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On the Spanning Property of Risk Bonds Priced by Equilibrium

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We propose a method of pricing financial securities written on nontradable underlyings such as temperature or precipitation levels. To this end, we analyze a financial market where agents are exposed to financial and nonfinancial risk factors. The agents hedge their financial risk in the stock market and trade a risk bond issued by an insurance company. From the issuer's point of view the bond's primary purpose is to shift insurance risks related to noncatastrophic weather events to financial markets. As such, its terminal payoff and yield curve depend on an underlying climate or temperature process whose dynamics are independent of the randomness driving stock prices. We prove that if the bond's payoff function is monotone in the external risk process, it can be priced by an equilibrium approach. The equilibrium market price of climate risk and the equilibrium price process are characterized as solutions of nonlinear backward stochastic differential equations (BSDEs). Transferring the BSDEs into partial differential equations (PDEs), we represent the bond prices as smooth functions of the underlying risk factors. Our analytical results make the model amenable to a numerical analysis.

Key words: backward stochastic differential equations; climate risk; partial equilibrium; pricing in illiquid financial markets
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1. Introduction and overview. In recent years an array of new financial products emerged at the interface of finance and insurance whose primary purpose is to shift insurance risks to capital markets. Capital market products such as weather derivatives and risk bonds are end-products of a financial process called securitization, that transforms nontradable risk into tradable financial securities. Developed in the U.S. mortgage markets in the 1970s, securitization has long become a key component driving the convergence of financial and insurance markets. In this paper we propose an equilibrium approach to pricing financial securities written on nontradable underlyings such as temperature or precipitation indices.

Weather conditions affect a large number of industries including retail, travel, entertainment, agriculture, and the energy sector. Oil and natural gas producers face at least two types of weather-related risks. One is the *catastrophic* risk arising from severe weather events such as hurricanes or tropical storms that result in loss of production by shutting down drilling operations or damaging platforms. The other stems from *noncatastrophic* variations in weather conditions. Power producers, for instance, face weather risks that range from shutdowns of power plants due to extreme conditions, to variabilities on the demand side that arise from temperatures departing form the norm. An above-normal temperature pattern in winter can result in a significant reduction in load and revenues while below-normal temperatures can adversely affect the extent of demand that has to be met at pre-contracted prices, resulting in opportunity losses. In agriculture and fishing, large accumulative losses are caused by the periodically occurring event of the El Niño Southern Oscillation (ENSO), a random periodic event of a rise of the sea surface temperature in the Eastern Pacific.

There are basically two approaches taken to mitigate or manage weather and climate risk exposures. In the first, a company identifies the catastrophic events that have an impact on its revenue stream and arranges for an appropriate hedge. Insurance companies have long provided protection against extreme meteorological events. While these companies have a variety of instruments available to hedge their asset portfolios and interest-rate sensitive liabilities, until 1993 the only way to hedge their underwriting risk was to lay off a part of it with reinsurers. The second type of risk mitigation seeks to provide protection against variances in profit and revenue streams resulting from weather events that are not necessarily extreme, but deviations from the norm and which could still devastate earnings reports or cripple seasonal cashflows.

Financial products with payments determined by weather conditions offer companies the chance to temper the economic consequences of risks arising from noncatastrophic weather events. Coverage against nonextremal deviations from the norm is usually structured as a derivative contract with a specified tenor and extent of coverage in return for a fixed premium. The first insurance derivatives, futures and options, were launched by the Chicago Board of Trade (CBOT) in December 1992. The market for insurance- and index-linked securities has experienced significant growth rates ever since; see for instance a recent report of The World Bank [40] for a discussion of the role of financial securities written on weather indices in mitigating agricultural production risk. Securities such as the New York Heating Degree Day (HDD) swap or index insurance contracts whose payoff depends on temperatures at specified locations have a low correlation with stock and bond indices. Hence they are acquired as additions to diversified portfolios by institutional investors.

Pricing weather derivatives does not follow actuarial approaches characteristic of insurance contracts, but is rather based on financial valuation principles; see Schweizer [37] for a discussion of the differences between financial and actuarial valuation principles. It is often assumed that a valuation technique akin to that for pricing options and other claims on market assets can be employed. In a frictionless financial market a plain vanilla option on stocks or foreign currency can be hedged by continuously rebalancing a suitable hedge portfolio made up of the tradable underlying and a riskless bond. There are, however, several important distinctions between weather derivatives and other, more traditional, derivative securities that render the standard Black-Scholes framework an inappropriate benchmark model for pricing weather-sensitive securities. Most importantly, weather derivatives are written on nontradable assets such as temperature indices or precipitation levels. Hence the market is incomplete and standard replication arguments do not apply.

Techniques for pricing and hedging on incomplete financial markets have become a major topic in economics and finance. They include super-replication (El Karoui and Quenez [14]), quantile hedging (Föllmer and Leukert [16, 17]), mean variance hedging (Schweizer [36], Duffie and Richardson [13]), methods of local risk minimization (Föllmer and Sondermann [18]), dynamic risk measures for the optimal design of derivative securities (Barrieu and El Karoui [1, 2]) and, in particular, utility indifference arguments (Davis [11, 12], Becherer [3, 4], Moeller [29], Musiela and Zariphopoulou [31, 32], Henderson and Hobson [20]).

In this paper we suggest a dynamic equilibrium model for pricing risk bonds in illiquid markets. The problem of reallocating a perishable good by trading a financial security in zero net supply has been solved in great generality by Karatzas et al. [23]. We extend existence and uniqueness results of equilibrium in exchange economies by Chaumont et al. [7] and Hu et al. [21]. They analyzed an equilibrium framework for pricing nonfinancial risk. Their approach is based on two concepts that will also be key to this paper: *market completion* and *partial equilibrium*. In their model, agents exposed to climate- or weather-related risk factors exchange individual risk exposures by trading a "fictitious" insurance asset. The asset is a mere vehicle to reallocate risk and does not pay any dividends or interest. In their framework the possible equilibrium price processes can readily be identified with a class of diffusion processes with nonzero volatility so the asset necessarily completes the market in equilibrium. This paper extends the models in Hu et al. [21] beyond the benchmark case of a pure exchange economy. In our model the redistribution of risk takes place through financial markets. Such an approach is of particular interest when agents with negatively correlated risk exposures are located far away from each other.

We consider a situation where the agents can trade both a stock and a risk bond. The risk bond is issued by a financial institution such as an investment bank or re-insurance company and is in fixed supply. From the issuer's point of view its primary purpose is to shift insurance risks related to noncatastrophic weather events to financial markets. As such, its terminal payoff and yield curve depend on an underlying climate or temperature process whose dynamics are independent of the randomness that drives stock prices. The demand for the risk bond comes from energy producers, orange farmers, or travel agents that face temperature-dependent income streams or from institutional investors who seek to diversify their portfolios. While our setup closely follows Hu et al. [21] the dependence of the bond's payoff on the external risk factor renders our analysis much more involved. With a climate-dependent payoff at maturity, there is no a priori reason to assume that the bond completes the market. In fact, our goal is to identify sufficient conditions for market completeness.

The agents act as price takers in the stock and bond market and maximize their expected terminal utility. To prove existence and uniqueness (in a certain class) of an equilibrium bond price process, we characterize a set of linear pricing schemes that are consistent with the assumption of no-arbitrage in the stock and bond market. Following the approach in Hu et al. [21] we show that the set of linear pricing measures can be parameterized by the market price of climate risk θ. For any such θ the bond price process will be given by the bond's discounted conditional expected payoff under some pricing measure $\mathbb{P}^\theta$.

In a first step we solve the agents' optimization problem under $\mathbb{P}^\theta$ assuming that the bond completes the market; this assumption will be justified in a subsequent step. The equilibrium market price of risk will then be determined by the market clearing condition of zero total excess demand in the bond market. In a second step we apply a method introduced in Müller [30]: we characterize the equilibrium market price of climate risk in terms of the solution to a backward stochastic differential equation (BSDE) with quadratic growth. With the backward equation we associate a linear-quadratic partial differential equation (PDE). The key is to obtain a sufficiently smooth solution. While an array of existence and uniqueness results for nonlinear PDEs on bounded domains is available, the treatment of unbounded domains is considerably more involved.

In a second step we therefore deal with diffusion models on bounded domains. Under suitable regularity conditions on the diffusion coefficients we establish the existence of a three times continuously differentiable solution. This allows us to derive a PDE for equilibrium bond prices. At the same time the bond price process can be represented as a stochastic integral with respect to the Brownian motion driving the external risk factor. We show that the integrand can also be written in terms of a solution to a PDE, namely the derivative of the PDE that describes the bond price. Associated with this PDE is a *linear* BSDE. Its explicit solution allows us to prove the main contribution of this paper: market completeness if the bond's terminal payoff and yield curve are monotone in the climate process. In this case the bond can be priced within a general equilibrium framework. It turns out that the bond price process can also be obtained by solving the dual problem of a suitably defined representative agent that pools all incomes and trades optimally in the financial market. As such, our equilibrium dynamics differ from the price dynamics obtained by a dynamic indifference valuation in a representative agent model.

In a third step we provide an extension of our pricing method to unbounded domains for a benchmark geometric Brownian motion model. This extension uses mainly probabilistic arguments. When asset prices follow a geometric Brownian motion and the external risk process has an additive noise term we prove that the equilibrium BSDE is Malliavin differentiable. Differentiability of BSDEs with Lipschitz continuous driver and deterministic terminal times has been established by El Karoui et al. [15]. To the best of our knowledge no extension to quadratic BSDEs is yet available. Cheridito et al. [9] consider a system of BSDEs where the second component may be viewed as the derivative of the first. However, to prove the existence of a solution to their BSDEs they, too, refer to PDE methods. Our differentiability results allow us to prove the existence of a bounded smooth solution to the associated PDE based on probabilistic arguments. In a subsequent step we prove a decomposition result for Malliavin derivatives of linear BSDEs. This yields an alternative approach to proving market completeness under monotonicity conditions that uses only probabilistic methods.

The remainder of this paper is organized as follows. In §2 we introduce our notion of a risk bond, characterize the set of admissible linear pricing rules, solve the agents' optimization problem, and characterize the equilibrium market price of climate risk. In §3 we derive the PDE for the equilibrium price process. Section 4 derives an explicit formula for the integrand of the BSDE for the bond price and proves that the risk bond completes the market if its payoff function is monotone in the external risk factors. While the approach of §4 studies diffusion processes on bounded domains, §5 provides a partial extension to unbounded domains. We illustrate our approach to pricing risk bonds by means of two examples in §6.

2. The microeconomic setup and the main results.

We consider an economy with a finite set $\mathbb{A}$ of *agents*. The agents can trade a stock and a financial security whose payoff depends on nonfinancial risk factors. We think of the latter security as a *risk bond* issued by an insurance company. The bond is in fixed supply. Its primary purpose is to shift insurance risks related to weather and climate phenomena to the capital market. As such, it transforms nontradable risk factors described by some climate (risk) process R into a tradable financial asset. The bond will be priced by equilibrium considerations. Stock prices, on the other hand, follow an exogenous diffusion:

$$dS_t = \mu^S(t, S_t)\,dt + \sigma^S(t, S_t)\,dW_t^1. \tag{1}$$

They do not depend on the demand of the agents $a \in \mathbb{A}$. This assumption is justified if the set of agents that are interested in trading the risk bond is a small subset of an otherwise large set of traders that are active in the financial market.

Since our focus is on climate-related financial losses due to nonextreme deviations from the norm, we assume that the dynamics of the risk process R can also be modelled by a diffusion:

$$dR_t = \mu^R(t, R_t)\,dt + \sigma^R(t, R_t)\,dW_t^2. \tag{2}$$

The standard Brownian motions W^1 and W^2 driving asset prices and the external risk factor are assumed to be independent and defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Remark 2.1. Dynamics of the form (2) include simple stochastic models for the El Niño Southern Oscillation, a disruption of the ocean-atmosphere system in the Tropical Pacific. Among these models are mean-reverting Ornstein-Uhlenbeck processes and bistable diffusion models with a time-periodic potential function with two minima, the depths of which fluctuate periodically; see Chaumont et al. [7] for a more detailed discussion of low dimensional stochastic climate models and a report by Swiss Re [38] for a discussion of the consequences of the El Niño phenomenon for the insurance industry. The dynamics in (2) also include Davis' [10] "accumulated heating degree days" and indices such as average temperatures and area crop yields. We refer to a recent report by the World Bank [40] for a discussion of indexing-insurance products to mitigate agricultural risk.

The market participants are exposed to both financial and climate risk. The dependence of the bond's terminal payoff h^I and its yield curve φ^I on nonfinancial risk factors allows the insurance company to adjust its payments in reaction to portfolio losses due to climate or weather events. Since the bond is in fixed supply, accumulated coupon payments cannot be reinvested in the bond market in the same way interest payments are usually reinvested in government bonds or a savings account. Under the simplifying assumption that the risk-free interest rate equals zero, the bond's accumulated payoff H^I up to maturity is thus given by

$$H^I = h^I(\tau, S_\tau, R_\tau) + \int_0^\tau \varphi^I(u, S_u, R_u)\, du. \tag{3}$$

We assume throughout that h^I and φ^I are bounded and that τ is a stopping time with values in $[0, T]$. In §4 we deal with the case where τ is the first exit time of the process (S, R) from some bounded domain. For the special case where asset prices follow a geometric Brownian motion and the external risk process has an additive noise term, we provide an extension to unbounded domains in §5.

The notion of a risk bond will be used as a terminology for a derivative security with a climate-dependent payoff. For instance, for $\varphi^I \equiv 0$, the structure (3) of the terminal payoff H^I contains certain classes of European call options and forward contracts as special cases while an approximation of an HDD swap is obtained for $h^I = 0$ and a payoff rate

$$\varphi^I(t, S_t, R_t) = \max\{(K - R_t)^+, C\} \quad \text{with strike } K \text{ and cap } C > 0. \tag{4}$$

2.1. Pricing rules and the market price of climate risk. While the stock price process is *exogenous*, the bond price process will be derived *endogenously* by an equilibrium condition. Pricing the bond by market considerations requires the price process to be determined by a *linear* pricing rule; otherwise the agents have an incentive to trade the bond on a secondary market. By the Riesz representation theorem a continuous positive linear pricing rule

$$l\colon L^2(\mathbb{P}) \to \mathbb{R} \tag{5}$$

can be identified with a square integrable random variable G in the sense that

$$l(F) = \mathbb{E}[G \cdot F] \quad \text{for all } F \in L^2(\mathbb{P}).$$

The no-arbitrage condition implies that l assigns the price one to a riskless security that pays one dollar in every state of the world. Hence G is a $\mathbb{P}$-a.s. strictly positive density function so that every pricing measure can be identified with a probability measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$. Since $\mathbb{Q}$ needs to be consistent with the assumption of no arbitrage in the financial market, the (discounted) asset price has to be a $\mathbb{Q}$-martingale. This means that the restriction of $\mathbb{Q}$ on the σ-field $\mathcal{F}^1$ generated by the Brownian motion W^1 is given in terms of the *market price of financial risk*,

$$\theta_t^S = \frac{\mu^S(t, S_t)}{\sigma^S(t, S_t)}.$$

The set $\mathcal{P}$ of *admissible pricing rules* is thus given by the set of equivalent probability measures with a given restriction to the σ-field $\mathcal{F}^1$:

$$\mathcal{P} := \{\mathbb{Q}\colon \mathbb{Q} \approx \mathbb{P} \text{ and } S \text{ is a } \mathbb{Q}\text{-martingale}\}. \tag{6}$$

For any $\mathbb{Q} \in \mathcal{P}$, the density process $d\mathbb{Q}/d\mathbb{P}|_{\mathcal{F}_t} = Z_t$ is an almost surely strictly positive uniformly integrable Brownian martingale. By Theorem V.3.5 and Proposition VIII.1.6 in Revuz and Yor [35], the process (Z_t) can thus be written as a stochastic exponential of some local martingale. Representing this martingale as an integral with respect to the Brownian motion $W = (W^1, W^2)$ yields a two-dimensional predictable process $\theta = (\theta_t)$ such that

$$Z_t = \exp\left(-\int_0^t \theta_s\, dW_s - \frac{1}{2}\int_0^t |\theta_s|^2\, ds\right). \tag{7}$$

The first component of θ equals the market price of financial risk θ^S. The second component θ^R will be referred to as the *market price of climate risk*. The set of linear pricing rules on $L^2(\mathbb{P})$ can thus be identified with the class of predictable processes θ^R that make the process (Z_t) *defined* by Equation (7) with $\theta = (\theta^S, \theta^R)$ a uniformly integrable martingale. For any such process we write

$$\left.\frac{d\mathbb{P}^\theta}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t^\theta = \exp\left(-\int_0^t \begin{pmatrix}\theta_s^S\\ \theta_s^R\end{pmatrix} d\begin{pmatrix}W_s^1\\ W_s^2\end{pmatrix} - \frac{1}{2}\int_0^t |\theta_s|^2\, ds\right) \tag{8}$$

and introduce a two-dimensional Brownian motion W^θ with respect to $\mathbb{P}^\theta$ by

$$W_t^\theta = W_t + \int_0^t \theta_s \, ds.$$

REMARK 2.2. A sufficient condition for uniform integrability of $\{Z_t\}$ is that $\int \theta_s \, dW_s$ is a BMO-martingale; see Kazamaki [24] for details. The BMO-property holds automatically if θ is bounded. BMO will also guarantee that the agents optimal expected utility is finite. We prove existence of an equilibrium market price of climate risk that satisfies the BMO-property.

For a given market price of climate risk θ^R and hence a given market price of risk $\theta = (\theta^S, \theta^R)$, the initial bond price is $B_0^\theta = \mathbb{E}^\theta[H^I]$. This expectation makes sense because H^I is bounded. To exclude arbitrage opportunities in the bond market, bond prices need to be defined as the conditional discounted expected payoffs under $\mathbb{P}^\theta$ so that

$$B_t^\theta = \mathbb{E}^\theta\left[h^I(\tau, S_\tau, R_\tau) + \int_0^\tau \varphi^I(s, S_s, R_s)\, ds \,\middle|\, \mathcal{F}_t\right]. \tag{9}$$

Representing the random variables B_t^θ as stochastic integrals with respect to the $\mathbb{P}^\theta$-Brownian motion W^θ yields a two-dimensional adapted process $\kappa^\theta = (\kappa^S, \kappa^R)$ such that

$$\begin{aligned} B_t^\theta &= \mathbb{E}^\theta[H_I] + \int_0^t \kappa_s^\theta \, dW_s^\theta \\ &= \mathbb{E}^\theta[H_I] + \int_0^t \kappa_s^S \,(dW_s^1 + \theta_s^S ds) + \int_0^t \kappa_s^R \,(dW_s^2 + \theta_s^R ds). \end{aligned} \tag{10}$$

The *equilibrium price process* will be determined by the market clearing condition of zero excess demand in the bond market. For this we first solve the agents' utility optimization problem.

2.2. Characterizing the equilibrium market price of risk. Throughout this section we fix a market price θ and assume that θ has the BMO property, i.e., that $\int \theta_s \, dW_s$ is a BMO-martingale. We also assume the risk bond completes the market.[1] By Lemma 26 in Müller [30] the assumption of market completeness is equivalent to

$$\kappa_t^R \neq 0 \quad \mathbb{P} \otimes \lambda\text{-a.s.} \tag{11}$$

We assume that an agent's preferences can be described by an exponential utility function with coefficient of risk aversion α_a and that her income H^a over the period $[0, \tau]$ is exposed to financial and nonfinancial risk. More precisely,

$$H^a = h^a(\tau, S_\tau, R_\tau) + \int_0^\tau \varphi^a(s, S_s, R_s)\, ds \tag{12}$$

for bounded terminal payoffs h^a and income rates φ^a. We denote by $\pi_t^{a,S}$ and $\pi_t^{a,B}$ her holdings of stocks and bonds at time t so her wealth from trading is given by

$$V_t^{a,\theta}(\pi) = \int_0^t \pi_s^{a,S} \, dS_s + \int_0^t \pi_s^{a,B} \, dB_s^\theta \quad (0 \le t \le \tau). \tag{13}$$

DEFINITION 2.1. The trading strategy $\pi^a = \{(\pi_t^S, \pi_t^B)\}$ is admissible if it is adapted, satisfies

$$\int_0^\tau |\pi_s|^2 \, ds < \infty \quad \mathbb{P}\text{-a.s.}$$

and if $(V_t^{a,\theta}(\pi))$ is a $\mathbb{P}^\theta$-supermartingale. The set of all admissible trading strategies is denoted $\mathcal{A}^\theta$.

The agent's goal is to maximize her expected terminal utility from trading in the stock and bond market. Her optimization problem is thus given by

$$\max_{\pi \in \mathcal{A}^\theta} \mathbb{E}[-\exp(-\alpha_a[V_\tau^{a,\theta}(\pi) + H^a])]. \tag{14}$$

[1] We prove in §4.3 that the assumption of market completion holds if the bond's payoff functions are monotone in the external risk factor while the BMO property will follow from results established in §4.2.

Since the bond is assumed to complete the market, the agent can replicate any contingent claim that is no more expensive than her climate-dependent income. The optimization problem over trading strategies can thus be transformed into an equivalent optimization problem over contingent claims:

$$\max_{\xi} \ \mathbb{E}[-\exp(-\alpha_a \xi)]$$
$$\text{subject to } \ \mathbb{E}^{\theta}[\xi] \le \mathbb{E}^{\theta}[H^a] \quad \text{and} \quad \xi \text{ is } \mathcal{F}_{\tau}\text{-measurable.} \tag{15}$$

Its solution can be given in closed form; for a proof we refer to Müller [30, Chapter 4.2.],

Theorem 2.1. *Let $\theta = (\theta^S, \theta^R)$ be a market price of risk that has the BMO property. If the bond completes the market, the agent's maximal utility is finite and the optimizing claim is given by*

$$\xi_a^{\theta} = c_a^{\theta} + \frac{1}{\alpha_a}\left\{\int_0^{\tau} \theta_u^S \left(dW_u^1 + \frac{1}{2}\theta_u^S du\right) + \int_0^{\tau} \theta_u^R \left(dW_u^2 + \frac{1}{2}\theta_u^R du\right)\right\}. \tag{16}$$

The constant c_a^{θ} is defined through the budget constraint

$$\mathbb{E}^{\theta}[\xi_a^{\theta}] = \mathbb{E}^{\theta}\left[H^a + \int_0^{\tau} \varphi^a(u, S_u, B_u^{\theta})\, du\right].$$

We refer to ξ_a^{θ} as the agent's optimal contingent claim under the pricing measure $\mathbb{P}^{\theta}$. Under θ the optimal claims of different agents differ only by the constants c_a^{θ} and the factors α_a. With $1/\alpha := \sum_{a\in\mathbb{A}}(1/\alpha_a)$, the combined optimal claim is given by

$$\sum_{a\in\mathbb{A}} \xi_a^{\theta} = \sum_{a\in\mathbb{A}} c_a^{\theta} + \frac{1}{\alpha}\int_0^{\tau} \theta_t\, dW_t + \frac{1}{2\alpha}\int_0^{\tau} |\theta_t|^2\, dt. \tag{17}$$

The role of the exponential utility function is further discussed in Remark 3.1 below.

Since the bond completes the market, the excess demand $\xi_a^{\theta} - H^a$ can be replicated by trading in the stock and bond market:

$$\xi_a^{\theta} - H^a = \int_0^{\tau} \pi_u^{a,S}\, dS_u + \int_0^{\tau} \pi_u^{a,B^{\theta}}\, dB_u^{\theta} \tag{18}$$

for an admissible trading strategy π^a. We call $\pi^{a,B^{\theta}}$ the agent's optimal trading strategy in the bond market and say that $\bar{\theta}$ is an equilibrium market price of risk if the demand for the risk bond satisfies the market clearing condition of zero total excess demand.

Definition 2.2. A market price of risk $\bar{\theta}$ will be called an *equilibrium*, if

$$\sum_{a\in\mathbb{A}} \pi_t^{a,B^{\bar{\theta}}} = 1 \qquad \mathbb{P}^{\bar{\theta}} \otimes \lambda\text{-a.s.} \tag{19}$$

Notice that if the insurance company were allowed to actively trade the risk bond it issued, i.e., if the insurance company belongs to the set $\mathbb{A}$, then the equilibrium condition would read

$$\sum_{a\in\mathbb{A}} \pi_t^{a,B^{\bar{\theta}}} = 0.$$

Such an equilibrium condition corresponds to the model of *risk sharing* studied by Chaumont et al. [7]. In their model the problem of market completeness is trivial. One can immediately solve for equilibrium prices in the class of diffusion models with nonvanishing volatility. The problem is "buy backs." Within the framework of Hu et al. [21] there is no reason to expect that the insurance company does *not* buy back some of the bonds in equilibrium. This is an undesirable feature in our context where the focus is on *risk transfer* rather than *risk sharing*. Buying back the bond or, equivalently, short selling less than one unit of the bond would contradict the assurer's goal to lay off its risk with the financial markets. It is the insurer's aim to transfer rather than share its risk that renders our analysis more involved. At the same time it is a property of equilibrium that some of the *agents* may short the bond. To some extent this puts them on the same foot as the original issuer. The difference is that an agents' short position depends on their preferences and endowments and is endogenously determined by an equilibrium constraint. The insurance company, on the other hand, sells a fixed and exogenously given number of bonds.

2.3. Equilibrium market prices of climate risk. For our analysis of equilibrium we assume that the risk bond matures the first time the two-dimensional diffusion process

$$X_t = (S_t, R_t)$$

leaves some open domain $\mathcal{O}$. When $\mathcal{O}$ is bounded, this means that the bond shares some features of a barrier option. Specifically we assume that

$$\tau := \inf\{t\colon X_t \notin \mathcal{O}\} \wedge T. \tag{20}$$

Our approach to equilibrium bond pricing is based on a characterization result of the equilibrium market price of climate risk as the solution to a BSDE and the interplay between BSDEs and PDEs. We shall associate a nonlinear PDE with the equilibrium BSDE. The key is to obtain a classical solution. If such a solution exists, then $\bar{\theta}^R$ can be represented in terms of its gradient.

While existence and uniqueness results for the equilibrium PDE on bounded domains are available (Ladyzenskaja et al. [26]), the case $\mathcal{O} = \mathbb{R}^2$ is more involved. We deal with general diffusions on bounded domains in §4 and study a benchmark geometric Brownian motion model on unbounded domains in §5. When asset prices follow a geometric Brownian motion and the risk process has an additive noise term, we prove existence and uniqueness of a smooth global solution to the equilibrium PDE on $\mathcal{O} = \mathbb{R}^2$ by probabilistic methods. The following subsection summarizes our main findings.

2.3.1. Diffusion models on bounded domains. In order to guarantee the existence of a smooth solution to the equilibrium PDE when $\mathcal{O}$ is bounded, we need to impose some regularity conditions on the drift and volatility coefficients

$$\mu(t, X_t) = \begin{pmatrix} \mu^S(t, S_t) \\ \mu^R(t, R_t) \end{pmatrix} \qquad \text{and} \qquad \sigma(t, X_t) = \begin{pmatrix} \sigma^S(t, S_t) & 0 \\ 0 & \sigma^R(t, R_t) \end{pmatrix}$$

of the process $\{X_t\}$ along with smoothness conditions on the payoff functions. To this end, we denote for any $l \in$ and $\beta \in (0, 1)$ by

$$\mathcal{H}^{l+\beta} = \mathcal{H}^{l+\beta}([0, T] \times \bar{\mathcal{O}})$$

the Banach space of all functions $u(t, x)$ that are bounded and continuous in $[0, T] \times \bar{\mathcal{O}}$ together with all derivatives of the form $D_t^r D_x^s$ for $2r + s \le l$ and whose derivatives of order l are Hölder continuous with coefficient β. We say that the domain $\mathcal{O}$ belongs to class $\mathcal{H}^l$ if its boundary can be described by a Hölder continuous function of order l; see Ladyzenskaja et al. [26, p. 9] for details.

ASSUMPTION 2.1. *There exists $\beta \in (0, 1)$ such that*:
(i) *The terminal payoff functions h^a and h^I belong to $\mathcal{H}^{3+\beta}$ while φ^a and φ^I belong to $\mathcal{H}^{1+\beta}$.*
(ii) *The drift belongs to $\mathcal{H}^{1+\beta}$, the volatility functions are strictly positive, and $\sigma \in \mathcal{H}^{2+\beta}$.*
(iii) *The domain $\mathcal{O}$ is bounded and of class $\mathcal{H}^{2+\beta}$.*
(iv) *The terminal payoff h^I and the yield curve φ^I are monotone in the risk process.*

We have seen in the previous section that the agents' optimization problems can be solved in closed form when the bond completes the market. In a subsequent chapter we show that the condition of market completion holds under Assumption 2.1. Under a suitable consistency condition that links the drift and volatility coefficients and the payoff functions at maturity, the equilibrium market price of climate risk can then be described by the solution of a nonlinear PDE. More precisely, we have the following theorem which also fills a gap in Chaumont et al. [7]. Its proof requires some preparation and will be carried out in §4.2.

THEOREM 2.2. *Suppose that Assumption* 2.1(i)–(iii) *is satisfied and that in equilibrium the climate bond completes the market. Under a suitable consistency condition the equilibrium market price of climate risk $\bar{\theta}^R$ satisfies*

$$\bar{\theta}_t^R = v(t, X_t) \quad \textit{for some function } v \in \mathcal{H}^{2+\beta}.$$

As an intermediate step to market completion, we prove in §4.3.1 that the equilibrium bond price process is a smooth function of the underlying risk factors.

THEOREM 2.3. *Suppose that Assumption* 2.1(i)–(iii) *is satisfied and that in equilibrium the climate bond completes the market. Under a suitable consistency condition on the diffusion coefficients and the bond's payoff functions, the equilibrium bond price process $\{B_t^{\bar{\theta}}\}$ satisfies*

$$B_t^{\bar{\theta}} = u(t, X_t) \quad \textit{for some function } u \in \mathcal{H}^{3+\beta}.$$

The smoothness properties of the solution u allow us to represent the integrand κ^R in (10) as the solution of a linear backward equation. The solution can be given in closed form. It turns out that the market is complete if the bond payoff functions are monotone in the external source of risk. The following result is obtained in §4.3.2.

THEOREM 2.4. *Suppose that Assumption* 2.1(i)–(iv) *is satisfied. Under a suitable consistency condition there exists a unique* (*in a certain class*) *equilibrium market price of climate risk.*

2.3.2. A benchmark diffusion model on unbounded domains. The case $\mathcal{O} = \mathbb{R}^2$ poses additional challenges because no general existence and uniqueness result for our equilibrium PDE on unbounded domains is available. In §5 we prove such a result for a benchmark geometric Brownian motion model using probabilistic arguments. The key is to establish Malliavin differentiability of the associated quadratic BSDE. The differentiability result does not carry over to bounded domains as this would involve taking the derivative of random stopping times which are typically not Malliavin differentiable. More specifically we have the following theorem.

THEOREM 2.5. *Let* $\mathcal{O} = \mathbb{R}^2$. *Suppose that the asset price process follows a geometric Brownian motion with possibly time-dependent drift and volatility functions and that the risk process has an additive noise term. Under Assumption* 2.1(iv) *and under suitable conditions on the diffusion coefficients and payoff functions, there exists a unique* (*in a certain class*) *equilibrium market price of climate risk.*

To summarize, we prove the existence of a unique smooth equilibrium market price of climate risk under the assumption that the bond's payoff structure is monotone in the external risk factor. For bounded domains the consistency conditions will be introduced when needed. Both the equilibrium market price of risk and the bond price process are given in terms of a classical solution of a PDE. This makes our model amenable to a numerical analysis of equilibrium.

3. The equilibrium market price of climate risk. This section characterizes all equilibrium market prices of risk θ that have the BMO-property under the assumption that the risk bond completes the market. Specifically, we characterize the equilibrium as a solution to a BSDE. A BSDE is an equation of the type

$$Y_t = H - \int_t^\tau Z_s\,dW_s + \int_t^\tau F(s, Y_s, Z_s)\,ds \quad (0 \le t \le \tau) \tag{21}$$

where W is a standard n-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the standard Brownian filtration $(\mathcal{F}_t)$. The finite stopping τ is called the *terminal time*, while the random function F and the $\mathcal{F}_\tau$-adapted random variable H are usually referred to as the *coefficient* and *terminal value*, respectively.

A *solution* consists of an adapted process Y and an adapted integrand Z that satisfy the integral equation (21). Solutions exist under suitable integrability and continuity conditions on H and F. For instance, if $H \in L^2(\mathbb{P})$ and F is Lipschitz continuous in the variables Y and Z, then a solution exists where both processes are continuous, adapted (thus predictable), and square integrable in $[0, \tau] \times \Omega$; see Pardoux and Peng [34] or El Karoui et al. [15]. For bounded random variables H and coefficients that are quadratic in Z, existence of a solution $(Y, Z) \in \mathbb{H}^\infty \times \mathbb{H}^2$ has been shown by Kobylanski [25]. Here

$$\mathbb{H}^\infty \quad \text{and} \quad \mathbb{H}^2$$

denote the space of continuous adapted bounded and square integrable processes, respectively. Her results have recently been extended to exponentially integrable terminal values by Briant and Hu [6].

We illustrate the idea of a backward equation by means of the following example; it will be useful for our subsequent analysis.

EXAMPLE 3.1 KOBYLANSKI [25]. Consider the backward equation

$$Y_t = \xi + \int_t^\tau \frac{1}{2}|Z_s|^2\,ds - \int_t^\tau Z_s\,dW_s$$

for some bounded random variable ξ. Using the exponential change of variable $y = \exp(Y)$ we can transform this equation into the linear backward equation

$$y_t = e^\xi - \int_t^\tau z_s\,dW_s.$$

The unique square integrable solution of the linear equation can be given explicitly:

$$y_t = \mathbb{E}[\exp(\xi) \mid \mathcal{F}_t],$$

while z is given by the theorem of representation of continuous martingales. A solution of the nonlinear equation is then given by

$$Y_t = \ln(y_t) \qquad \text{and} \qquad Z_t = \frac{z_t}{y_t}.$$

The progressively measurable process Y turns out to be bounded while Z is square integrable.

In order to find the bond price process we first characterize equilibria in terms of an equivalent condition on the agents' combined optimal terminal claims. In equilibrium, the agents buy exactly one unit of the bond. They hence receive the payoff $B_\tau^{\bar\theta} = H^I$ from the issuer at maturity while the issuer receives the initial bond price $B_0^{\bar\theta}$ at time $t=0$. The agents hedge their financial risk by trading in the stock market. The accumulated gains or losses from trading in the stock are given by a stochastic integral with respect to the price process. This suggests that in equilibrium a condition of the form

$$\sum_{a\in\mathbb{A}} (\xi_a^{\bar\theta} - H^a) = H^I - c^{\bar\theta} + \int_0^\tau \phi\,(dW_t^1 + \theta_t^S\,dt) \tag{22}$$

holds for some adapted integrable process ϕ and some constant $c^{\bar\theta}$ and vice versa. The following proposition confirms our intuition.

Proposition 3.1. *Let $\bar\theta$ be a market price of risk such that $\int \theta_s\,dW_s$ is a BMO-martingale and assume that the risk bond completes the market. The optimal trading strategies for the bond satisfy the market clearing condition (19) if and only if the utility-maximizing claims combined satisfy Equation (22) for some ϕ and a constant $c^{\bar\theta}$. In this case we get*

$$c^{\bar\theta} = B_0^{\bar\theta} \qquad \text{and} \qquad \phi = \sigma^S \sum_a \bar\pi^{S,a}. \tag{23}$$

Proof. In view of (18) the total optimal excess demand satisfies

$$\sum_{a\in\mathbb{A}} (\xi_a^\theta - H_a) = \int_0^\tau \sum_{a\in\mathbb{A}} \bar\pi_u^{S,a}\,dS_u + \int_0^\tau \sum_{a\in\mathbb{A}} \bar\pi_u^{B^{\bar\theta},a}\,dB_u^{\bar\theta}. \tag{24}$$

If the market clearing condition is satisfied, then

$$\int_0^\tau \sum_{a\in\mathbb{A}} \bar\pi_u^{B^{\bar\theta},a}\,dB_u^{\bar\theta} = B_\tau^{\bar\theta} - B_0^{\bar\theta}$$

and hence (22) holds. At the same time (22) and (24) represent the agents' aggregate excess demand in terms of a stochastic integral with respect to the stock and bond price process. Hence uniqueness of the integrand yields (19). Equation (23) is also a consequence of this uniqueness. □

The equilibrium condition (22) along with the specific structure of the agents' optimal claims (16) allows us to find a BSDE that simultaneously characterizes the market price of climate risk $\bar\theta^R$, the aggregate trading strategy $\sigma^S\phi$ for the financial asset, and the initial bond price $c^{\bar\theta}$. To this end, observe first that ϕ is an integrator for the first component W^1 of the Brownian motion W, due to (22). Equation (16), on the other hand, shows that $\bar\theta^R$ integrates W^2. To transform these variables into the integrand of a BSDE we recall that $\alpha^{-1} = \sum_a \alpha_a^{-1}$ and introduce the quantities

$$H = \alpha\left(\sum_{a\in\mathbb{A}} h_a + h_I\right), \qquad \varphi_t = \alpha\left(\sum_{a\in\mathbb{A}} \varphi_a + \varphi_I\right), \qquad Y_0 = \alpha\left(B_0^\theta + \sum_{a\in\mathbb{A}} c_a^\theta\right)$$

and

$$z_t = (z_t^S, z_t^R) \quad \text{where } z_t^S = \theta_t^S - \alpha\sigma_t^S\phi_t \text{ and } z_t^R = \theta_t^R. \tag{25}$$

Plugging (17) into (22) and rearranging terms shows that the equilibrium condition (22) can be rewritten as

$$Y_0 = H - \int_0^\tau \begin{pmatrix} z_s^S \\ z_s^R \end{pmatrix} d(W_s^1, W_s^2) + \frac{1}{2}\int_0^\tau [-(z_s^R)^2 + (\theta_s^S)^2 - 2\theta_s^S z_s^S + \varphi_s]\,ds. \tag{26}$$

Replacing the initial time 0 by a time $t\in[0,\tau]$ in (26) we obtain the BSDE

$$Y_t = H - \int_t^\tau \begin{pmatrix} z_s^S \\ z_s^R \end{pmatrix} d(W_s^1, W_s^2) + \frac{1}{2}\int_t^\tau [-(z_s^R)^2 + (\theta_s^S)^2 - 2\theta_s^S z_s^S + \varphi_s]\,ds \tag{27}$$

for the equilibrium market price of climate risk. Specifically, the following result holds; we sketch its proof and refer to Hu et al. [21] for details.

THEOREM 3.1. *The backward Equation (27) has a unique solution $(Y, (z^S, z^R))$. Under the assumption that the bond completes the market, z^R is an equilibrium market price of climate risk.*

PROOF. Existence and uniqueness of a solution (Y, Z) to (27) where Y is bounded and Z is square integrable has been established by Kobylanski [25]. In view of (25) the integrand (z^S, z^R) can be transformed in the processes $\bar{\theta}^R$ and ϕ. Calculating the combined optimal claims $\sum_a \xi_a^{\bar{\theta}}$ for $\bar{\theta} = (\theta^S, z^R)$, we see that (22) is satisfied for

$$c^{\bar{\theta}} = \frac{1}{\alpha} Y_0 - \sum_a c_a^{\bar{\theta}}.$$

Hence the equilibrium property follows from Proposition 3.1 because $(\int_0^t \theta_s \, dW_s)$ is a BMO-martingale, due to Lemma 3.4 in Hu et al. [21]. □

The previous result establishes the existence of an equilibrium market price of risk under the assumption that, in equilibrium, the risk bond completes the market. It turns out that the equilibrium dynamics can also be viewed as the equilibrium dynamics of a suitable representative agent model. More precisely, our pricing measure is the dual measure of a representative agent model and price dynamics coincide with the price dynamics when the representative agent pools all incomes and trades optimally in the financial market. As such, our equilibrium prices differ from the utility indifference price process of Mania and Schweizer [28].

Our main contribution is to prove completeness in equilibrium. This will be achieved in the next section where we represent the solution of our BSDE in terms of a nonlinear PDE.

REMARK 3.1. Our BSDE characterization of the equilibrium market price of climate risk uses the explicit structure of the agents' excess demand. Due to the exponential utility function, the integrand in the dW-integral of the aggregate demand is the market price of risk. This allows us to translate the equilibrium condition into a BSDE. Such a characterization is not possible for other utility functions. For a simple one-period model, the problem of finding an equilibrium price on a segment of the market is solved in Müller [30] in greater generality.

4. Market completion in equilibrium. In this section we consider general diffusion models on bounded domains. We first characterize the equilibrium market price of risk in terms of a nonlinear PDE. In a subsequent step this result is applied to establish market completeness in equilibrium.

4.1. Linking PDEs and BSDEs. Our approach to equilibrium bond pricing is based on the interplay between PDEs and BSDEs. To illustrate the PDE-BSDE link let us introduce some bounded open domain

$$\mathscr{D} \subset \mathbb{R}^d \qquad \text{and put} \qquad \partial\mathscr{D}_\Lambda = ((0, T) \times \partial\mathscr{D}) \cup (\{\Lambda\} \times \mathscr{D}) \quad \text{for } \Lambda \in \{0, T\}.$$

We denote by $\mathscr{C}^{1,2}((0, T) \times \mathscr{D})$ the class of all continuous functions u: $(0, T) \times \mathscr{D} \to \mathbb{R}$ that are continuously differentiable with respect to the time variable and twice continuously differentiable with respect to the state variable, and let

$$u_t := \frac{\partial u}{\partial t}, \qquad u_i := \frac{\partial u}{\partial x_i}, \qquad \nabla_x u = (u_i)_i, \qquad u_{i,j} = \frac{\partial^2 u}{\partial x_i \partial x_j} \qquad \text{and} \qquad \nabla_x^2 u = (u_{i,j})_{i,j}.$$

Given a d-dimensional Brownian motion W and a pair $(t, x) \in [0, T] \times \mathscr{D}$, we denote by $X^{t,x}$ the solution to the stochastic differential equation (SDE)

$$\begin{cases} dX_s^{t,x} = \mu(s, X_s^{t,x})dt + \sigma(s, X_s^{t,x})dW_s & \text{for } t \le s \le \tau, \\ X_t^{t,x} = x, \end{cases} \tag{28}$$

where τ denotes the infimum of the exit time of $X^{t,x}$ from $\mathscr{D}$ and some given time horizon $T > 0$. We assume that the drift and volatility functions are smooth with linear growth at infinity and that the generator $\mathscr{L}$ associated with $X^{t,x}$ is uniformly elliptic. Under these assumptions, the Feynman-Kac formalism asserts that the unique classical solution $u \in \mathscr{C}^{1,2}$ of the linear boundary value problem

$$\begin{cases} -u_t - \mathscr{L}u + f - hu = 0 & \text{on } \mathscr{D}, \\ u = g & \text{on } \partial\mathscr{D}_T, \end{cases} \tag{29}$$

for Lipschitz continuous functions $f(t,\cdot), h(t,\cdot)\colon \mathscr{D}\to\mathbb{R}$ with linear growth at infinity can be represented in probabilistic terms as

$$u(t,x)=\mathbb{E}\left[\int_t^\tau f(s,X_s^{t,x})e^{-\int_s^\tau h(\tau,X_r^{t,x})\,dr}\,ds+g(\tau,X_\tau^{t,x})e^{-\int_t^\tau h(\tau,X_r^{t,x})\,dr}\right].$$

A nonlinear extension of the Feynman-Kac formula allows us to represent the solution of the BSDE

$$\begin{cases}-dY_s^{t,x}=F(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})ds-Z_s^{t,x}dW_s & \text{for } t\le s\le \tau,\\ Y_\tau^{\tau,x}=g(\tau,X_\tau^{t,x}),\end{cases}\tag{30}$$

driven by the forward process $X^{t,x}$ in terms of a nonlinear PDE. More specifically, an application of Itô's formula yields the following result.

Proposition 4.1. *Let $F\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ and $g\colon [0,T]\times\mathbb{R}^d\to\mathbb{R}$ be bounded measurable functions. If the nonlinear boundary value problem*

$$\begin{cases}-u_t+\mathscr{L}u-F(t,x,u,\sigma(s,x)\cdot\nabla_x u)=0 & \text{on } \mathscr{D},\\ u=g & \text{on } \partial\mathscr{D}_T,\end{cases}\tag{31}$$

has a classical solution $u\in\mathscr{C}^{1,2}$, then a solution $(Y^{t,x},Z^{t,x})$ of the BSDE (30) is given by

$$\begin{cases}Y_s^{t,x}=u(s,X_s^{t,x}),\\ Z_s^{t,x}=\sigma(s,X_s^{t,x})\cdot\nabla_x u(s,X_s^{t,x}).\end{cases}\tag{32}$$

While an array of "small-T-solutions" is available that guarantees the existence of a smooth classical solution of (31) up to some "explosion time," the existence of a global solution on the entire time interval $[0,T]$ is typically not guaranteed. This needs to be established on a case-by-case basis. It typically requires a couple of consistency conditions that link the partial derivatives of the terminal data with the diffusion coefficients.

The generalized Feynman-Kac formalism carries over to unbounded domains with obvious modifications. Unbounded domains are studied in, for instance, Chaumont et al. [7]. However, their Theorem 1.4 only yields a solution for small times or, equivalently, a global solution for sufficiently small initial data.

4.2. The PDE for the market price of climate risk. In this section we represent the equilibrium market price of climate risk in terms of a sufficiently regular solution of a nonlinear PDE. For this we denote by $\langle,\rangle$ the standard inner product and recall that the strictly elliptic generator $\mathscr{L}$ of the diffusion process $\{X_t\}=\{(S_t,R_t)\}$ acts on sufficiently smooth functions $u(t,x)$ according to

$$\mathscr{L}u=-\frac{1}{2}\sigma\sigma^t\nabla_x^2u-\langle\mu,\nabla_x u\rangle.\tag{33}$$

For a function $g(t,\cdot)$ on $\mathbb{R}^2$, the first coordinate x_1 throughout stands for an asset price S while the second coordinate x_2 stands for a state R of the external risk process; $\{X_s^{t,x}\}$ denotes the process $\{X_s\}$ that starts in $x=(x_1,x_2)$ at time t. Finally, we recall the definition of the processes z^S and z^R from (25) and put

$$Z_s^{x,t}=(z^S(s,X^{x,t}),z^R(s,X^{t,x}))\equiv(z_s^S,z_s^R).$$

In terms of this notation, the backward stochastic Equation (27) is of the form (30) with coefficient

$$G(s,X_s^{x,t},Y_s^{x,t},Z_s^{x,t})=\frac{1}{2}[-(z_s^R)^2+(\theta^S(s,S_s))^2-2\theta^S(s,S_s)z_s^S+2\varphi(s,X_s^{t,x})].$$

It follows from Proposition 4.1 that if the associated nonlinear boundary value problem

$$\begin{cases}-u_t+\mathscr{L}u-G(t,x,u,\sigma(t,x)\cdot\nabla_x u)=0 & \text{on } \mathscr{O},\\ u=h & \text{on } \partial\mathscr{O}_T,\end{cases}\tag{34}$$

has a sufficiently regular solution u, then the equilibrium market price of climate risk is given in terms of the derivative of u with respect to the climate component

$$\bar{\theta}_t^R = \sigma^R(t, X_t^{0,x}) u_2(t, X_t^{0,x}).$$

To establish the existence of a smooth solution, we first transform our terminal value problem into an initial value problem. For this we put $\hat{\theta}^S(t,\cdot) = \theta^S(T-t,\cdot)$ and introduce the corresponding time-reversed functions $\hat{\theta}^R$, $\hat{\mu}^S$, $\hat{\mu}^R$, $\hat{\varphi}$, and $\hat{h}$ along with the second order differential operator

$$\hat{\mathscr{L}}u = -\frac{1}{2}\hat{\sigma}\hat{\sigma}^t \nabla_x^2 u - \langle \hat{\mu}, \nabla_x u\rangle. \tag{35}$$

The terminal value problem (34) is then equivalent to the initial value problem

$$\begin{cases} u_t + \hat{\mathscr{L}}u - G(t, x, u, \hat{\sigma}(t,x)\cdot\nabla_x u) = 0 & \text{on } \mathscr{O}, \\ u = \hat{h} & \text{on } \partial\mathscr{O}_0. \end{cases} \tag{36}$$

In order to apply Theorem V.6.1 in Ladyzenskaja et al. [26] to (36) we bring our initial value problem in the form of their Equation (6.1) of Chapter V. To this end, we rewrite the PDE in (36) as

$$u_t - \sum_{i=1}^2 \frac{d}{dx_i}(a_i(t, x, u, \nabla_x u)) + a(y, x, u, \nabla_x u) = 0, \tag{37}$$

where

$$a_1(t, x, u, \nabla_x u) = \frac{1}{2}(\hat{\sigma}^S(t, x_1))^2 u_1, \qquad a_2(t, x, u, \nabla_x u) = \frac{1}{2}(\hat{\sigma}^R(t, x_2))^2 u_2,$$

and

$$a(t, x, u, \nabla_x u) = -\langle \hat{\mu}, \nabla_x u\rangle + \frac{1}{2}(\hat{\sigma}^R)^2 u_2^2 - \hat{\varphi} + \hat{\sigma}^S \hat{\sigma}_1^S u_1 + \hat{\sigma}^R \hat{\sigma}_2^R u_2 + \theta^S \sigma^S u_1 - \frac{1}{2}(\theta^S)^2.$$

Theorem 4.1. *Suppose that Assumption 2.1 is satisfied and that the first order compatibility condition*

$$\hat{h}_t - \sum_{i=1}^2 \frac{d}{dx_i}(a_i(0, x, \hat{h}, \nabla_x \hat{h})) + a(0, x, \hat{h}, \nabla_x \hat{h}) = 0 \quad \text{on } \partial\mathscr{O}_0 \tag{38}$$

holds. Then the boundary value problem (34) has a unique solution $u \in \mathscr{H}^{2+\beta}$.

Proof. The proof follows from Theorem V.6.1 in Ladyzenskaja et al. [26] upon verification of the respective assumptions. To this end, we first bring Equation (37) in the form of Equation (6.4) of Chapter V in Ladyzenskaja et al. [26]:

$$\hat{\mathscr{L}}u \equiv u_t - \sum_{i=1}^2 \frac{\partial a_i(t, x, u, \nabla_x u)}{\partial u_i} u_{i,i} + A(t, x, u, \nabla_x u) = 0, \tag{39}$$

where

$$A(t, x, u, \nabla_x u) = a(t, x, u, \nabla_x u) - \sum_{i=1}^2 \frac{\partial a_i(t, x, u, \nabla_x u)}{\partial x_i}.$$

Since $A(t, x, u, 0) = a(t, x, u, 0) = -(1/2)(\theta^S(t, x_1))^2 - \hat{\varphi}(t, x)$ is independent of u and because both $\hat{\varphi}$ and θ^S are bounded, there exist nonnegative constants b_1 and $b_2 \geq 0$ such that

$$A(t, x, u, 0)u \geq -b_1 u^2 - b_2 \quad \text{for all } u \in \mathbb{R}, \quad t \in [0, T], \quad \text{and } x \in \bar{\mathscr{O}}.$$

Observe now that $a_i(x, t, u, \nabla_x u)$ depends on $\nabla_x u$ only through u_i and that $\partial a_i(x, t, u, \nabla_x u)/\partial u_i > 0$. Hence, condition (a) of Theorem V.6.1 in Ladyzenskaja et al. [26] is satisfied. Conditions (b)–(d) hold because the functions a_i and a are smooth and Hölder continuous while assumptions (e) and (f) are satisfied due to Assumption 2.1(i) and (iii) and the first order consistency condition (38). As a result, (34) has a unique solution from the class $\mathscr{H}^{2+\beta}$. □

Additional regularity of the solution is guaranteed due to our regularity assumptions on the coefficients of $\mathscr{L}$ and the boundary condition. In fact, by (39) the function u solves the *linear* boundary value problem

$$\begin{cases} w_t - \sum_{i=1}^{2} \dfrac{\partial a_i(t,x,u,\nabla_x u)}{\partial u_i} w_{i,i} + A(t,x,u,\nabla_x u) = 0 & \text{on } \mathscr{O}, \\ w = \hat{h} & \text{on } \partial\mathscr{O}_0. \end{cases} \tag{40}$$

Since $u \in H^{2+\beta}$, we see that its derivative $\nabla_x u$ belongs to $\mathscr{H}^{1+\beta}$ so Assumption 2.1 guarantees that

$$a_i(t,x,u(t,x),\nabla_x u(t,x)) \in H^{1+\beta} \qquad \text{and} \qquad A(t,x,u(t,x),\nabla_x u(t,x)) \in H^{1+\beta}. \tag{41}$$

Hence Theorem IV.5.2 in Ladyzenskaja et al. [26] asserts the existence of a unique solution w of (40) from the class $H^{3+\beta}$ upon verification of the following first order compatibility condition:

$$\hat{h}_t(0,x) = \sum_{i=1}^{2} \frac{\partial a_i(0,x,u(0,x),\nabla_x u(0,x))}{\partial u_i} \hat{h}_{x,x}(0,x) - A(0,x,u(0,x),\nabla_x u(0,x)) \quad \text{on } \partial\mathscr{O}_0.$$

Since u satisfies the boundary condition $u(0,x) = \hat{h}(0,x)$, this is nothing but Equation (38). Viewing the coefficients (41) of the linear problem as functions from the class $\mathscr{H}^{\beta}$, we see that w is also the unique solution from the class $H^{2+\beta}$. Along with the fact that $u \in \mathscr{H}^{2+\beta}$ solves (40), this yields $w = u$. Hence we have shown the following result.

Theorem 4.2. *Under the assumptions of Theorem 4.1, the nonlinear boundary value problem (34) has a unique solution $u \in \mathscr{H}^{3+\beta}$.*

4.3. Market dynamics in equilibrium. The previous section characterized the equilibrium market price of climate risk in terms of a smooth solution to a nonlinear boundary value problem. Our analysis of equilibrium is hence complete once we can prove that under the candidate pricing measure the market is complete. This is achieved by showing that the corresponding price process has the representation property.

In this section we give an explicit formula of the integrand κ^R in (10). To this end, we will first find a backward equation for the bond prices. The solution to the BSDE simultaneously defines the bond prices and κ^R. In a subsequent step we derive a PDE-representation of κ^R. It will turn out that this PDE can be associated with a linear BSDE. Its solution κ can hence be given in closed form as the conditional expectation of a strictly positive (negative) random variable. As such it is almost surely different from zero and hence the market is complete.

4.3.1. The PDE for the bond price process. We recall from (10) that the random variables B_t^θ can be represented as stochastic integrals with respect to the $\mathbb{P}^\theta$-Brownian motion W^θ. Uniqueness of the representation of continuous martingales shows that the risk bond B^θ along with the integrand κ^θ constitute the unique pair of adapted processes that satisfy the backward equation

$$\begin{aligned} Y_t &= H^I - \int_t^\tau Z_s \, dW_s^\theta \\ &= h^I - \int_t^\tau Z_s \, dW_s - \int_t^\tau \{\langle Z_s, \theta_s\rangle - \varphi_s^I\} \, ds; \end{aligned} \tag{42}$$

see also Example 3.1. The market price of risk $\theta = (\theta^S, \theta^R)$ is a given function of time and space with smooth derivatives up to order three due Theorem 4.2. Hence the PDE

$$\begin{cases} -v_t + \mathscr{L}v + \langle \sigma \cdot \nabla_x v, \theta\rangle - \varphi^I = 0 & \text{on } \mathscr{O}, \\ v = h^I & \text{on } \partial\mathscr{O}_T, \end{cases} \tag{43}$$

associated with (42) has smooth coefficients. If this boundary value problem has a sufficiently regular solution v, then the solution to (42) and hence the quantities B_t^θ and κ_t are given by

$$B_t^\theta = v(t,S_t,R_t), \qquad \kappa_t^S = \sigma^S(t,S_t) v_1(t,S_t,R_t), \qquad \text{and} \qquad \kappa_t^R = \sigma^R(t,R_t) v_2(t,S_t,R_t).$$

In view of Theorem 4.2, the coefficients of the linear PDE belong to $\mathscr{H}^{2+\beta}$. Transforming the terminal value problem into an initial value problem along the lines of the previous section shows that the time-transformed coefficients satisfy the assumptions of Theorem IV.5.2 in Ladyzenskaja et al. [26] with $l = 2$. This yields the following result.

Theorem 4.3. *Under the assumptions of Theorem 4.1 the linear PDE (43) has a unique solution $v \in \mathscr{H}^{3+\beta}$ if the following compatibility condition holds*:

$$(\mathscr{L}h^I)(T,x) + \left\langle \sigma(T,x)\cdot\nabla_x h^I(T,x), \begin{pmatrix}\theta^S(T,x)\\ \nabla_x h^I(T,x)\end{pmatrix}\right\rangle - \varphi^I(T,x) = h^I_t(T,x) \quad \text{on } \partial\mathscr{O}. \tag{44}$$

The consistency condition links the coefficients of the diffusion process X with the bond's payoff structure. If, at maturity, the bond simply pays off its face value, then the terminal payoff function h^I is constant. In such a situation the consistency condition reduces to

$$\varphi^I(T,x) = 0 \quad \text{for } x \in \partial\mathscr{O}. \tag{45}$$

4.3.2. The PDE for the integrand. Integrants of stochastic BSDEs have previously been identified by El-Karoui et al. [15] under Lipschitz conditions on the coefficients. Lazrak [27] used their explicit solution to show that the integrand of some Lipschitz BSDEs is bounded away from zero. This section gives an explicit representation of the integrand κ^R in the martingale representation for the bond price process. By Theorem 4.3 we can derive a PDE representation of the function κ^R. Specifically, $w = v_2$ with $\nabla_x w = (v_{1,2}, v_{2,2})$ satisfies a linear PDE of the form

$$\begin{cases} -w_t + \mathscr{L}w - \varphi^I_2 - Aw - \langle \sigma\cdot\nabla_x w, B\rangle = 0 & \text{on } \mathscr{O},\\ w = h^I_x & \text{on } \partial\mathscr{O}_T. \end{cases} \tag{46}$$

The functions A and B are defined in terms of the market price of risk, its derivative with respect to the climate variable, and the drift and volatility coefficients by

$$A(t,S_t,R_t) = \mu^R_2(t,S_t,R_t) - (\sigma^R\cdot\theta^R)_2(t,S_t,R_t) \qquad \text{and} \qquad B(t,S_t,R_t) = (-\theta^S, \sigma^R_2 - \theta^R)(t,S_t,R_t).$$

Our previous results show that the equilibrium market price of risk and its derivative with respect to the climate component are bounded on $[0,\tau]\times\bar{\mathscr{O}}$. As a result, A and B are bounded.

With the PDE (46) we can associate the following linear backward stochastic differential equation with a bounded coefficient:

$$\kappa^R_t = h^I_x - \int_t^\tau \alpha_u\,dW_u - \int_t^\tau \{A(u,S_u,R_u)\kappa^R_u + \langle B(u,S_u,R_u), \alpha_u\rangle + \varphi^I_2\}\,du.$$

By the representation theorem of linear BSDEs as a conditional expectation (see, e.g., El Karoui et al. [15]) the integrand κ^R can be given in closed form:

$$\kappa^R_t = \mathbb{E}\left[\Upsilon^\tau_t h^I_2(\tau,S_\tau,R_\tau) + \int_t^\tau \Upsilon^u_t \varphi^I_2(u,S_u,R_u)\,du \,\middle|\, \mathscr{F}_t\right] \tag{47}$$

where the strictly positive quantities Υ^u_t are given by

$$\Upsilon^u_t = \exp\left(\int_t^u B(r,S_r,R_r)\,dW_r - \frac{1}{2}\int_t^u |B(r,S_r,R_r)|^2\,dr - \int_t^u A(r,S_r,R_r)\,dr\right).$$

This function is well defined since A and B are bounded. From (47) we see that $\kappa^R_t \neq 0$ almost surely if the bond's terminal payoff and its yield curve payoff are monotone in the climate variable and strictly monotone on a set of positive Lebesque measure on $(0,T)\times\mathbb{R}^2$. Hence we have the following result.

Theorem 4.4. *Suppose that Assumption 2.1 is satisfied and that the payoff functions φ^I and h^I are monotone in the risk process and strictly monotone on a set of positive measure. If the compatibility conditions (38) and (44) are satisfied, then there exists a unique equilibrium market price of climate risk in the class of twice continuously differentiable functions.*

5. A benchmark model on unbounded domains. In the previous sections we provided an equilibrium approach to pricing risk bonds for diffusion models on bounded domains. The approach was based on a PDE method to solving the equilibrium BSDE. The lack of existence and uniqueness results for solution of nonlinear

PDEs that yield sufficiently integrable stochastic processes when applied to the forward diffusion was the reason for restricting ourselves to bounded domains.

An extension to unbounded domains is straightforward if the payoff functions are independent of the financial market dynamics. In this case the equilibrium market price of climate risk is given by a one-dimensional quadratic PDE. This PDE can be transformed into a linear PDE for which existence and uniqueness results for bounded solutions on unbounded domains are readily available. If the payoffs depend on stock prices in a nontrivial manner, such a transformation is not possible because the PDE is linear in the first component of the gradient and quadratic in the second component.

In the sequel we provide an extension to $\mathscr{O} = \mathbb{R}^2$ for a benchmark geometric Brownian motion model. Our arguments use Malliavin calculus and generalize earlier results of Chaumont et al. [8]; for the theory of Malliavin derivatives we refer to the book by Nualart [33].

Throughout this section we assume that asset prices follow a geometric Brownian motion and that the external risk process has an additive noise term, a feature shared by many climate models. Adjusting the payoff functions accordingly we may as well consider the logarithm of the stock price, a Brownian motion with drift. Hence our forward processes are

$$dS_t = \left(\mu(t) - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)\,dW_t^1 \qquad \text{and} \qquad dR_t = m(t, R_t)dt + dW_t^2. \tag{48}$$

We also assume that the diffusion coefficients and payoff functions are sufficiently smooth with bounded derivatives. To obtain bounded Malliavin derivatives we also impose some conditions on the diffusion coefficients and the payoff functions.

Assumption 5.1. (i) *The diffusion coefficients μ and σ are bounded* (*time-dependent*) *deterministic functions with bounded derivatives, and σ and σ' are bounded away from* 0.

(ii) *The function m is differentiable with respect to the space variable with bounded derivative.*

(iii) *The aggregate terminal payoff function h as well as the income rate φ are twice continuously differentiable with bounded first and second derivatives.*

Restricting ourselves to a simple benchmark model with a deterministic market price of financial risk we are able to prove Malliavin differentiability of the quadratic BSDE for the equilibrium market price of climate risk. Differentiability of stochastic backward equations with Lipschitz continuous driver and deterministic terminal times has been established in El Karoui et al. [15], but to the best of our knowledge no extension to quadratic BSDEs is available.

Armed with our differentiability result we prove subsequently that the PDE associated with the equilibrium BSDE has a smooth bounded solution with bounded derivatives. This yields that the Malliavin derivatives of the equilibrium market price of climate risk are bounded. In a final step we prove a decomposition result for Malliavin derivatives of *linear* BSDEs from which we deduce market completeness. The proof of market completeness is based solely on probabilistic arguments and does not need PDE methods.

Remark 5.1. (i) The approach of this section does *not* apply to the case of barrier options studied in the previous section. Working with random exit times instead of deterministic terminal times would involve taking the Malliavin derivative of stopping times. Stopping times, however, are typically not Malliavin differentiable.

(ii) The approach of this section suggests a probabilistic reason why the treatment of models with general diffusions on unbounded domains may currently be beyond the reach of the PDE approach: the lack of a general existence result for Malliavin derivatives of quadratic BSDEs. Such a result would yield existence and uniqueness results for the equilibrium PDEs. The existence of a smooth solution to this equation with bounded derivatives, on the other hand, would yield differentiability of our quadratic BSDE; see Proposition 1.5.1 in Nualart [33].

Before proceeding with the differentiability properties of our equilibrium BSDE we briefly recall the definition of a Malliavin derivative of a diffusion process; see §2.2 in Nualart [33] for details. Let (M_t) be defined by the SDE

$$dM_t = b(t, M_t)\,dt + \sigma(t, M_t)\,dW_t \tag{49}$$

where W is an m-dimensional vector of independent Brownian motions, $b(t,\cdot) \in \mathbb{R}^m$ and $\sigma(t,\cdot) = \operatorname{diag}(\sigma^1(t,\cdot), \ldots, \sigma^m(t,\cdot))$ is a diagonal matrix. We assume that the diffusion coefficients are differentiable in x with bounded derivatives and fix a differentiable function $g\colon \mathbb{R}^m \to \mathbb{R}$. With

$$\Delta_{i,t} := \exp\left(\int_0^t \sigma_x^i(s, M_s^i)\,dW_s^i + \int_0^t \left\{ b_x^i(s, M_s^i) - \frac{1}{2}(\sigma_x^i)^2(s, M_s^i)\right\} ds\right), \tag{50}$$

the Malliavin derivative $D_u^i g(M_t)$ of $g(M_t)$ at time u with respect to the ith Wiener process is given by

$$D_u^i g(M_t) = \mathbf{1}_{\{t\geq u\}} g_x^i(M_t)\sigma(u, M_u)\Delta_{i,t}\Delta_{i,u}^{-1}.$$

For our process $X_t = (S_t, R_t)$ this means that

$$D_u^1 X_t = \mathbf{1}_{\{t\geq u\}}\sigma(u) \qquad \text{and} \qquad D_u^2 X_t = \mathbf{1}_{\{t\geq u\}} \exp\left(\int_u^t m_x(s, R_s)\, ds\right). \tag{51}$$

The boundedness of the Malliavin derivatives D^1X and D^2X and the fact that the market price of financial risk is deterministic allows us to take the Malliavin derivatives of the quadratic BSDE for the equilibrium market price of climate risk.

5.1. The equilibrium market price of risk as a Malliavin derivative. In this section we identify the equilibrium market price of climate risk in terms of the Malliavin derivative of the quadratic BSDE (27). The assumptions of a deterministic market price of financial risk and an additive noise dynamic for the external risk process are key.

THEOREM 5.1. *Under Assumption 5.1, the solution of the BSDE (27) is Malliavin differentiable. The derivative processes D^1Y and D^2Y of Y with respect to W^1 and W^2 satisfy the BSDE*

$$D_u^i Y_t = D_u^i h(S_T, R_T) - \int_t^T D_u^i Z_s\, dW_s + \int_t^T \{-z_s^R D_u^i z_s^R - 2\theta_s^S D_u^i z_s^S + D_u^i \varphi(s, X_s)\}\, ds.$$

Furthermore, there exists a bounded version of Z. More precisely, there exists a constant $C > 0$ such that Z satisfies for every $t \in [0, T]$ $\mathbb{P}$-a.s.

$$|z_t^S| \leq C(\|h_1\|_\infty + \|\varphi_1\|_\infty) \qquad \text{and} \qquad |z_t^R| \leq C(\|h_2\|_\infty + \|\varphi_2\|_\infty). \tag{52}$$

PROOF. Let us introduce the truncated quadratic functions

$$g_n(x) = \begin{cases} x^2 & |x| \leq n, \\ n^2 & |x| > n, \end{cases}$$

and consider the family of backward equations

$$Y_t^n = h(X_T) - \int_t^T Z_s^n\, dW_s + \int_t^T \left\{-\frac{1}{2} g_n(z_s^{R,n}) + \frac{1}{2}(\theta_s^S)^2 - 2\theta^S z_s^{S,n} + \varphi(s, X_s)\right\} ds. \tag{53}$$

All these equations have Lipschitz continuous drivers. Hence they are Malliavin differentiable due to Proposition 5.3 in El Karoui et al. [15]. Their derivatives satisfy

$$D_u^i Y_t^n = D_u^i h(X_T) - \int_t^T D_u^i Z_s^n\, dW_s - \int_t^T \{\mathbf{1}_{\{|z_s^{R,n}|\}\leq n} z_s^{R,n} D_u^i z_s^{R,n} + 2\theta_s^S D_u^i z_s^{S,n} + D_u^i \varphi(s, X_s)\}\, ds$$

because θ^S is deterministic so that its Malliavin derivative vanishes and the Brownian motions driving asset prices and the external risk factor are independent.

According to a convergence result by Kobylanski [25], the solutions (Y^n, Z^n) converge to (Y, Z) in $\mathbb{H}^\infty \times \mathbb{H}^2$ where $\mathbb{H}^\infty$ and $\mathbb{H}^2$ are the spaces of continuous adapted bounded and square integrable processes, respectively. In fact, all the integrands Z^n have a uniformly bounded version. In order to see this, recall that the solutions of the differentiated BSDE can be given in closed form: in terms of

$$\mathscr{E}_s^{t,n} := \mathscr{E}\left(-\int_t^s \begin{pmatrix} \theta_u^S \\ \mathbf{1}_{\{|z_u^{R,n}|\leq n\}} z_u^{R,n} \end{pmatrix} dW_u\right), \qquad t \leq s \leq T,$$

the solutions take the form

$$D_u^i Y_t^n = \mathbb{E}\left[\mathscr{E}_T^{t,n} D_u^i h(X_T) + \int_t^T \mathscr{E}_s^{t,n} D_u^i \varphi(s, X_s)\, ds \,\Big|\, \mathscr{F}_t\right].$$

In particular, $D_u^i Y_t^n$ is given as the conditional expected value of $D_u^i h(X_T)$ where the expectation is taken with respect to the probability measure equivalent to $\mathbb{P}$ with density $\mathscr{E}_T^{0,n}$. In view of (51) the quantities

$$D_u^1 h(X_T) = h_1(X_T) D_u^1 S_T \qquad \text{and} \qquad D_u^2 h(X_T) = h_2(X_T) D_u^2 R_T,$$

as well as the Malliavin derivatives $D_u^i \varphi(t, X_t)$ are bounded so the derivatives $D_u^i Y_t^n$ are uniformly bounded by some constant N. Since the respective Malliavin traces $(D_t^1 Y_t^n)$ and $(D_t^2 Y_t^n)$ are versions of $z^{S,n}$ and $z^{R,n}$, we see that all the integrands have uniformly bounded versions. By Kobylanski's [25] convergence result this yields

$$Y = Y^n \qquad \text{and} \qquad Z = Z^n$$

for all $n > N$. In particular (Y, Z) solves (53) for all $n \geq N$ so Y is Malliavin differentiable with a bounded derivative. □

The previous theorem allows us to prove the existence of a classical solution to the terminal value problem

$$\begin{cases} -u_t + \mathscr{L}u - \dfrac{1}{2}u_2^2 + \dfrac{1}{2}(\theta^S)^2 - \left(\mu - \dfrac{1}{2}\sigma^2\right)u_1 + \varphi = 0, \\ u(T, x) = h(x), \end{cases} \tag{54}$$

associated with the equilibrium BSDE. After time reflection (54) can be identified with an initial value problem so Proposition 15.1.1 in Taylor [39] yields a solution up to some explosion time where either the solution itself or its gradient explodes. Using our probabilistic representation of the solution we show that both the solution and its gradient are uniformly bounded in the time variable. This will then allow us to extend the "small-t" solution to a global solution.

Lemma 5.1. *Suppose that the initial value problem* (54) *has a classical solution* $u \in \mathscr{C}^{1,2}$ *on some time interval* $[0, t_0]$. *Then* u *as well as its gradient* $\nabla u(t, \cdot)$ *is uniformly bounded with a bound that does not depend on* t_0.

Proof. Consider the reflected PDE (54) in the interval $[0, t_0]$ and the quadratic BSDE (27) with terminal time t_0. In terms of the notation of §4.1 it follows from (32) that

$$u(t_0 - t, x) = Y_t^{x,t}, \qquad u_1(t_0 - t, x) = \frac{1}{\sigma} z_t^{x,t,S}, \qquad \text{and} \qquad u_2(t_0 - t, x) = z_t^{x,t,R} \tag{55}$$

almost surely. Applying an a priori estimate (Corollary 2.2 in Kobylanski [25]) yields

$$|Y_t^{x,t}| \leq \|h\|_\infty + T\|\varphi\|_\infty$$

for all starting points $x \in \mathbb{R}^2$ and starting times $t \in [0, T]$ of the forward process. Hence the solution u is bounded by the terminal payoff function. The estimates (52) for the integrand of the BSDE yield a bound for the gradient of $u(t, \cdot)$ in terms of the derivatives of the payoff functions:

$$|z_t^{x,t,S}| \leq C(\|h_1\|_\infty + \|\varphi_1\|_\infty) \qquad \text{and} \qquad |z_t^{x,t,R}| \leq C(\|h_2\|_\infty + \|\varphi_2\|_\infty) \quad \text{for all } t \in [0, T] \quad \mathbb{P}\text{-a.s.} \quad \square$$

As an immediate corollary from the previous result we obtain a classical global solution to the terminal value problem (54).

Corollary 5.1. *Suppose that Assumption* 5.1 *holds. Then, for all* $T > 0$, *the PDE* (54) *with terminal condition* h *has a unique solution*

$$u \in \mathscr{C}([0, T], C^1(\mathbb{R}^2)) \cap \mathscr{C}^2([0, T] \times \mathbb{R}^2).$$

The solution is bounded with bounded first derivatives.

Proof. By Proposition 15.1.1 in Taylor [39] the terminal value problem (54) has a solution

$$u \in \mathscr{C}([0, t_0], \mathscr{C}^1(\mathbb{R}^2))$$

up to some "explosion time" t_0. The explosion time can be estimated from below by the $\mathscr{C}^1$-norm of the initial data. Lemma 5.1 shows that u with its first derivative is uniformly bounded on $[0, t_0]$ with a bound that does not depend on t_0. Hence a standard argument yields a global solution

$$u \in \mathscr{C}([0, T], \mathscr{C}^1(\mathbb{R}^2))$$

because the terminal data is twice continuously differentiable. To obtain additional regularity we recall that the strongly continuous semi-group $e^{t\mathscr{L}}$ associated with the generator $\mathscr{L}$ of the forward process (48) satisfies

$$\|e^{t\mathscr{L}}\|_{L(\mathscr{C}, \mathscr{C}^s)} \leq C_s t^{-s/2} \quad (0 \leq t \leq 1)$$

for all $s > 0$. In particular, $e^{t\mathscr{L}}$ maps the class of continuous functions into a class of differentiable functions with Hölder continuous derivatives. Hence the semi-group representation

$$u(t, \cdot) = e^{t\mathscr{L}} h + \int_0^t e^{(t-s)\mathscr{L}} \left(-\frac{1}{2} u_2^2(s, \cdot) + \frac{1}{2} (\theta_s^S)^2 - \left(\mu - \frac{1}{2} \sigma^2 \right) u_1(s, \cdot) + \varphi(s, \cdot) \right) ds$$

of u shows that the gradients of $u(t, \cdot)$ belong to the class $\mathscr{H}^\beta$ for all $\beta \in [0, 1)$. Hence a standard result on smoothing properties of heat kernels discussed in, e.g., §4 of Ladyzenskaja et al. [26] shows that $u(t, \cdot)$ is twice continuously differentiable for all $0 \leq t \leq T$ since the terminal condition is sufficiently smooth. □

The proof of the previous corollary shows that the gradient of u is Hölder continuous with constant $\beta < 1$. In a subsequent step we extend this result showing that u_2 is Hölder continuous with constant $\beta = 1$, i.e., Lipschitz continuous. In particular, the second derivative of u with respect to the climate component is bounded. In view of the representation

$$z_t^R = u_2(t, X_t^{x, t})$$

for the market price of climate risk, this shows that its Malliavin derivative with respect to the risk process R exists and is bounded. Specifically, for all $0 \leq u \leq t \leq T$, Equation (51) yields

$$D_t^2 z_u^R = u_{22}(t, X_t^{x, t}) \exp\left(\int_u^t m_x(s, R_s)\, ds \right).$$

Proposition 5.1. *The solution u of the terminal value problem (54) has a uniformly bounded second derivative with respect to the second space variable. In particular, the Malliavin derivatives $(D_t^2 z_u^R)_{u, t \in [0, T]}$ are uniformly bounded.*

Proof. Let us first show that u_2 is Lipschitz continuous. To this end, we differentiate (54) with respect to the second space variable. By Corollary 5.1 the function u_2 is the unique $\mathscr{C}^{1,1}$ solution to the initial value problem

$$\begin{cases} v_t + \sigma^2 v_{11} + v_{22} + (m - x_2) v_2 - v_2 - v v_2 - 2\left(\mu - \frac{1}{2} \sigma^2 \right) v_1 + \varphi_2 = 0, \\ v(0, x) = h_2(x). \end{cases} \tag{56}$$

We may view u_2 as a bounded continuous viscosity solution v of (56) so our goal is to prove Lipschitz continuity of a viscosity solution to a nonlinear PDE. For this, we apply Theorem 3.3 of Jakobsen and Karlsen [22]. It states that $v(t, \cdot)$ is Lipschitz continuous with a constant that is time dependent, but bounded on bounded intervals. To apply their result we have to check their conditions (C1)–(C4) as well as (C6) and (C8). In terms of their notation,

$$-tr[A^\theta(t, x, D_v) D^2 v] = +\sigma^2 v_{11} + v_{22}$$

and

$$f^\theta(t, x, r, p, X) = +(m - x_2) p_2 - p_2 - r p_2 - 2\left(\mu - \frac{1}{2} \sigma^2 \right) p_1 + \varphi_2.$$

Here θ belongs to some set Θ of external parameters and $r \in \mathbb{R}$, $p \in \mathbb{R}^2$, and $X \in \mathbb{R}^{2,2}$ stand for the function v and its first and second derivatives, respectively. In our application, f does not depend on X and Θ is a singleton. Hence condition (C1) as well as the first part of Assumption (C2) are void. The second assumption in (C2) is satisfied with $\gamma_R = -p_2$. Condition (C3) holds for

$$a^\theta = \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$$

while (C4) is satisfied because $f^\theta(t,x,0,0)=0$. For the case of Lipschitz continuity, Assumption (C6) requires that for each $R>0$, there exists a constant $C_R^f>0$ such that

$$|f^\theta(t,x,r,p,X)-f^\theta(t,y,r,p,X)|\le C_R^f|p||x-y|$$

for all $\theta\in\Theta$, each $t\in[0,T]$, all $|r|<R$, $x,y,p\in\mathbb{R}^2$, and every symmetric matrix X. In our case,

$$|f^\theta(t,x,r,p,X)-f^\theta(t,y,r,p,X)|\le|v_2||x_2-y_2|,$$

so (C6) holds with $C_R^f=1$. Assumption (C8) is a condition on the matrix a^θ that is trivially satisfied for constant matrices. As a result, the viscosity solution $v=u_2$ of (5.1) is Lipschitz continuous with constant $L(T)$ on $[0,T]\times\mathbb{R}^2$. By Corollary 5.1 this implies that $u_{2,2}$ is bounded. Hence $|D_t^2 z_u^R|$ is $\mathbb{P}$-a.s. bounded because the function m has a bounded derivative with respect to the space variable. □

Our results show that the solution to the quadratic BSDE that characterizes the equilibrium market price can be represented as a smooth function of the forward process and that it can be viewed as a BSDE with a Lipschitz continuous driver. This makes it amenable to a numerical analysis. To the best of our knowledge all numerical methods such as Bouchard and Touzi [5] for BSDEs assume a Lipschitz driver and no general scheme for quadratic BSDEs is yet available.

5.2. A Malliavin approach to market completeness. Establishing the existence of a smooth global solution to the initial value problem associated with the equilibrium BSDE was key to proving market completeness in the previous sections. Armed with Proposition 5.1 we could now prove existence and uniqueness results for the *linear* BSDE that characterizes the integrand κ^R in the martingale representation of the bond price process using the PDE methods of the previous section. In this section we propose an alternative approach based solely on probabilistic methods. This does not render the results of §4.3.2 redundant because this alternative method does not apply to pricing problems with random terminal times. This would involve Malliavin derivatives of stopping times so for barrier options we still need to rely on PDE methods.

5.2.1. Decomposition of the Malliavin derivative of a linear BSDE. In this section we prove a decomposition property of the Malliavin derivative of a linear BSDE

$$B_t=g(M_T)-\int_t^T Z_s\,dW_s-\int_t^T f(s,M_s)Z_s\,ds. \tag{57}$$

We present the argument for a one-dimensional Brownian motion W and a BSDE depending on a one-dimensional forward diffusion M as defined in (49). In Remark 5.4 we explain how to extend this result to our model. We assume that $g(\cdot)$ and $f(t,\cdot)$ are bounded continuously differentiable functions, and that $g(\cdot)$ has a bounded derivative. Proposition 5.4 in El Karoui et al. [15] guarantees that the solution (B,Z) of (57) is Malliavin differentiable with the derivative processes given by

$$D_uB_t=D_ug(M_T)-\int_t^T D_uZ_s\,dW_s-\int_t^T\{Z_sD_uf(s,M_s)+f(s,M_s)D_uZ_s\}\,ds. \tag{58}$$

Remark 5.2. Notice that (58) yields a family of BSDEs depending on the time parameter $u\in[0,T]$. For a given u the solution $(D_uB_t,D_uZ_t)_{u\le t\le T}$ of the corresponding BSDE can be given in closed form. However, the integrand $Z_t=D_tB_t$ $(t\in[0,T])$ of the BSDE (57) cannot be obtained as the solution to a single BSDE (58). In order to obtain a closed form representation of Z_t we are going to decompose the process $(D_uB_t,D_uZ_t)_{u\le t\le T}$ into the product of an $\mathcal{F}_u$-measurable random variable and a process $(\zeta_t,\alpha_t)_{0\le t\le T}$ that does not depend on u. Specifically we identify $(\zeta_t,\alpha_t)_{0\le t\le T}$ as the solution of the linear BSDE (61) below.

Plugging the derivatives of $g(M_T)$ and $f(t,M_t)$ into (58) yields

$$\begin{aligned}D_uB_t&=g_x(M_T)\Delta_T\Delta_u^{-1}\sigma(M_u)-\int_t^T D_uZ_s\,dW_s\\&\quad-\int_t^T\{Z_sf_x(s,M_s)\Delta_s\Delta_u^{-1}\sigma(M_u)+f(s,M_s)D_uZ_s\}\,ds.\end{aligned} \tag{59}$$

Introducing the processes $\mathscr{E}_s^t:=\mathscr{E}(-\int_t^s f_x\,dW_x)$, a standard argument yields a representation of Z:

$$Z_t=D_tB_t=E\left[\mathscr{E}_T^tg_x(M_T)\sigma(M_t)\Delta_T\Delta_t^{-1}+\int_t^T\mathscr{E}_s^tZ_sf_x(s,M_s)\Delta_s\Delta_t^{-1}\sigma(M_t)\,ds\,\middle|\,\mathcal{F}_t\right]. \tag{60}$$

Multiplying the last equation by $\Delta_t/\sigma(M_t)$ we obtain

$$\frac{\Delta_t}{\sigma(M_t)} Z_t = \mathbb{E}\left[\mathscr{E}_T^t h_x(M_T)\Delta_T - \int_t^T \mathscr{E}_s^t Z_s \frac{\Delta_s}{\sigma(M_s)} f_x(s, M_s)\sigma(M_s)\, ds \,\middle|\, \mathscr{F}_t\right].$$

The conditioned expectation describes the solution $\zeta_t = (\Delta_t/\sigma(M_t))Z_t$ of a linear BSDE. The following lemma provides the decomposition of $(D_u B_t, D_u Z_t)_{u\le t\le T}$.

Lemma 5.2. *Let $(\zeta, \alpha) \in \mathbb{H}^2 \times \mathbb{H}^2$ be the unique solution of the linear BSDE*

$$\zeta_t = g_x(M_T)\Delta_T - \int_t^T \alpha_s\, dW_s - \int_t^T \{\alpha_s f(s, M_s) + \zeta_s \sigma(M_s) f_x(s, M_s)\}\, ds \tag{61}$$

driven by a one-dimensional Wiener process. If the processes

$$\left\{\left(\frac{\sigma(M_u)}{\Delta_u}\zeta_t, \frac{\sigma(M_u)}{\Delta_u}\alpha_t\right)\right\}_{0\le t\le T} \quad \textit{belong to } \mathbb{H}^2 \times \mathbb{H}^2,$$

then we have the following representation of the Malliavin derivative of the solutions to the linear BSDEs (58):

$$D_u B_t = \frac{\sigma(M_u)}{\Delta_u}\zeta_t \quad \textit{and} \quad D_u Z_t = \frac{\sigma(M_u)}{\Delta_u}\alpha_t \qquad \mathbb{P}\textit{-a.s.}$$

In particular, the Malliavin derivatives $D_u B_t$ are strictly positive (negative) if the derivatives of g is positive (negative) and strictly positive (negative) on a set of positive measure.

Proof. The assertion follows by direct verification taking into account the uniqueness of the solution of (58) in $\mathbb{H}^2 \times \mathbb{H}^2$ and that $(\sigma(M_u)\Delta_u^{-1}\zeta_t, \sigma(M_u)\Delta_u^{-1}\alpha_t)_{0\le t\le T}$ is a solution. Positivity (negativity) can be seen by using the explicit solution of (61). □

Remark 5.3. If the PDE associated with (57) has a classical solution with bounded derivatives, it is possible to use the solution of the Feynman-Kac PDE to prove the decomposition of $(D_u B_t, D_u Z_t)_{u\le t\le T}$ into the solution of a BSDE (ζ_t, α_t) multiplied by $\sigma(M_u)/\Delta_u$. In this case

$$B_t = u(t, M_t) \qquad \text{and} \qquad Z_t = \sigma(M_t) u_x(t, M_t)$$

where the forward process M starts at time $t = 0$ in some $x \in \mathbb{R}$. If u is bounded with bounded first and second derivatives, we apply Proposition 1.5.1:

$$D_u Z_t = (\sigma(M_t) u_{xx}(t, M_t) + \sigma(M_t) u_x(t, M_t))\Delta_t \Delta_u^{-1}\sigma(M_u)$$

and

$$D_u B_t = u_x(t, M_t)\Delta_t \Delta_u^{-1}\sigma(X_u).$$

Hence

$$\zeta_s = u_x(s, M_s)\Delta_s \qquad \text{and} \qquad \alpha_s = (\sigma(M_s) u_{xx}(s, M_s) + \sigma_x(M_s) u_x(s, M_s))\Delta_s.$$

Lemma 5.2 yields the representation of the integrand of a class of linear BSDE driven by a one-dimensional Brownian motion. The following remark outlines how to extend this result to multiple dimensions. This can be viewed as an extension of a result of Lazrak [27]. He showed that the integrand of some class of Lipschitz continuous BSDEs is bounded away from zero. We obtain an explicit representation of the integrand for linear BSDEs.

Remark 5.4. Let W be an m-dimensional Brownian motion and consider the linear BSDE

$$B_t = g(M_T) - \int_t^T Z_s\, dW_s - \int_t^T \{f(s, M_s) Z_s + \varphi(s, M_s)\}\, ds \tag{62}$$

driven by a diffusion process $M = (M^1, \ldots, M^m)$. Assume that each component M^j is a diffusion

$$dM_t^j = b^j(t, M_t^j)dt + \sigma^j(t, M_t^j)dW_t^j$$

that depends only on W^j. In this case $D^i M^j \equiv 0$ for $i \neq j$ and $D_u^j M_t^j = \Delta_t^j(\Delta_u^j)^{-1}\sigma^j(M_u^j)$. The Malliavin derivatives $(D_u^j B_t, D_u^j Z_t)_{u\le t\le T}$ have almost surely the following representations:

$$D_u^j B_t = \frac{\sigma^j(M_u^j)}{\Delta_u^j}\zeta_t^j \qquad \text{and} \qquad D_u^j Z_t^k = \frac{\sigma^j(M_u^j)}{\Delta_u^j}\alpha_t^{j,k} \quad (k = 1, \ldots, m).$$

Here $(\zeta^j, \alpha^j) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^m)$ is a solution of the BSDE

$$\zeta_t^j = g_j(M_T)\Delta_T^j - \int_t^T \alpha_s^j\, dW_s - \int_t^T \{\alpha_s^j f_j(s, M_s) + \zeta_s \sigma^j(M_s^j) f_j(s, M_s) + \varphi_j(s, M_s)\Delta_s^j\}\, ds, \tag{63}$$

where g_j, f_j, and φ_j denote the derivative with respect to x_j. Hence we get a separate BSDE for each component D^j of the derivative.

5.2.2. Market completeness in equilibrium. This section finishes our alternative proof of market completion in equilibrium. For this we first recall that the equilibrium market price of risk process takes the form

$$\theta_t(X_t) = \begin{pmatrix} \theta_t^S \\ \theta_t^R(X_t) \end{pmatrix}.$$

By Lemma 5.1 the function θ_t^R is continuously differentiable with bounded derivative with respect to the climate component. The equilibrium bond price process B^θ along with the integrand κ^θ constitute the unique pair of adapted processes that satisfy the backward equation (42). The components of the forward process X satisfy the conditions of Remark 5.4, so κ^R is equal to

$$\kappa_t^R = \zeta_t^R \Delta_t^R,$$

where ζ^R solves the BSDE

$$\zeta_t^R = h_2^I(X_T) - \int_t^T \alpha_s\, dW_s - \int_t^T \{\langle \alpha_s, \theta_s\rangle + \zeta_s^R \theta_2^R(s, X_s) + \varphi_2^I(s, X_s)\Delta_s^R\}\, ds.$$

Since all the functions θ, θ_2, φ_2, and Δ^R all are bounded, the BSDE has a unique solution in $\mathcal{H}^2 \times \mathcal{H}^2$. Using the explicit solution for ζ, we may write κ_t^R as a conditioned expectation. For this, let $(\Gamma_s^t)_s$ be the solution of

$$\frac{d\Gamma_s^t}{\Gamma_s^t} = -\theta(t)dW_s - \theta_2(s, X_s)ds, \quad \Gamma_t^t = 1.$$

We obtain

$$\kappa_t^R = \Delta_t^R \mathbb{E}\left[\Gamma_T^t h_2^I(X_T) + \int_t^T \Gamma_s^t \varphi_2^I(s, X_s)\Delta_s^R\, ds \,\Big|\, \mathcal{F}_t\right].$$

The following result summarizes our results.

Theorem 5.2. *Let h^I, φ^I, and the coefficients of the forward process X be given according to Assumption 5.1. Then the two-dimensional process* $(\exp(S), B^\theta)$ *has the martingale representation property: every local* $(\mathbb{P}^\theta, \mathcal{F})$*-martingale can be represented as a stochastic integral with respect to* $(\exp(S), B^\theta)$ *where $\mathcal{F}$ is the $\mathbb{P}^\theta$-augmentation of the filtration generated by W.*

6. Examples and applications. This section illustrates our results in benchmark models where the market participants are merely exposed to nonfinancial risk factors. Hence all payoff functions are independent of stock prices. Our focus is on climate risk related to the ENSO phenomenon; see a recent report by Swiss Re [38], the re-insurance company, for a discussion of its climatic impacts and possible consequences for the insurance industry. We believe that the framework is flexible enough to cover a variety of applications ranging from energy to agricultural production risk.

6.1. Temperature risk and the ENSO-phenomenon. Let us first study a toy model for the transfer of climate risk related to the ENSO, a periodic event of an anomalous rise of the sea surface temperature in the Eastern Pacific. El Niño strikes in a random period every three to eight years around Christmas. Among its economic and environmental impacts in South America are a significant drop in the catch rates for several species of fish such as the big tuna eye; see Gaol and Manurung [19] for a discussion of the impact of the El Niño oscillation on tuna fishing.

Chaumont et al. [7] develop a model of an exchange economy within which local farmers and fishers redistribute their risk exposures by trading fictitious insurance contracts. They assume that the evolution of the sea surface temperature follows an Ornstein-Uhlenbeck dynamic with constant volatility:

$$dR_t = \mu^R(t, R_t)dt + \sigma^R dW_t^R. \tag{64}$$

Our results carry beyond the benchmark of an exchange economy. This is of particular relevance when agents with negatively correlated risk exposures are located away from each other. The drop in catch rates in South America caused by the El Niño phenomenon, for instance, is typically accompanied by increasing rates in the same species in some parts of Asia. When agents with complementary risk exposures are located in different countries or continents it is natural to assume that risk sharing takes place through intermediaries or financial markets. Let us therefore consider a situation where an insurance company sells coverage against ENSO-related economic damages to South American fishers. The insurance company then transfers some of its risk to Asian financial markets by issuing a risk bond in Indonesia. The bond's payoff reflects the insurance companies' expected financial losses in the South American market.

Let us denote by φ^{Fisher} and φ^{Farmer} the income rates of the fisher and farmer, respectively and assume that their terminal payoffs are linear in the external risk factor:

$$h^{\text{Fisher}}(t,x)=m_1(t)x+b_1(t) \qquad \text{and} \qquad h^{\text{Fisher}}(t,x)=m_2(t)x+b_2(t).$$

Suppose further that the yield curve is given as a convex combination of some possibly time-dependent benchmark rate $r(t)$ and a floating component $g(t,R_t)$ whose dynamics depend on the evolution of the external risk factor

$$\varphi^I(t,R_t)=(1-\alpha_t)r(t)+\alpha_t g(t,R_t) \quad \text{for } \alpha_t\in[0,1].$$

The yield curve $g(t,\cdot)$ is assumed to be decreasing. This reflects the increased risk the insurance company faces in the South American market. Thus, Assumption 2.1(iv) is satisfied if, for instance,

$$h^I(t,x)=m(t)x+b(t) \quad (m(t)\le 0).$$

In our current one-dimensional setting, the domain $\mathcal{O}$ is given by the finite interval. The nonlinear PDE (34) for the equilibrium market price of climate risk hence reduces to

$$\begin{cases} -v_t+\mathcal{L}v+\sigma^2 v_x^2=\varphi & \text{on } \mathcal{O}, \\ v=h^I & \text{on } \{T\}\times\partial\mathcal{O}. \end{cases} \tag{65}$$

The equilibrium bond price process is given in terms of the solution of the linear boundary value problem

$$\begin{cases} -w_t+\mathcal{L}w+\sigma w_x v_x=\varphi^I & \text{on } \mathcal{O}, \\ w=h & \text{on } \{T\}\times\partial\mathcal{O}. \end{cases} \tag{66}$$

Assuming that $\sigma^R=1$ and that $m'(T)=b'(T)=0$, the consistency conditions (38) along with the monotonicity condition on the yield rate requires $\varphi^I(T,\cdot)$ to be constant:

$$\varphi^I(T,\cdot)\equiv m^2(T) \quad \text{so that} \quad \alpha_T=0 \quad \text{and} \quad r(T)=m^2(T).$$

For given dynamics of the risk process and for given income rates, the consistency condition (44) imposes an additional restriction on the derivatives of the coefficients $m_1(T)$ and $m_2(T)$. For instance, if $m_i'(T)=b_i'(T)=0$ for $i=1,2$, then we have the following condition on $\partial\mathcal{O}$:

$$-\mu^R(T,x)[m_1(T)+m_2(T)]+\frac{1}{2}[m_1(T)+m_2(T)+m(T)]^2-\varphi(T,x)=0.$$

Remark 6.1. The Ornstein-Uhlenbeck process (64) satisfies the assumptions of §5. Hence if asset prices follow a diffusion with a deterministic market price of financial risk, the assumption of a bounded domain can be dropped.

6.2. Energy bonds and portfolio diversification. Let us now consider an operator of a power plant that operates on crude oil. The operator faces a temperature-dependent demand for electricity and random price fluctuations in the oil market. We assume that oil prices follow a diffusion process $\{S_t\}$ of the form (1) while the dynamics of the temperature process are modelled by an Ornstein-Uhlenbeck process of the form (64). The revenue from selling electricity at some point t during the summer season $[0,T]$ is increasing as the temperature rises due to increasing household demand and, thus, increasing retail prices. It is decreasing in the oil price as crude oil constitutes an input factor of the production process. To hedge the temperature-dependent risk, the energy supplier may choose to issue a bond whose yield curve is linked to revenues generated from producing and selling electricity or, equivalently, to the evolution of oil prices and outside temperatures. In such a situation the yield curve takes the form

$$\varphi^I(t,S_t,R_t)$$

for some smooth function that is increasing in the third variable. For simplicity we may also assume that the bond pays back its face value at maturity so that $h^I\equiv 1$.

The demand for such climate bonds comes from institutional investors that seek to diversify their portfolios by investing in financial securities written on nonfinancial risk factors. A benchmark model where the institutional investors are not exposed to climate risk and perfectly hedge their financial risk is captured by constant payoff functions $h^a \equiv 0$ and $\varphi^a \equiv 0$. The PDEs for the equilibrium market price of climate risk and the bond price process are then given by

$$\begin{cases} -v_t + \mathscr{L}v + \sigma^2 v_x^2 = \varphi^I & \text{on } \mathscr{O}, \\ v = 1 & \text{on } \mathscr{O}_T, \end{cases} \quad \text{and} \quad \begin{cases} -w_t + \mathscr{L}w + \sigma w_x v_x = \varphi^I & \text{on } \mathscr{O}, \\ w = 1 & \text{on } \mathscr{O}_T, \end{cases} \tag{67}$$

respectively where $\mathscr{L}$ denotes the generator of the diffusion process $\{(S_t, R_t)\}$. In a model of pure portfolio diversification $\varphi = \varphi^I$ and the consistency condition means that the yield at maturity equals the risk-less rate:

$$\varphi^I(T, \cdot) = 0 \quad \text{on } \partial\mathscr{O}.$$

Hence if the yield curve is smooth enough, the institutional investors buy the bond to diversify their portfolios while the energy supplier successfully transfers some of its nonfinancial risk into a tradable financial asset. In our benchmark geometric Brownian motion model where we have no boundary and consistency condition we can also price an approximation of an HDD with payoff rate (4).

7. Conclusion. In this paper we propose an equilibrium approach to pricing and hedging in illiquid financial markets. Our approach is based on the idea of market completion and partial equilibrium introduced in Müller [30]. It extends earlier results reported in Chaumont et al. [7, 8] and Hu et al. [21]. In a first step we solve an individual agents optimization problem under the assumption that the risk bond completes the market. In a second step we characterize the equilibrium market price of risk in terms of a solution to a nonlinear PDEs. Finally, we show that the assumption of market completion is indeed satisfied if the bond payoff functions are monotone in the external risk factor. While the first approach for bounded domains is based entirely on PDE methods, a second approach for a benchmark geometric Brownian motion model uses mainly probabilistic arguments. In this case we prove that the quadratic BSDE for the equilibrium market price of risk can be viewed as a BSDE with a Lipschitz continuous driver. This has two immediate consequences: first, the BSDE is Malliavin differentiable; second, it can be simulated using standard numerical methods. To the best of our knowledge no numerical scheme is available for BSDE with non-Lipschitz drivers.

Several avenues are open for future research. Most importantly, the compatibility condition linking the diffusion coefficients and the payoff functions at maturity is somewhat artificial and imposes an unnecessary restriction. The alternative approach proposed in §5 suggests that it is possible to drop this assumption in a general diffusion model once we can establish differentiability of quadratic BSDEs. Representing the market price of climate risk and the bond price process in terms of either the solution to a PDE or a BSDE with Lipschitz continuous driver makes our results amenable to a numerical analysis. It would also be desirable to extend our results beyond the case of monotone payoff functions and to prove that the assumption of market completeness holds for a larger class of payoff profiles. Finally, we assume that all the agents share common risk factors. A more realistic approach would allow for a basis risk where the distribution of an agent's payoff rather than the payoff itself depends on a common risk process. Such an extension is not covered by our methods. It might instead be captured by a blend of our approach and that pioneered by Barrieu and El Karoui [2]. We leave this for future research.

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