$ such that

$$\mathbb{P}[V_{t-1}\left(\lambda_{t-1} V_t(\lambda_t X)\right) \leq -n] > 0.$$

By concavity of V_t, one has

$$V_t(\lambda_{t-1}\lambda_t X) \leq \lambda_{t-1} V_t\left(\lambda_t X\right),$$

and it follows that

$$\mathbb{P}\left[V_{t-1}(\lambda_{t-1}\lambda_t X) \leq -n\right] \geq \mathbb{P}\left[V_{t-1}(\lambda_{t-1} V_t(\lambda_t X)) \leq -n\right] > 0$$

for all λ_{t-1} and λ_t sufficiently large. Iterating this argument yields (2.13). □

B Proofs of Section 3

Proof of Theorem 3.1
Let us first assume that the bounded adapted process $(S_t)_{t=0}^T$ and the strategies $(\hat{\vartheta}_t^a)_{t=1}^T$, $a \in \mathbb{A}$, form an equilibrium. Then condition (iii) holds by definition. Moreover, the associated continuation value processes (H_t^a) are bounded from below and one obtains

$$\begin{aligned}
-\hat{u}_t^*(S_t) &= \operatorname*{ess\,sup}_{x \in L^0(\mathcal{F}_t)^K} \{\hat{u}_t(x) - x \cdot S_t\} \\
&= \operatorname*{ess\,sup}_{\vartheta^a \in L^0(\mathcal{F}_t)^{J+K}} \sum_{a \in \mathbb{A}} \left\{u_t^a(\vartheta^a) - \vartheta^{a,S} \cdot S_t\right\} \\
&= \operatorname*{ess\,sup}_{\vartheta^a \in C_{t+1}^a} \sum_{a \in \mathbb{A}} U_t^a\left(H_{t+1}^a + \vartheta^{a,R} \cdot \Delta R_{t+1} + \vartheta^{a,S} \cdot \Delta S_{t+1}\right) \\
&= \sum_{a \in \mathbb{A}} U_t^a\left(H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot \Delta S_{t+1}\right) \\
&= \sum_{a \in \mathbb{A}} U_t^a\left(H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot S_{t+1}\right) - n \cdot S_t \\
&\leq \hat{u}_t(n) - n \cdot S_t.
\end{aligned} \tag{B.1}$$

Since $\hat{u}_t(n) + \hat{u}_t^*(S_t) \leq n \cdot S_t$, the inequality in (B.1) must be an equality, and it follows that

$$S_t \in \partial \hat{u}_t(n) \quad \text{as well as} \quad \sum_a U_t^a\left(H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot S_{t+1}\right) = \hat{u}_t(n),$$

which shows that conditions (i) and (ii) hold.

For the reverse implication, assume that (i)–(iii) are satisfied. Then the market clearing condition holds, and one has for all admissible trading strategies $(\vartheta_t^a)_{t=1}^T$, $a \in \mathbb{A}$,

$$\begin{aligned}
&\sum_{a \in \mathbb{A}} U_t^a(H_{t+1}^a + \vartheta_{t+1}^{a,R} \cdot \Delta R_{t+1} + \vartheta_{t+1}^{a,S} \cdot \Delta S_{t+1}) \\
=\ &\sum_{a \in \mathbb{A}} U_t^a(H_{t+1}^a + \vartheta_{t+1}^{a,R} \cdot \Delta R_{t+1} + \vartheta_{t+1}^{a,S} \cdot S_{t+1}) - \vartheta_{t+1}^{a,S} \cdot S_t \\
\leq\ &\hat{u}_t\left(\sum_{a \in \mathbb{A}} \vartheta_{t+1}^{a,S}\right) - \sum_{a \in \mathbb{A}} \vartheta_{t+1}^{a,S} \cdot S_t \\
\leq\ &-\hat{u}_t^*(S_t) = \hat{u}_t(n) - n \cdot S_t \\
=\ &\sum_{a \in \mathbb{A}} U_t^a(H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot \Delta S_{t+1}).
\end{aligned}$$

From here it follows by backwards induction that $(\hat{\vartheta}_t^a)_{t=1}^T$ is an optimal strategy for each agent $a \in \mathbb{A}$. □

Proof of Proposition 3.2
Suppose there exists no probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ which satisfies (3.5). Then it follows from the Dalang–Morton–Willinger theorem (Dalang et al. 1990) that there exists a $t \leq T-1$ and a one-step trading strategy $\vartheta_{t+1} \in L^0(\mathcal{F}_t)^{J+K}$ such that $\vartheta_{t+1}^R \cdot \Delta R_{t+1} + \vartheta_{t+1}^S \cdot \Delta S_{t+1}$ is non-negative and strictly positive with positive probability. The same is true for $\varepsilon_t(\vartheta_{t+1}^R \cdot \Delta R_{t+1} + \vartheta_{t+1}^S \cdot \Delta S_{t+1})$ for arbitrary $\mathcal{F}_t$-measurable $\varepsilon_t > 0$. But this means that there can exist no optimal trading strategies for the agents with strictly monotone preference functionals and open trading constraints, a contradiction to the assumption that the market is in equilibrium. □

Proof of Theorem 3.4

Set $S_T = S$ and $H^a_T = H^a$, $a \in \mathbb{A}$. Then existence of an equilibrium follows from Theorem 3.1 if we can show that for every $t \le T-1$, $S_{t+1} \in L^\infty(\mathcal{F}_{t+1})^K$ and bounded from below $H^a_{t+1} \in L^0(\mathcal{F}_{t+1})$, $a \in \mathbb{A}$, the following hold:

(a) there exist one-step strategies $\hat{\vartheta}^a_{t+1} \in C^a_{t+1}$, $a \in \mathbb{A}$, such that

$$\sum_{a \in \mathbb{R}} \hat{\vartheta}^{a,S}_{t+1} = n \quad \text{and} \quad \sum_{a \in \mathbb{A}} U^a_t(H^a_{t+1} + \hat{\vartheta}^{a,R}_{t+1} \cdot \Delta R_{t+1} + \hat{\vartheta}^{a,S}_{t+1} \cdot S_{t+1}) = \hat{u}_t(n)$$

(b) there exists $S_t \in \partial \hat{u}_t(n) \cap L^\infty(\mathcal{F}_t)^K$.

(a) follows from Lemma B.1 below and (b) will be shown in Lemma B.2. To prove Lemmas B.1 and B.2, we need the following concepts from Cheridito et al. (2011):

We call a subset C of $L^0(\mathcal{F})^d$ $\mathcal{F}$-linear if

$$\lambda x + y \in C$$

for all $x, y \in L^0(\mathcal{F})^d$ and $\lambda \in L^0(\mathcal{F})$. We call C $\mathcal{F}$-convex if

$$\lambda x + (1-\lambda) y \in C$$

for all $x, y \in L^0(\mathcal{F})^d$ and $\lambda \in L^0(\mathcal{F})$ such that $0 \le \lambda \le 1$. We say C is $\mathcal{F}$-polyhedral if it is of the form

$$C = \left\{ x \in L^0(\mathcal{F})^d : x \cdot a_i \le \alpha_i,\ i = 1, \dots, I \right\}$$

for random vectors $a_1, \dots, a_I \in L^0(\mathcal{F})^d$ and $\alpha_1, \dots, \alpha_I \in L^0(\mathcal{F})$. A mapping $f : L^0(\mathcal{F})^d \to L^0(\mathcal{F})^m$ is $\mathcal{F}$-linear if

$$f(\lambda x + y) = \lambda f(x) + f(y)$$

for all $x, y \in L^0(\mathcal{F})^d$ and $\lambda \in L^0(\mathcal{F})$. If $m = 1$, we say f is $\mathcal{F}$-convex ($\mathcal{F}$-concave) if

$$f(\lambda x + (1-\lambda) y) \le (\ge) \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in L^0(\mathcal{F})^d$ and $\lambda \in L^0(\mathcal{F})$ satisfying $0 \le \lambda \le 1$. By $\mathbb{N}(\mathcal{F})$ we denote the set of all $\mathcal{F}$-measurable random variables taking values in $\mathbb{N} = \{1, 2, ...\}$. For a sequence $(x_m)_{m \in \mathbb{N}}$ in $L^0(\mathcal{F})^d$ and $M \in \mathbb{N}(\mathcal{F})$, we define $x_M := \sum_{m \in \mathbb{N}} 1_{\{M = m\}} x_m$.

Lemma B.1 *Let $t \le T-1$, $S_{t+1} \in L^\infty(\mathcal{F}_{t+1})^K$ and $H^a_{t+1} \in L^0(\mathcal{F}_{t+1})$, $a \in \mathbb{A}$, all bounded from below. Assume the sets C^a_{t+1}, $a \in \mathbb{A}$, factorize and are sequentially closed. If there exists a (possibly empty) subset $\mathbb{A}' \subset \mathbb{A}$ such that*

(i) U^a_0 is sensitive to large losses for all $a \in \mathbb{A}'$ and

(ii) C^a_{t+1} is $\mathcal{F}_t$-bounded for all $a \in \mathbb{A} \setminus \mathbb{A}'$,

then $\hat{u}_t(x) < \infty$ for all $x \in L^0(\mathcal{F}_t)^K$, the set

$$\Theta = \left\{ x \in L^0(\mathcal{F})^K : \hat{u}_t(x) \in L^0(\mathcal{F}_t) \right\}$$

is $\mathcal{F}_t$-polyhedral, and for all $x \in \Theta$, there exist one-step trading strategies $\vartheta^a_{t+1} \in C^a_{t+1}$, $a \in \mathbb{A}$, such that

$$\sum_{a \in \mathbb{R}} \vartheta^{a,S}_{t+1} = x \quad \text{and} \quad \sum_{a \in \mathbb{A}} U^a_t \left(H^a_{t+1} + \vartheta^{a,R}_{t+1} \cdot \Delta R_{t+1} + \vartheta^{a,S}_{t+1} \cdot S_{t+1} \right) = \hat{u}_t(x). \tag{B.2}$$

In particular, $\hat{u}_t(n) \in L^0(\mathcal{F}_t)$ and there exist one-step trading strategies $\hat{\vartheta}^a_{t+1} \in C^a_{t+1}$, $a \in \mathbb{A}$, such that

$$\sum_{a \in \mathbb{R}} \hat{\vartheta}^{a,S}_{t+1} = n \quad \text{and} \quad \sum_{a \in \mathbb{A}} U^a_t \left(H^a_{t+1} + \hat{\vartheta}^{a,R}_{t+1} \cdot \Delta R_{t+1} + \hat{\vartheta}^{a,S}_{t+1} \cdot S_{t+1} \right) = \hat{u}_t(n).$$

Proof. It is clear that

$$f(\prod_{a\in\mathbb{A}} C^a_{t+1}) \subset \tilde{\Theta} := \left\{x \in L^0(\mathcal{F})^K : \hat{u}_t(x) > -\infty\right\},$$

where $f : L^0(\mathcal{F}_t)^{|\mathbb{A}|(J+K)} \to L^0(\mathcal{F}_t)^K$ is the $\mathcal{F}_t$-linear mapping given by

$$f((\vartheta^a)_{a\in\mathbb{A}}) = \sum_{a\in\mathbb{A}} \vartheta^{a,S}.$$

On the other hand, it follows from the definition of $\hat{u}_t$ that for every $x \in \tilde{\Theta}$, there exists a sequence A_n, $n \in \mathbb{N}$, of $\mathcal{F}_t$-measurable events together with random vectors $\vartheta^a_n \in C^a_{t+1}$ such that $A_n \uparrow \Omega$ almost surely and

$$\sum_{a\in\mathbb{A}} \vartheta^{a,S}_n = x \quad \text{on the set } A_n \setminus A_{n-1}, \quad \text{where } A_0 = \emptyset.$$

Since the sets C^a_{t+1} are $\mathcal{F}_t$-convex and sequentially closed, the strategy

$$\vartheta^a = \sum_{n\in\mathbb{N}} 1_{A_n\setminus A_{n-1}} \vartheta^a_n \quad \text{belongs to } C^a_{t+1} \text{ for all } a \in \mathbb{A},$$

and one has $\sum_{a\in\mathbb{A}} \vartheta^{a,S} = x$. This shows that $f(\prod_{a\in\mathbb{A}} C^a_{t+1}) = \tilde{\Theta}$. Since $\prod_{a\in\mathbb{A}} C^a_{t+1}$ is $\mathcal{F}_t$-polyhedral, it follows from Cheridito et al. (2011) that $\tilde{\Theta}$ is again $\mathcal{F}_t$-polyhedral. By condition (C1), n belongs to $\tilde{\Theta}$. So if we can show (B.2) for all $x \in \tilde{\Theta}$, it follows that $\tilde{\Theta} = \Theta$, and the lemma is proved.

To do this, fix $x \in \tilde{\Theta}$. Since the price process $(R_t)_{t=0}^T$ satisfies (NA), one obtains from the Dalang–Morton–Willinger theorem (Dalang et al. 1990) that there exists an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$ such that $R_t = \mathbb{E}_{\mathbb{Q}}\left[R_{t+1} \mid \mathcal{F}_t\right]$. Set $W = S_{t+1} - \mathbb{E}_{\mathbb{Q}}\left[S_{t+1} \mid \mathcal{F}_t\right]$. There exist one-step strategies $\vartheta^a_{t+1} \in C^a_{t+1}$, $a \in \mathbb{A}$, satisfying (B.2) if and only if the conditional optimization problem

$$\operatorname*{ess\,sup}_{\eta\in B} g(\eta) \tag{B.3}$$

has an optimal solution, where $g : L^0(\mathcal{F}_t)^{|\mathbb{A}|(J+K)} \to L^0(\mathcal{F}_t)$ is the $\mathcal{F}_t$-concave mapping given by

$$g(\eta) = \sum_{a\in\mathbb{A}} U^a_t \left(H^a_{t+1} + \eta^{a,R} \cdot \Delta R_{t+1} + \eta^{a,S} \cdot W\right)$$

and B is the $\mathcal{F}_t$-polyhedral set

$$B := \left\{\eta = (\eta^a)_{a\in\mathbb{A}} \in \prod_{a\in\mathbb{A}} C^a_{t+1} \,:\, \sum_{a\in\mathbb{A}} \eta^{a,S} = x\right\}.$$

Introduce the $\mathcal{F}_t$-linear set

$$E := \left\{\theta \in L^0(\mathcal{F}_t)^{(J+K)} : \theta^R \cdot \Delta R_{t+1} + \theta^S \cdot W = 0\right\}$$

and denote by Π the $\mathcal{F}_t$-conditional projection from $L^0(\mathcal{F}_t)^{|\mathbb{A}|(J+K)}$ to $(E^\perp)^{|\mathbb{A}'|} \times L^0(\mathcal{F}_t)^{|\mathbb{A}\setminus\mathbb{A}'|(J+K)}$. Since B is $\mathcal{F}_t$-polyhedral, it follows from Cheridito et al. (2011) that $\Pi(B)$ is $\mathcal{F}_t$-polyhedral too. Fix $\overline{\eta} \in \prod_{a\in\mathbb{A}} C^a_{t+1}$ such that $\sum_{a\in\mathbb{A}} \overline{\eta}^{a,S} = x$. Then the $\mathcal{F}_t$-convex set

$$C := \{\eta \in \Pi(B) : g(\eta) \geq g(\overline{\eta})\}$$

is sequentially closed, and (B.3) has an optimal solution if and only if

$$\operatorname*{ess\,sup}_{\eta\in C} g(\eta)$$

has one. Next we show that C is $\mathcal{F}_t$-bounded. If C contains only elements $\eta = (\eta^a)_{a\in\mathbb{A}}$ such that $\eta^a = 0$ for all $a \in \mathbb{A}'$, then this is a direct consequence of the assumption that C^a_{t+1} is $\mathcal{F}_t$-bounded for all $a \in \mathbb{A} \setminus \mathbb{A}'$. On the other hand, if C contains an element $\eta = (\eta^a)_{a\in\mathbb{A}}$ such that $\eta^a \neq 0$ for some $a \in \mathbb{A}'$, then there exists a set $A \in \mathcal{F}_t$ with $\mathbb{P}[A] > 0$ and a non-empty subset $\mathbb{A}''$ of $\mathbb{A}'$ such that

$$\mathbb{P}[\eta^{a,R} \cdot \Delta R_{t+1} + \eta^{a,S} \cdot W \neq 0 \mid \mathcal{F}_t] > 0 \quad \text{on} \quad A \quad \text{for all } a \in \mathbb{A}''$$

and

$$\eta^{a,R} \cdot \Delta R_{t+1} + \eta^{a,S} \cdot W = 0 \quad \text{on} \quad A \quad \text{for all } a \in \mathbb{A}' \setminus \mathbb{A}''$$

Since ΔR_{t+1} and W satisfy the no-arbitrage condition (NA), $\mathbb{P}[\vartheta^{a,R} \cdot \Delta R_{t+1} + \vartheta^{a,S} \cdot W < 0 \mid \mathcal{F}_t]$ must be strictly positive on A for all $a \in \mathbb{A}''$. So it follows from the sensitivity to large losses of the functionals U^a_0 that

$$\lim_{m\to\infty} U^a_t \left(H^a_{t+1} + m \left(\eta^{a,R} \cdot \Delta R_{t+1} + \eta^{a,S} \cdot W\right)\right) \to -\infty \quad \text{almost surely on } A \quad \text{for all } a \in \mathbb{A}''. \tag{B.4}$$

Indeed, assume to the contrary that there exists $a \in \mathbb{A}''$, $A' \in \mathcal{F}_t$ with $A' \subset A$ and $\mathbb{P}[A'] > 0$ such that

$$\lim_{l\to\infty} \sup_{m\geq l} U^a_t \left(H^a_{t+1} + m \left(\eta^{a,R} \cdot \Delta R_{t+1} + \eta^{a,S} \cdot W\right)\right) > -\infty \quad \text{almost surely on } A'.$$

Then there exist $c \in \mathbb{R}$ and $A'' \in \mathcal{F}_t$ with $A'' \subset A'$ and $\mathbb{P}[A''] > 0$ such that

$$\lim_{l\to\infty} \sup_{m\geq l} U^a_t \left(H^a_{t+1} + m \left(\eta^{a,R} \cdot \Delta R_{t+1} + \eta^{a,S} \cdot W\right)\right) \geq c \quad \text{almost surely on } A''.$$

It follows that there exists a sequence $(M_l)_{l\in\mathbb{N}}$ in $\mathbb{N}(\mathcal{F}_t)$ such that $M_{l+1} \geq M_l \geq l$ for all $l \in \mathbb{N}$ and

$$U^a_t \left(H^a_{t+1} + M_l \left(\eta^{a,R} \cdot \Delta R_{t+1} + \eta^{a,S} \cdot W\right)\right) \geq c - 1 \quad \text{on } A''.$$

But since U^a_t is $\mathcal{F}_t$-concave, this implies

$$U^a_t \left(H^a_{t+1} + l \left(\eta^{a,R} \cdot \Delta R_{t+1} + \eta^{a,S} \cdot W\right)\right) \geq c - 1 \quad \text{on } A'' \cap \{l \geq M_1\} \text{ for all } l \in \mathbb{N}. \tag{B.5}$$

Choose $l_0 \in \mathbb{N}$ such that $\mathbb{P}[A'''] > 0$ for $A''' = A'' \cap \{l_0 \geq M_1\}$. Since H^a_{t+1} is bounded from below, there exists $m \in \mathbb{N}$ such that

$$\mathbb{P}[1_{A'''}((H^a_{t+1})^+ + m(\eta^{a,R} \cdot \Delta R_{t+1} + \eta^{a,S} \cdot W)) < 0] > 0.$$

So for $l \geq m$,

$$\begin{aligned} & U^a_0 \left(1_{A'''}(H^a_{t+1} + l(\eta^{a,R} \cdot \Delta R_{t+1} + \eta^{a,S} \cdot W))\right) \\ \leq \quad & U^a_0 \left(\frac{l}{m} 1_{A'''} \left((H^a_{t+1})^+ + m(\eta^{a,R} \cdot \Delta R_{t+1} + \eta^{a,S} \cdot W)\right)\right) \to -\infty \quad \text{as } l \to \infty. \end{aligned}$$

and hence,

$$\begin{aligned} & U^a_0 \circ U^a_t \left(1_{A'''}(H^a_{t+1} + l(\eta^{a,R} \cdot \Delta R_{t+1} + \eta^{a,S} \cdot W))\right) \\ = \quad & U^a_0 \left(1_{A'''}(H^a_{t+1} + l(\eta^{a,R} \cdot \Delta R_{t+1} + \eta^{a,S} \cdot W))\right) \to -\infty \quad \text{for } l \to \infty. \end{aligned}$$

But this contradicts (B.5). So (B.4) must be true. Since $g(\overline{\eta}) \in L^0(\mathcal{F}_t)$, it follows that there exists an $m \in \mathbb{N}$ such that $m\eta \notin C$. Hence, we obtain from Cheridito et al. (2011) that C is $\mathcal{F}_t$-bounded and there exists a $\vartheta \in C$ such that

$$g(\vartheta) = \operatorname*{ess\,sup}_{\eta\in C} g(\eta).$$

□

Lemma B.2 *Under the assumptions of Lemma B.1 there exists a random vector S_t in $\partial\hat{u}_t(n)\cap L^\infty(\mathcal{F}_t)^K$.*

Proof. It follows from Lemma B.1 that $\hat{u}_t(n) \in L^0(\mathcal{F}_t)$. In the next step we show that

$$\hat{u}_t(n+x) - \hat{u}_t(n) \le L||x||_{\mathcal{F}_t} \quad \text{for all } x \in L^0(\mathcal{F}_t)^K, \tag{B.6}$$

where $L = \left(\sum_{k=1}^K ||S^k_{t+1}||^2_\infty\right)^{1/2}$. To do that let $x \in L^0(\mathcal{F}_t)^K$ such that $\hat{u}_t(n+x) > -\infty$ (inequality (B.6) holds trivially on the event $\{\hat{u}_t(n+x) = -\infty\}$). By Lemma B.1, there exist one-step strategies $\vartheta^a_{t+1} \in C^a_{t+1}$, $a \in \mathbb{A}$, such that

$$\sum_{a\in\mathbb{A}} \vartheta^{a,S}_{t+1} = n + x \quad \text{and} \quad \sum_{a\in\mathbb{A}} U^a_t\left(H^a_{t+1} + \vartheta^{a,R}_{t+1}\cdot\Delta R_{t+1} + \vartheta^{a,S}_{t+1}\cdot S_{t+1}\right) = \hat{u}_t(n+x).$$

Since the sets C^a_{t+1} factorize and there exist one-step strategies $\hat{\vartheta}^a_{t+1} \in C^a_{t+1}$, $a \in \mathbb{A}$, satisfying

$$\sum_{a\in\mathbb{A}} \hat{\vartheta}^{a,S}_{t+1} = n \quad \text{and} \quad \sum_{a\in\mathbb{A}} U^a_t\left(H^a_{t+1} + \hat{\vartheta}^{a,R}_{t+1}\cdot\Delta R_{t+1} + \hat{\vartheta}^{a,S}_{t+1}\cdot S_{t+1}\right) = \hat{u}_t(n),$$

there exist one-step strategies $\eta^a_{t+1} \in C^a_{t+1}$ such that for all $a \in \mathbb{A}$,

$$\eta^{a,R}_{t+1} = \vartheta^{a,R}_{t+1}, \quad \operatorname{sign}(\vartheta^{a,S,k}_{t+1} - \eta^{a,S,k}_{t+1}) = \operatorname{sign}(x^k) \quad \text{for every } k = 1,\dots,K, \quad \text{and} \quad \sum_{a\in\mathbb{R}} \eta^{a,S}_{t+1} = n.$$

It follows that

$$\begin{aligned}
\hat{u}_t(n) &\ge \sum_{a\in\mathbb{A}} U^a_t\left(H^a_{t+1} + \eta^{a,R}_{t+1}\cdot\Delta R_{t+1} + \eta^{a,S}_{t+1}\cdot S_{t+1}\right)\\
&\ge \sum_{a\in\mathbb{A}} U^a_t\left(H^a_{t+1} + \vartheta^{a,R}_{t+1}\cdot\Delta R_{t+1} + \vartheta^{a,S}_{t+1}\cdot S_{t+1}\right) - \sum_{k=1}^K |x^k|||S^k_{t+1}||_\infty\\
&\ge \hat{u}_t(n+x) - ||x||_{\mathcal{F}_t}\left(\sum_{k=1}^K ||S^k_{t+1}||^2_\infty\right)^{1/2},
\end{aligned}$$

and one obtains (B.6).

Now define for every $x \in L^0(\mathcal{F}_t)^K$,

$$p(x) = \sup_{m\in\mathbb{N}} m[\hat{u}_t(n + x/m) - \hat{u}_t(n)].$$

It follows from (B.6) and $\mathcal{F}_t$-concavity that

$$L||x||_{\mathcal{F}_t} \ge p(x) \ge \hat{u}_t(n+x) - \hat{u}_t(n).$$

In particular, p maps $L^0(\mathcal{F}_t)^K$ to $\underline{L}^0(\mathcal{F}_t)$ and it is enough to show that there exists $y \in L^\infty(\mathcal{F}_t)^K$ such that

$$x\cdot y \ge p(x) \quad \text{for all } x \in L^0(\mathcal{F}_t)^K. \tag{B.7}$$

Note that p satisfies

$$\begin{aligned}
p(\lambda x) &= \lambda p(x) \text{ for all } x \in L^0(\mathcal{F}_t)^K \text{ and } \lambda \in L^0_+(\mathcal{F}_t) &\text{(B.8)}\\
p(x+z) &\ge p(x) + p(z) \text{ for all } x, z \in L^0(\mathcal{F}_t)^K. &\text{(B.9)}
\end{aligned}$$

In particular,

$$p(1_A x) = 1_A p(x) \quad \text{for all } A \in \mathcal{F}_t \text{ and } x \in L^0(\mathcal{F}_t)^K \tag{B.10}$$

and

$$0 \geq p(x) + p(-x) \quad \text{for all } x \in L^0(\mathcal{F}_t)^K. \tag{B.11}$$

Denote

$$\alpha = \operatorname*{ess\,sup}_{x \in L^0(\mathcal{F}_t)^K,\, ||x||_{\mathcal{F}_t}=1} p(x) \leq L.$$

It follows from (B.10) that there exists a sequence $(x_m)_{m\in\mathbb{N}}$ in $L^0(\mathcal{F}_t)^K$ such that $||x_m||_{\mathcal{F}_t} = 1$ and $p(x_m) \uparrow \alpha$ almost surely. It follows from Cheridito et al. (2011) that there exists a sequence $(M_m)_{m\in\mathbb{N}}$ in $\mathbb{N}(\mathcal{F}_t)$ such that $M_{m+1} \geq M_m \geq m$ for all $m \in \mathbb{N}$ and $x_{M_m} \to z$ almost surely for some $z \in L^0(\mathcal{F}_t)^K$ with $||z||_{\mathcal{F}_t} = 1$. One deduces from the fact that Θ is $\mathcal{F}_t$-polyhedral that $p(z) = \alpha$. So z is a direction of steepest ascent for the function p at 0. The random vector

$$y = \alpha^+ z$$

satisfies $||y||_{\mathcal{F}_t} \leq L$. It remains to show that it fulfills (B.7). It clearly does so on the set $\{\alpha \leq 0\}$. So to finish the proof, we can assume that $\mathbb{P}[\alpha > 0] = 1$. The subset

$$E = \left\{\lambda z : \lambda \in L^0(\mathcal{F}_t)\right\} \subset L^0(\mathcal{F}_t)^K$$

is $\mathcal{F}_t$-linear and $\mathcal{F}_t$-closed. So it follows from Cheridito et al. (2011) that

$$L^0(\mathcal{F}_t)^K = E + E^\perp \quad \text{and} \quad E \cap E^\perp = \{0\}$$

for the conditional orthogonal complement

$$E^\perp = \left\{x \in L^0(\mathcal{F}_t)^K : x \cdot v = 0 \text{ for all } v \in E\right\}.$$

If we can show that

$$p(\lambda z + x) \leq \lambda p(z) \quad \text{for all } \lambda \in L^0(\mathcal{F}_t) \text{ and } x \in E^\perp, \tag{B.12}$$

it follows that

$$p(\lambda z + x) \leq \lambda \alpha \leq (\lambda z + x) \cdot y \quad \text{for all } \lambda \in L^0(\mathcal{F}_t) \text{ and } x \in E^\perp,$$

and the proof is complete. By (B.8) and (B.10), inequality (B.12) follows if we can show it for the three special cases $\lambda = 0$, $\lambda = 1$ and $\lambda = -1$. If $\lambda = 0$, it is enough to show that

$$p(x) \leq 0 \quad \text{for all } x \in E^\perp \text{ such that } ||x||_{\mathcal{F}_t} = 1. \tag{B.13}$$

Assume by way of contraction that there exists an $x \in E^\perp$ such that $||x||_{\mathcal{F}_t} = 1$ and $\mathbb{P}[p(x) > 0] > 0$. Then on the set $\{p(x) > 0\}$, one has

$$p\left(\lambda z + \sqrt{1-\lambda^2}x\right) \geq \lambda p(z) + \sqrt{1-\lambda^2} p(x) > p(z)$$

for the random variable

$$\lambda = \sqrt{\frac{p^2(z)}{p^2(z) + p^2(x)}}.$$

But since $||\lambda z + \sqrt{1-\lambda^2}x||_{\mathcal{F}_t} = 1$, this contradicts the fact that z is a direction of steepest ascent of p at 0. So (B.13) must be true. Next, note that for all $x \in E^\perp$,

$$p(z+x) - p(z) \leq \lim_{m\to\infty} \frac{p(z + x/m) - p(z)}{1/m} \leq \lim_{m\to\infty} \frac{\sqrt{1 + ||x||^2_{\mathcal{F}_t}/m^2} - 1}{1/m} p(z) = 0,$$

and therefore,

$$p(z+x) \leq p(z).$$

Finally, assume there exists an $x \in E^\perp$ such that $\mathbb{P}[p(-z+x) > -p(z)] > 0$. Then one has

$$p(x/2) \geq \frac{p(-z+x) + p(z)}{2} > 0$$

on the set $\{p(-z+x) > -p(z)\}$. But this contradicts (B.13). So (B.12) is proved. □

C Proofs of Section 4

Proof of Proposition 4.2

Suppose there exists an equilibrium price process $(S_t)_{t=0}^T$ and equilibrium trading strategies $(\hat{\vartheta}_t^a)_{t=1}^T$, $a \in \mathbb{A}$. Let $a \in \mathbb{A}$ such that U_t^a satisfies (D) and C_{t+1}^a is $\mathcal{F}_t$-open for all $t \leq T-1$. $\hat{\vartheta}_{t+1}^a$ maximizes

$$U_t^a\left(H_{t+1}^a + \vartheta_{t+1}^R \cdot \Delta R_{t+1} + \vartheta_{t+1}^S \cdot \Delta S_{t+1}\right)$$

over all $\vartheta_{t+1} \in C_{t+1}^a$. Fix $j \in \{1, \ldots, J\}$ and choose an $\mathcal{F}_t$-measurable random variable $\varepsilon > 0$ such that $\hat{\vartheta}_{t+1}^a + \varepsilon e^j \in C_{t+1}^a$, where e^j is the j-th standard vector in $\mathbb{R}^{J+K}$. Then one obtains

$$\begin{aligned}
& \varepsilon \mathbb{E}\left[\Delta R_{t+1}^j \tilde{\nabla} U_t^a(H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot \Delta S_{t+1}) \mid \mathcal{F}_t\right] \\
= & \lim_{m \to \infty} \frac{U_t^a(H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot \Delta S_{t+1} + \varepsilon \Delta R_{t+1}^j / m) - U_t^a(H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot \Delta S_{t+1})}{1/m} \\
= & \ 0.
\end{aligned}$$

This shows that

$$R_t^j = \mathbb{E}\left[R_{t+1}^j \tilde{\nabla} U_t^a(H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot \Delta S_{t+1}) \mid \mathcal{F}_t\right],$$

and one obtains by backwards induction that

$$\begin{aligned}
R_t^j &= \mathbb{E}\left[R_T^j \prod_{s=t}^{T-1} \tilde{\nabla} U_s^a(H_{s+1}^a + \hat{\vartheta}_{s+1}^{a,R} \cdot \Delta R_{s+1} + \hat{\vartheta}_{s+1}^{a,S} \cdot \Delta S_{s+1}) \mid \mathcal{F}_t\right] \\
&= \mathbb{E}\left[R_T^j \prod_{s=t}^{T-1} \tilde{\nabla} U_s^a \left(U_{s+1}^a \left(H^a + \sum_{r=s+1}^T \hat{\vartheta}_r^{a,R} \cdot \Delta R_r + \hat{\vartheta}_r^{a,S} \cdot \Delta S_r\right)\right) \mid \mathcal{F}_t\right] \\
&= \mathbb{E}\left[R_T^j \prod_{s=t}^{T-1} \tilde{\nabla} U_s^a \left(U_{s+1}^a \left(H^a + \sum_{r=1}^T \hat{\vartheta}_r^{a,R} \cdot \Delta R_r + \hat{\vartheta}_r^{a,S} \cdot \Delta S_r\right)\right) \mid \mathcal{F}_t\right] \\
&= \mathbb{E}\left[R_T^j \nabla U_t^a \left(H^a + \sum_{r=1}^T \hat{\vartheta}_r^{a,R} \cdot \Delta R_r + \hat{\vartheta}_r^{a,S} \cdot \Delta S_r\right) \mid \mathcal{F}_t\right].
\end{aligned}$$

The second equality is a consequence of the definition of the process $(H_t^a)_{t=0}^T$, the third holds because U_s^a and U_{s+1}^a have the translation property (T), and the fourth one follows from formula (4.2). Analogously, one obtains

$$S_t^k = \mathbb{E}\left[S_T^k \nabla U_t^a \left(H^a + \sum_{r=1}^T \hat{\vartheta}_r^{a,R} \cdot \Delta R_r + \hat{\vartheta}_r^{a,S} \cdot \Delta S_r\right) \mid \mathcal{F}_t\right] \quad \text{for all } k.$$

That

$$\frac{d\mathbb{Q}_t^a}{d\mathbb{P}} = \nabla U_t^a \left(H^a + \sum_{s=1}^T \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s\right)$$

defines a probability measure $\mathbb{Q}^a$ follows from the fact that U_t^a has the properties (M) and (T). If U_0^a is strictly monotone, one has

$$\nabla U_0^a \left(H^a + \sum_{r=1}^T \hat{\vartheta}_r^{a,R} \cdot \Delta R_r + \hat{\vartheta}_r^{a,S} \cdot \Delta S_r\right) > 0.$$

So $\mathbb{Q}^a = \mathbb{Q}_0^a$ is equivalent to $\mathbb{P}$ and one obtains (4.5). $\square$

Proof of Proposition 4.4
Assume there exist two optimal admissible trading strategies $(\hat{\vartheta}^a_t)_{t=1}^T$ and $(\vartheta^a_t)_{t=1}^T$ for agent a and a time s such that

$$\hat{\vartheta}^{a,R}_s \cdot \Delta R_s + \hat{\vartheta}^{a,S}_s \cdot \Delta S_s \neq \vartheta^{a,R}_s \cdot \Delta R_s + \vartheta^{a,S}_s \cdot \Delta S_s. \tag{C.1}$$

Then it follows by backwards induction that the strategy $(\tilde{\vartheta}^a_t)_{t=1}^T$ given by

$$\tilde{\vartheta}^a_t = \begin{cases} \hat{\vartheta}^a_t & \text{if } t \neq s \\ \vartheta^a_t & \text{if } t = s \end{cases}$$

is again optimal. The two sides of (C.1) cannot differ only by a constant, for otherwise one of the two strategies would be better than the other one. Therefore, strict concavity modulo translation implies

$$U^a_0\left(H^a + \sum_{t=1}^T \frac{\hat{\vartheta}^{a,R}_t + \tilde{\vartheta}^{a,R}_t}{2} \cdot \Delta R_t + \frac{\hat{\vartheta}^{a,R}_t + \tilde{\vartheta}^{a,S}_t}{2} \cdot \Delta S_t\right) > U^a_0\left(H^a + \sum_{t=1}^T \hat{\vartheta}^{a,R}_t \cdot \Delta R_t + \hat{\vartheta}^{a,S}_t \cdot \Delta S_t\right),$$

a contradiction to the optimality of $(\hat{\vartheta}^a_t)_{t=1}^T$. □

D Proofs of Section 5 and 6

Proof of Theorem 5.1
By assumption, there exist optimal admissible trading strategies $(\hat{\vartheta}^a_t)_{t=1}^T$ for all agents $a \in \mathbb{A}$, and the optimal one-step trading gains

$$\hat{\vartheta}^{a,R}_t \cdot \Delta R_t + \hat{\vartheta}^{a,S}_t \cdot \Delta S_t, \quad t \geq 1,$$

are unique. Since $C^a_{t+1} = C_{t+1}/\gamma^a - \eta^a_{t+1}$, the strategy $\tilde{\vartheta}^a_{t+1} := \gamma^a(\hat{\vartheta}^a_{t+1} + \eta^a_{t+1})$ belongs to C_{t+1} for all $a \in \mathbb{A}$ and $t \leq T-1$. Moreover, since $U_t(X) = \gamma^a U^a_t(X/\gamma^a)$, one has

$$U_t\left(\sum_{s=t+1}^T \tilde{\vartheta}^{a,R}_s \cdot \Delta R_s + \tilde{\vartheta}^{a,S}_s \cdot \Delta S_s\right) = \operatorname{ess\,sup}_{\vartheta_s \in C_s} U_t\left(\sum_{s=t+1}^T \vartheta^R_s \cdot \Delta R_s + \vartheta^S_s \cdot \Delta S_s\right)$$

and

$$\tilde{\vartheta}^{a,R}_t \cdot \Delta R_t + \tilde{\vartheta}^{a,S}_t \cdot \Delta S_t = \tilde{\vartheta}^{b,R}_t \cdot \Delta R_t + \tilde{\vartheta}^{b,S}_t \cdot \Delta S_t \quad \text{for all } a, b \in \mathbb{A} \text{ and } t \geq 1.$$

It follows that the strategy

$$(\hat{\vartheta}^R_t, \hat{\vartheta}^S_t) = \sum_{a \in \mathbb{A}} \frac{\gamma}{\gamma^a}(\tilde{\vartheta}^{a,R}_t, \tilde{\vartheta}^{a,S}_t) = \gamma\left(\sum_{a \in \mathbb{A}} \hat{\vartheta}^{a,R}_t + \eta^{a,R}_t, n + \eta^S_{t+1}\right), \quad t \geq 1,$$

satisfies

$$U_t\left(\sum_{s=t+1}^T \hat{\vartheta}^R_s \cdot \Delta R_s + \hat{\vartheta}^S_s \cdot \Delta S_s\right) = \operatorname{ess\,sup}_{\vartheta_s \in C_s} U_t\left(\sum_{s=t+1}^T \vartheta^R_s \cdot \Delta R_s + \vartheta^S_s \cdot \Delta S_s\right)$$

for all $t \geq 1$. This shows (5.2) and (5.3) because U_t is of the form $U_t(X) = \gamma^a U^a_t(X/\gamma^a)$.

The rest of the theorem follows from Proposition 4.2 by noting that if U_t satisfies (D) for all $t \leq T-1$, then

$$\begin{aligned}
&\nabla U_t\left(\sum_{s=1}^T \hat{\vartheta}^R_s \cdot \Delta R_s + \gamma\left(n + \eta^S_t\right) \cdot \Delta S_s\right) \\
=\;& \nabla U^a_t\left(\sum_{s=1}^T (\hat{\vartheta}^{a,R}_s + \eta^a_s) \cdot \Delta R_s + (\hat{\vartheta}^{a,S}_s + \eta^{a,S}_s) \cdot \Delta S_s\right) \\
=\;& \nabla U^a_t\left(H^a + \sum_{s=1}^T \hat{\vartheta}^{a,R}_s \cdot \Delta R_s + \hat{\vartheta}^{a,S}_s \cdot \Delta S_s\right)
\end{aligned}$$

for all $a \in \mathbb{A}$. $\square$

Proof of Proposition 6.1
It is enough to show that for every $t \leq T-h$, there exist random increments Δb^i_{t+h}, $i = d+1, \dots, D$ such that

$$\mathbb{P}[\Delta b^i_{t+h} = \pm\sqrt{h} \mid \mathcal{F}_t] = 1/2 \quad \text{for} \quad i = d+1, \dots, D, \tag{D.1}$$

$$\mathbb{E}\left[\Delta b^i_{t+h} \Delta b^l_{t+h} \mid \mathcal{F}_t\right] = 0 \quad \text{for all} \quad 1 \leq i \neq l \leq D, \tag{D.2}$$

and

$$\left\{x + z \cdot \Delta b_{t+h} : x \in L^0(\mathcal{F}_t), z \in L^0(\mathcal{F}_t)^D\right\} = L^0(\mathcal{F}_{t+h}). \tag{D.3}$$

By conditioning on $\mathcal{F}_t$, one can assume $t = 0$. Set

$$X^i = \frac{1}{\sqrt{h}} \Delta b^i_h, \quad i = 1, \dots, d.$$

The σ-algebra $\sigma(X^1, \dots, X^d)$ is generated by the 2^d atoms $\left\{X^1 = \pm 1, \dots, X^d = \pm 1\right\}$. Let us denote them by $A_1, \dots, A_{2^d}$. There are $D = 2^d - 1$ random variables of the form $\prod_{i \in \mathcal{I}} X^i$, where $\mathcal{I}$ is a non-empty subset of $\{1, \dots, d\}$. $X^1, \dots, X^d$ are d of them. Let us denote the remaining ones by $X^{d+1}, \dots, X^D$. Each X^i can only take the values ± 1, and one obtains by symmetry that

$$\mathbb{P}[X^i = \pm 1] = 1/2 \quad \text{for all } i = 1, \dots, D. \tag{D.4}$$

Moreover,

$$\mathbb{E}\left[X^m X^n\right] = 0 \quad \text{for all } m \neq n. \tag{D.5}$$

This holds because $X^m = \prod_{i \in \mathcal{I}} X^i$ and $X^n = \prod_{j \in \mathcal{J}} X^j$, where $\mathcal{I}$ and $\mathcal{J}$ are two different non-empty subsets of $\{1, \dots, d\}$. Therefore, there exists an i which is in $\mathcal{I}$ but not in $\mathcal{J}$ or the other way around. It then follows by independence that

$$\mathbb{E}\left[X^m X^n\right] = \mathbb{E}\left[X^i\right] \mathbb{E}\left[X^m X^n / X^i\right] = 0.$$

(D.4) and (D.5) show that $1, X^1, \dots, X^D$ are orthogonal. So they span the 2^d-dimensional space of all $\sigma(X^1, \dots, X^d)$-measurable random variables $\left\{\sum_{i=1}^{2^d} \lambda^i 1_{A_i} : \lambda \in \mathbb{R}^{2^d}\right\}$. If one sets $\Delta b^i_h = \sqrt{h} X^i$, $i = d+1, \dots, D$, then (D.1)–(D.3) are satisfied. $\square$

Proof of Proposition 6.2
For every $Y \in L^0(\mathcal{F}_t)^D$ and $m \in \mathbb{N}$, one has

$$\begin{aligned}
& f_t(nZ^S_{t+h} + Y/m, Z_{t+h}) - f_t(nZ^S_{t+h}, Z_{t+h}) \\
\leq \quad & \sum_{a \in \mathbb{A}} f^a_t \left(\frac{Y}{m\,|\mathbb{A}|} + Z^a_{t+h} + \hat{\vartheta}^{a,R}_{t+h} Z^R_{t+h} + \hat{\vartheta}^{a,S}_{t+h} Z^S_{t+h}\right) \\
& - \sum_{a \in \mathbb{A}} f^a_t \left(Z^a_{t+h} + \hat{\vartheta}^{a,R}_{t+h} Z^R_{t+h} + \hat{\vartheta}^{a,S}_{t+h} Z^S_{t+h}\right),
\end{aligned}$$

and therefore,

$$\begin{aligned}
& \limsup_{m \to \infty} m \left\{f_t(nZ^S_{t+h} + Y/m, Z_{t+h}) - f_t(nZ^S_{t+h}, Z_{t+h})\right\} \\
\leq \quad & Y \cdot \frac{1}{|\mathbb{A}|} \sum_{a \in \mathbb{A}} \nabla f^a_t \left(Z^a_{t+h} + \hat{\vartheta}^{a,R}_{t+h} Z^R_{t+h} + \hat{\vartheta}^{a,S}_{t+h} Z^S_{t+h}\right).
\end{aligned}$$

But since f_t is conditionally convex in v, the $\limsup$ is a $\lim$ and the inequality an equality. This shows that $\nabla^v f_t(nZ^S_{t+h}, Z_{t+h})$ exists and is equal to (6.7). $\square$

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