Risk Minimization and Optimal Derivative Design in a Principal Agent Game*

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September 4, 2008

Abstract

We consider the problem of Adverse Selection and optimal derivative design within a Principal-Agent framework. The principal's income is exposed to non-hedgeable risk factors arising, for instance, from weather or climate phenomena. She evaluates her risk using a coherent and law invariant risk measure and tries minimize her exposure by selling derivative securities on her income to individual agents. The agents have mean-variance preferences with heterogeneous risk aversion coefficients. An agent's degree of risk aversion is private information and hidden from the principal who only knows the overall distribution. We show that the principal's risk minimization problem has a solution and illustrate the effects of risk transfer on her income by means of two specific examples. Our model extends earlier work of Barrieu and El Karoui (2005) and Carlier, Ekeland and Touzi (2007).

AMS classification: 60G35, 60H20, 91B16, 91B70.

Keywords: Optimal derivative design, structured securities, adverse selection, risk transfer.

*We thank Guillaume Carlier, Pierre-Andre Chiappori, Ivar Ekeland, Andreas Putz and seminar participants at various institutions for valuable comments and suggestions. Financial support through an NSERC individual discovery grant is gratefully acknowledged.

1 Introduction

In recent years there has been an increasing interest in derivative securities at the interface of finance and insurance. *Structured products* such as risk bonds, asset-backed securities and weather derivatives are end-products of a process known as *securitization* that transforms non-tradable risk factors into tradable financial assets. Developed in the U.S. mortgage markets, the idea of pooling and underwriting risk that cannot be hedged through investments in the capital markets alone has long become a key factor driving the convergence of insurance and financial markets.

Structured products are often written on non-tradable underlyings, tailored to the issuers specific needs and traded "over the counter". Insurance companies, for instance, routinely sell weather derivatives or risk bonds to customers that cannot go to the capital markets directly and/or seek financial securities with low correlation with stock indices as additions to diversified portfolios. The market for such claims is generally incomplete and illiquid. As a result, many of the standard paradigms of traditional derivative pricing theory, including replication arguments do not apply to structured products. In an illiquid market framework, preference-based valuation principles that take into account characteristics and endowment of trading partners may be more appropriate for designing, pricing and hedging contingent claims. Such valuation principles have become a major topic of current research in economics and financial mathematics. They include rules of Pareto optimal risk allocation ([12], [17]), market completion and dynamic equilibrium pricing ([15], [16]) and, in particular, utility indifference arguments ([2], [3], [5], [6], [9], ...). The latter assumes a high degree of market symmetry. For indifference valuation to be a *pricing* rather than *valuation* principle, the demand for a financial security must come from *identical* agents with known preferences and negligible market impact while the supply must come from a single principal. When the demand comes from *heterogeneous* individuals with hidden characteristics, indifference arguments do not always yield an appropriate pricing scheme.

In this paper we move away from the assumption of investor homogeneity and allow for heterogeneous agents. We consider a single principal with a random endowment whose goal is to lay off some of her risk with heterogeneous agents by designing and selling derivative securities on her income. The agents have mean variance preferences. An agent's degree of risk aversion is private information and hidden to the principal. The principal only knows the distribution of risk aversion coefficients which puts her at an informational disadvantage. If all the agents were homogeneous, the principal, when offering a structured product to a single agent, could (perhaps) extract the indifference (maximum) price from each trading partner. In the presence of agent heterogeneity this is no longer possible, either because the agents would hide their characteristics from the principal or prefer another asset offered by the principal but designed and priced for another customer.

The problem of optimal derivative design in a Principal-Agent framework with informed agents and an uninformed principal has first been addressed in a recent paper by Carlier, Ekeland

and Touzi [7]. With the agents being the informed party, theirs is a screening model. The literature on screening within the Adverse Selection framework can be traced back to Mussa and Rosen [19], where both the principal's allocation rule and the agents' types are one-dimensional. Armstrong [1] relaxes the hypothesis of agents being characterized by a single parameter. He shows that, unlike the one-dimensional case, "bunching" of the first type is robust when the types of the agents are multi-dimensional. In their seminal paper, Rochet and Choné [21] further extend this analysis. They provide a characterization of the contracts, determined by the (non-linear) pricing schedule, that maximize the principal's utility under the constraints imposed by the asymmetry of information in the models. Building on their work, Carlier, Ekeland and Touzi [7] study a Principal-Agent model of optimal derivative design where the agents' preferences are of mean-variance type and their multi-dimensional types characterize their risk aversions and initial endowments. They assume that there is a direct cost to the principal when she designs a contract for an agent, and that the principal's aim is to maximize profits.

We start from a similar set-up, but substitute the idea that providing products carries a cost for the idea that traded contracts expose the principal to additional risk - as measured by a convex risk measure - in exchange for a known revenue. This may be viewed as a partial extension of the work by Barrieu and El Karoui ([2],[3]) to an incomplete information framework.

The principal's aim is to minimize her risk exposure by trading with the agents subject to the standard incentive compatibility and individual rationality conditions on the agents' choices. In order to prove that the principal's risk minimization problem has a solution we first follow the seminal idea of Rochet and Choné [21] and characterize incentive compatible catalogues in terms of U-convex functions. When the impact of a single trade on the principal's revenues is linear as in Carlier, Ekeland and Touzi [7], the link between incentive compatibility and U-convexity is key to establish the existence of an optimal solution. In our model the impact is non-linear as a single trade has a non-linear impact on the principal's risk assessment. Due to this non-linearity we face a non-standard variational problem where the objective cannot be written as the integral of a given Lagrangian. Instead, our problem can be decomposed into a standard variational part representing the aggregate income of the principal, plus the minimization of the principal's risk evaluation, which depends on the aggregate of the derivatives traded. We state sufficient conditions that guarantee that the principal's optimization problem has a (unique) solution. We also show that all the optimal contracts can be chosen co-linear and illustrate the effect of risk transfer on her exposure by means of two specific examples.

The remainder of this paper is organized as follows. In Section 2 we formulate our Principal-Agent model and state the main result. The proof is given in Section 3. In Section 4 we illustrate the effects of risk transfer on the principal's position by two examples. In the first one we consider a situation where the principal restricts herself to type-dependent multiples of some benchmark claim. This case can be solved in closed form by means of a standard variational problem. The second example considers put options with type-dependent strikes. In both cases we assume that the principal's risk measure is Average Value at Risk. As a consequence the risk minimization problem can be stated in terms of a min-max problem; we provide an efficient numerical scheme for approximating the optimal solution. The code is available from the authors upon request.

2 The Microeconomic Setup

We consider an economy with a single *principal* whose income W is exposed to non-hedgeable risk factors rising from, e.g., climate or weather phenomena. The random variable W is defined on a standard, non-atomic, probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and it is square integrable:

$$W \in L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

The principal's goal is to lay off parts of her risk with individual *agents*. The agents have heterogenous mean-variance preferences[1] and are indexed by their coefficients of risk aversion $\theta \in \Theta$. Given a contingent claim $Y \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ an agent of type θ enjoys the utility

$$U(\theta, Y) = \mathbb{E}[Y] - \theta \operatorname{Var}[Y]. \tag{1}$$

Types are private information. The principal knows the distribution μ of types but not the realizations of the random variables θ. We assume that the agents are risk averse and that the risk aversion coefficients are bounded away from zero. More precisely,

$$\Theta = [a, 1] \qquad \text{for some } a > 0.$$

The principal offers a derivative security $X(\theta)$ written on her random income for every type θ. The set of all such securities is denoted by

$$\mathcal{X} := \left\{ X = \{X(\theta)\}_{\theta \in \Theta} \mid X \in L^2(\Omega \times \Theta, \mathbb{P} \otimes \mu),\, X \text{ is } \sigma(W) \times \mathcal{B}(\Theta) \text{ measurable} \right\}. \tag{2}$$

We refer to a list of securities $\{X(\theta)\}$ as a *contract*. A *catalogue* is a contract along with prices $\pi(\theta)$ for every available derivative $X(\theta)$. For a given catalogue (X, π) the indirect utility of the agent of type θ is given by

$$v(\theta) = \sup_{\theta' \in \Theta} \left\{ U(\theta, X(\theta')) - \pi(\theta') \right\}. \tag{3}$$

Remark 2.1 *No assumption will be made on the sign of $\pi(\theta)$; our model contemplates both the case where the principal takes additional risk in exchange of financial compensation and the one where she pays the agents to take part of her risk.*

[1] Our analysis carries over to preferences of mean-variance type with random initial endowment as in [7]; the assumption of simple mean-variance preferences is made for notational convenience.

A catalogue (X, π) will be called *incentive compatible* (**IC**) if the agents' interests are best served by revealing their types. This means that the indirect utility of an agent of type θ is achieved by the security $X(\theta)$:

$$U(\theta, X(\theta)) - \pi(\theta) \geq U(\theta, X(\theta')) - \pi(\theta') \quad \text{for all} \quad \theta, \theta' \in \Theta. \tag{4}$$

We assume that each agent has some outside option ("no trade") that yields a utility of zero. A catalogue is thus called *individually rational* (**IR**) if it yields at least the reservation utility for all agents, i.e., if

$$U(\theta, X(\theta)) - \pi(\theta) \geq 0 \quad \text{for all} \quad \theta \in \Theta. \tag{5}$$

Remark 2.2 *By offering only incentive compatible contracts, the principal forces the agents to reveal their types. Offering contracts where the IR constraint is binding allows the principal to exclude undesirable agents from participating in the market. It can be shown that under certain conditions, the interests of the principal are better served by keeping agents of "lower types" to their reservation utility; Rochet and Choné [21] have shown that in higher dimensions this is always the case.*

If the principal issues the catalogue (X, π), she receives a cash amount of $\int_\Theta \pi(\theta)\, d\mu(\theta)$ and is subject to the additional liability $\int_\Theta X(\theta)\mu(d\theta)$. She evaluates the risk associated with her overall position

$$W + \int_\Theta (\pi(\theta) - X(\theta)) d\mu(\theta)$$

via a coherent and law-invariant risk measure $\varrho : L^2(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R} \cup \{\infty\}$ that has the Fatou property. It turns out that such risk measures can be represented as robust mixtures of Average Value at Risk.[2] The principal's risk associated with the catalogue (X, π) is given by

$$\varrho\left(W + \int_\Theta (\pi(\theta) - X(\theta)) d\mu(\theta)\right). \tag{6}$$

Her goal is to devise a catalogue (X, π) that minimizes (6) subject to the incentive compatibility and individual rationality condition:

$$\inf\left\{\varrho\left(W + \int_\Theta (\pi(\theta) - X(\theta)) d\mu(\theta)\right) \mid X \in \mathcal{X},\, (X, \pi) \text{ is } \mathbf{IC} \text{ and } \mathbf{IR}\right\}. \tag{7}$$

We are now ready to state the main result of this paper. The proof requires some preparation and will be carried out in the following section.

[2] We review properties of coherent risk measures on L^p spaces in the appendix and refer to the textbook by Föllmer and Schied [13] and the paper of Jouini, Schachermayer and Touzi [18] for detailed discussion of law invariant risk measures.

Theorem 2.3 *If ϱ is a coherent and law invariant risk measure on $L^2(\mathbb{P})$ and if ϱ has the Fatou property, then the principal's optimization problem has a solution.*

The solution to the principal's problem will typically not be unique even if the risk measure ϱ is strictly convex. We comment on the issue of uniqueness in greater detail in Section 3.3 below. Typically, the solution cannot be given in closed form. However, its structure can be further characterized. More specifically, the following holds.

Proposition 2.4 *If $\{Y(\theta)\}$ is an optimal contract, then all the random variables $Y(\theta)$ are co-linear. More precisely, there exists a random variable $Z \in L^2(\mathbb{P})$ and a function $\alpha : \Theta \to \mathbb{R}$ such that almost surely*

$$Y(\theta, \omega) = \alpha(\theta) Z(\omega).$$

For notational convenience we establish all our results for the special case $d\mu(\theta) = d\theta$. The general case follows from straightforward modifications.

3 Proof of the Main Results

Let (X, π) be a catalogue. In order to prove our main result it will be convenient to assume that the principal offers any square integrable contingent claim and to view the agents' optimization problem as optimization problems over the set $L^2(\mathbb{P})$. This can be achieved by identifying the price list $\{\pi(\theta)\}$ with the pricing scheme

$$\pi : L^2(\mathbb{P}) \to \mathbb{R}$$

that assigns the value $\pi(\theta)$ to an available claim $X(\theta)$ and the value $\mathbb{E}[Y]$ to any other claim $Y \in L^2$. Note this is an application of the Taxation Principle ([14], [20]) in the particular case of Mean-Variance preferences. In terms of this pricing scheme the value function v defined in (3) satisfies, for any incentive compatible catalogue,

$$v(\theta) = \sup_{Y \in L^2(\mathbb{P})} \left\{ U(\theta, Y) - \pi(Y) \right\}. \tag{8}$$

For the remainder of this section we work with the value function (8). The function is U-convex[3] in the sense of Definition A.1. In fact, it turns out to be convex and non-increasing. Our goal is therefore to identify the class of **IC** and **IR** catalogues with a class of convex and non-increasing functions on the type space. To this end, we first recall the link between incentive compatible contracts and U-convex functions from Rochet and Choné [21] and Carlier, Ekeland and Touzi [7].

[3] We recall the notion of U-convexity along with some basic properties of U-convex functions in Appendix A.

Proposition 3.1 *([21], [7]) If a catalogue (X, π) is incentive compatible, then the function v defined by (3) is proper (i.e., never $-\infty$ and not identically $+\infty$) and U-convex and $X(\theta) \in \partial_U v(\theta)$. Conversely, any proper, U-sub-differentiable and U-convex function induces an incentive compatible catalogue.*

PROOF. Incentive compatibility of a catalogue (X, π) means that

$$U(\theta, X(\theta)) - \pi(\theta) \geq U(\theta, X(\theta')) - \pi(\theta') \quad \text{for all} \quad \theta,\, \theta' \in \Theta,$$

so $v(\theta) = U(\theta, X(\theta)) - \pi(\theta)$ is U-convex and $X(\theta) \in \partial_U v(\theta)$. Conversely, for a proper, U-convex function v and $X(\theta) \in \partial_U v(\theta)$ let

$$\pi(\theta) := U(\theta, X(\theta)) - v(\theta).$$

By the definition of the U-subdifferential, the catalogue (X, π) is incentive compatible.

□

The following lemma is key. It shows that the U-convex function v is convex and non-increasing and that any convex and non-increasing function is U-convex, i.e., it allows a representation of the form (8). This allows us to rephrase the principal's problem as an optimization problem over a compact set of convex functions.

Lemma 3.2 *(i) Suppose that the value function v as defined by (8) is proper. Then v is convex and non-increasing. Any optimal claim $X^*(\theta)$ is a U-subgradient of $v(\theta)$ and almost surely*

$$-\mathrm{Var}[X^*(\theta)] = v'(\theta).$$

(ii) If $\bar{v} : \Theta \to \mathbb{R}_+$ is proper, convex and non-increasing, then $\bar{v}$ is U-convex, i.e., there exists a map $\bar{\pi} : L^2(\mathbb{P}) \to \mathbb{R}$ such that

$$\bar{v}(\theta) = \sup_{Y \in L^2(\mathbb{P})} \{U(\theta, Y) - \bar{\pi}(Y)\}.$$

Furthermore, any optimal claim $\bar{X}(\theta)$, that is any claim for which the supremum above is attained, belongs to the U-subdifferential of $\bar{v}(\theta)$ and satisfies

$$-\mathrm{Var}[\bar{X}(\theta)] = \bar{v}'(\theta).$$

PROOF.

(i) Given a proper, U-convex function v, its U-conjugate is:

$$\begin{aligned} v^U(Y) &= \sup_{\theta \in \Theta} \{\mathbb{E}[Y] - \theta \mathrm{Var}[Y] - v(\theta)\} \\ &= \mathbb{E}[Y] + \sup_{\theta \in \Theta} \{\theta(-\mathrm{Var}[Y]) - v(\theta)\} \\ &= \mathbb{E}[Y] + v^*(-\mathrm{Var}[Y]), \end{aligned}$$

where v^* denotes the convex conjugate of v. By Proposition A.2, the map v is characterized by the fact that $v = (v^U)^U$. Thus

$$\begin{aligned} v(\theta) &= (v^U)^U(\theta) \\ &= \sup_{Y \in L^2(\Omega,\mathbb{P})} \{U(\theta, Y) - \mathbb{E}[Y] - v^*(-\mathrm{Var}[Y])\} \\ &= \sup_{Y \in L^2(\Omega,\mathbb{P})} \{\mathbb{E}[Y] - \theta \mathrm{Var}[Y] - \mathbb{E}[Y] - v^*(-\mathrm{Var}[Y])\} \\ &= \sup_{y \leq 0} \{\theta \cdot y - v^*(y)\} \end{aligned}$$

where the last equality uses the fact that the agents' consumption set contains claims of any variance. We deduce from the preceding representation that v is non-increasing. Furthermore $v = (v^*)^*$ so v is convex. To characterize $\partial_U v(\theta)$ we proceed as follows:

$$\begin{aligned} \partial_U v(\theta) &= \left\{Y \in L^2 \mid v(\theta) = U(\theta, X) - v^U(Y)\right\} \\ &= \left\{Y \in L^2 \mid v(\theta) = \mathbb{E}[Y] - \theta \mathrm{Var}[Y] - v^U(Y)\right\} \\ &= \left\{Y \in L^2 \mid v(\theta) = \mathbb{E}[Y] - \theta \mathrm{Var}[Y] - \mathbb{E}[Y] - v^*(-\mathrm{Var}[Y])\right\} \\ &= \left\{Y \in L^2 \mid v(\theta) = \theta(-\mathrm{Var}[Y]) - v^*(-\mathrm{Var}[Y])\right\} \\ &= \left\{Y \in L^2 \mid -\mathrm{Var}[Y] \in \partial v(\theta)\right\} \end{aligned}$$

The convexity of v implies it is a.e. differentiable so we may write

$$\partial_U v(\theta) := \left\{Y \in L^2 \mid v'(\theta) = -\mathrm{Var}[Y])\right\}.$$

(ii) Let us now consider a proper, non-negative, convex and non-increasing function $\bar{v} : \Theta \to \mathbb{R}$. There exists a map $f : \mathbb{R} \to \mathbb{R}$ such that $\bar{v}(\theta) = \sup_{y \leq 0} \{\theta \cdot y - f(y)\}$. Since $\bar{v}$ is non-increasing there exists a random variable $Y(\theta) \in L^2$ such that $-\mathrm{Var}[Y(\theta)] \in \partial \bar{v}(\theta)$ and the definition of the subgradient yields

$$\bar{v}(\theta) = \sup_{Y \in L^2} \{\theta(-\mathrm{Var}[Y]) - f(-\mathrm{Var}[Y])\}.$$

With the pricing scheme on $L^2(\mathbb{P})$ defined by

$$\bar{\pi}(Y) := -\mathbb{E}[Y] - f(-\mathrm{Var}[Y])$$

we obtain $\bar{v}(\theta) = \sup_{Y \in L^2} \{U(\theta, Y) - \bar{\pi}(Y)\}$. The characterization of the subdifferential follows by analogy to part (i).

□

The preceding lemma along with Proposition 3.1 shows that any convex, non-negative and non-increasing function v on Θ induces an incentive compatible catalogue (X, π) via

$$X(\theta) \in \partial_U v(\theta) \quad \text{and} \quad \pi(\theta) = U(\theta, X(\theta)) - v(\theta).$$

Here we may with no loss of generality assume that $\mathbb{E}[X(\theta)] = 0$. Due to the monetary property of ϱ, the principal could otherwise just charge an extra $\mathbb{E}[X(\theta)]$ dollars without changing the structure of the underlying optimization problem. In terms of the principal's choice of v her income is given by

$$I(v) = \int_\Theta \big(\theta v'(\theta) - v(\theta)\big)\, d\theta.$$

Since $v \geq 0$ is decreasing and non-negative the principal will only consider functions that satisfy the normalization constraint

$$v(1) = 0.$$

We denote the class of all convex, non-increasing and non-negative real-valued functions on Θ that satisfy the preceding condition by $\mathcal{C}$:

$$\mathcal{C} = \{v : \Theta \to \mathbb{R} \mid v \text{ is convex, non-increasing, non-negative and } v(1) = 0.\}$$

Conversely, we can associate with any **IC** and **IR** catalogue (X, π) a non-negative U-convex function of the form (8) where the contract satisfies the variance constraint $-\mathrm{Var}[X(\theta)] = v'(\theta)$. In view of the preceding lemma this function is convex and non-increasing so after normalization we may assume that v belongs to the class $\mathcal{C}$. We therefore have the following alternative formulation of the principal's problem.

Theorem 3.3 *The principal's optimization problem allows the following alternative formulation:*

$$\inf \left\{ \varrho \left(W - \int_\Theta X(\theta) d\theta \right) - I(v) \mid v \in \mathcal{C},\, \mathbb{E}[X(\theta)] = 0,\; -\mathrm{Var}[X(\theta)] = v'(\theta) \right\}.$$

In terms of our alternative formulation we can now prove a preliminary result. It states that a principal with no initial endowment will not issue any contracts.

Lemma 3.4 *If the principal has no initial endowment, i.e., if $W = 0$, then $(v, X) = (0, 0)$ solves her optimization problem.*

Proof. Since ϱ is a coherent, law invariant risk measure on $L^2(\mathbb{P})$ that has the Fatou property it satisfies

$$\varrho(Y) \geq -\mathbb{E}[Y] \quad \text{for all } Y \in L^2(\mathbb{P}). \tag{9}$$

For a given function $v \in \mathcal{C}$ the normalization constraint $\mathbb{E}[X(\theta)] = 0$ implies

$$\varrho\left(- \int_\Theta X(\theta) d\theta \right) - I(v) \geq \mathbb{E}\left[\int_\Theta X(\theta) d\theta \right] - I(v) = -I(v).$$

Since v is non-negative and non-increasing $-I(v) \geq 0$. Taking the infimum in the preceding inequality shows that $v \equiv 0$ and hence $X(\theta) \equiv 0$ is an optimal solution.

□

3.1 Minimizing the risk for a given function v

In the general case we approach the principal's problem in two steps. We start by fixing a function v from the class $\mathcal{C}$ and minimize the associated risk

$$\varrho\left(W - \int_{\Theta} X(\theta)d\theta\right)$$

subject to the moment conditions $\mathbb{E}[X(\theta)] = 0$ and $-\mathrm{Var}[X(\theta)] = v'(\theta)$. To this end, we shall first prove the existence of optimal contracts X_v for a relaxed optimization problem where the variance constraint is replaced by the weaker condition

$$\mathrm{Var}[X(\theta)] \leq -v'(\theta).$$

In a subsequent step we show that based on X_v the principal can transfer risk exposures among the agents in such a way that (i) the aggregate risk remains unaltered; (ii) the variance constraint becomes binding. We assume with no loss of generality that v does not have a jump at $\theta = a$.

3.1.1 The relaxed optimization problem

For a given $v \in \mathcal{C}$ we consider the convex set of derivative securities

$$\mathcal{X}_v := \left\{X \in \mathcal{X} \mid E[X(\theta)] = 0, \mathrm{Var}[X(\theta)] \leq -v'(\theta) \ \mu - a.e.\right\} \tag{10}$$

and call the function v acceptable for the principal if there exists some contract $X \in \mathcal{X}_v$ such that the implementation of (v, X) does not increase the principal's risk.

Lemma 3.5 *(i) All functions $v \in \mathcal{C}$ that are acceptable for the principal are uniformly bounded.*

(ii) Under the conditions of (i) the set $\mathcal{X}_v$ is closed and bounded in $L^2(\mathbb{P} \otimes \mu)$. More precisely,

$$\|X\|_2^2 \leq v(a) \quad \textit{for all} \quad X \in \mathcal{X}_v.$$

PROOF.

(i) If v is acceptable for the principal, then there exists some $X \in \mathcal{X}_v$ that satisfies

$$\varrho\left(W - \int_{\Theta} X(\theta)d\theta\right) - I(v) \leq \varrho(W).$$

From (9) and that fact that $\mathbb{E}[X(\theta)] = 0$ we deduce that

$$-\mathbb{E}[W] - I(v) \leq \varrho\left(W - \int_\Theta X(\theta)d\theta\right) - I(v) \leq \varrho(W)$$

so

$$-I(v) \leq \mathbb{E}[W] + \varrho(W) =: K.$$

Integrating by parts and using that v is non-increasing and $v(1) = 0$ we see that

$$K \geq -I(v) = av(a) + 2\int_a^1 v(\theta)d\theta \geq av(a).$$

This proves the assertion because $a > 0$.

(ii) For $X \in \mathcal{X}_v$ we deduce from the normalization constraint $v(1) = 0$ that

$$\|X\|_2^2 = \int\int X^2(\theta,\omega)d\mathbb{P}\,d\theta \leq -\int v'(\theta)d\theta \leq v(a)$$

so the assertion follows from part (i).

□

Since ϱ is a l.s.c. convex risk measure on $L^2(\mathbb{P})$ and because the set $\mathcal{X}_v$ of contingent claims is convex, closed and bounded in $L^2(\mathbb{P})$, a general result from the theory of convex optimization yields the following proposition.

Proposition 3.6 *If the function v is acceptable for the principal, then there exists a contract X_v such that*

$$\inf_{X \in \mathcal{X}_v} \varrho\left(W - \int_\Theta X(\theta)d\theta\right) = \varrho\left(W - \int_\Theta X_v(\theta)d\theta\right).$$

The contract X_v along with the pricing scheme associated with v does not yield an incentive compatible catalogue unless the variance constraints happen to be binding. However, as we are now going to show, based on X_v the principal can find a redistribution of risk among the agents such that the resulting contract satisfies our **IC** condition.

3.1.2 Redistributing risk exposures among agents

Let

$$\partial\mathcal{X}_v = \left\{X \in \mathcal{X}_v \mid E[X(\theta)] = 0,\, \mathrm{Var}[X(\theta)] = -v'(\theta),\, \mu - a.e.\right\}$$

be the set of all contracts from the class $\mathcal{X}_v$ where the variance constraint is binding. Clearly,

$$\varrho\left(W - \int_\Theta X_v(\theta)d\theta\right) \leq \inf_{X \in \partial\mathcal{X}_v} \varrho\left(W - \int_\Theta X(\theta)d\theta\right).$$

Let us then introduce the set of types

$$\Theta_v := \left\{\theta \in \Theta \mid \mathrm{Var}[X_v(\theta)] < -v'(\theta)\right\},$$

for whom the variance constraint is not binding. If $\mu(\Theta_v) = 0$, then X_v yields an incentive compatible contract. Otherwise, we consider a random variable $\tilde{Y} \in \mathcal{X}_v$, fix some type $\overline{\theta} \in \Theta$ and define

$$Y := \frac{\tilde{Y}(\overline{\theta})}{\sqrt{\mathrm{Var}[\tilde{Y}(\overline{\theta})]}}. \tag{11}$$

The purpose of introducing Y is to offer a set of structured products Z_v based on X_v, such that Z_v together with the pricing scheme associated with v yields an incentive compatible catalogue. To this end, we choose constants $\alpha(\theta)$ for $\theta \in \Theta_v$ such that

$$\mathrm{Var}[X_v(\theta) + \alpha(\theta)Y] = -v'(\theta).$$

This equation holds for

$$\alpha^{\pm}(\theta) = -\mathrm{Cov}[X_v(\theta), Y] \pm \sqrt{\mathrm{Cov}^2[X_v(\theta), Y] - v'(\theta) - \mathrm{Var}[X_v(\theta)]}.$$

For a type $\theta \in \Theta_v$ the variance constraint is not binding. Hence $-v'(\theta) - \mathrm{Var}[X_v(\theta)] > 0$ so that $\alpha^+(\theta) > 0$ and $\alpha^-(\theta) < 0$. An application of Jensen's inequality together with the fact that $\|X_v\|_2$ is bounded shows that $\alpha^{\pm}$ are μ-integrable functions. Thus there exists a threshold type $\theta^* \in \Theta$ such that

$$\int_{\Theta_v \cap (a,\theta^*]} \alpha^+(\theta)d\theta + \int_{\Theta_v \cap (\theta^*,1]} \alpha^-(\theta)d\theta = 0.$$

In terms of θ^* let us now define a function

$$\alpha(\theta) := \begin{cases} \alpha^+(\theta), & \text{if } \theta \leq \theta^* \\ \alpha^-(\theta), & \text{if } \theta > \theta^* \end{cases}$$

and a contract

$$Z_v := X_v + \alpha Y \in \partial\mathcal{X}_v. \tag{12}$$

Since $\int \alpha d\theta = 0$ the aggregate risks associated with X_v and Z_v are equal. As a result, the contract Z_v solves the risk minimization problem

$$\inf_{X \in \partial\mathcal{X}_v} \varrho\left(W - \int_\Theta X(\theta)\mu(d\theta)\right). \tag{13}$$

Remark 3.7 *In Section 4 we shall consider a situation where the principal restricts itself to a class of contracts for which the random variable X_v can be expressed in terms of the function v. In general such a representation will not be possible since v only imposes a restriction on the contracts' second moments.*

3.2 Minimizing the overall risk

In order to finish the proof of our main result it remains to be shown that the minimization problem

$$\inf_{v\in\mathcal{C}}\left\{\varrho\left(W-\int_\Theta Z_v(\theta)\mu(d\theta)\right)-I(v)\right\}$$

has a solution and the infimum is attained. To this end, we consider a minimizing sequence $\{v_n\}\subset\mathcal{C}$. The functions in $\mathcal{C}$ are locally Lipschitz continuous because they are convex. In fact they are *uniformly* locally Lipschitz: by Lemma 3.5 (i) the functions $v\in\mathcal{C}$ are uniformly bounded and non-increasing so all the elements of $\partial v(\theta)$ are uniformly bounded on compact sets of types. As a result, $\{v_n\}$ is a sequence of uniformly bounded and uniformly equicontinuous functions when restricted to compact subsets of Θ. Thus there exists a function $\bar{v}\in\mathcal{C}$ such that, passing to a subsequence if necessary,

$$\lim_{n\to\infty} v_n=\bar{v}\quad\text{uniformly on compact sets.}$$

A standard 3ϵ-argument shows that the convergence properties of the sequence $\{v_n\}$ carry over to the derivatives so that

$$\lim_{n\to\infty} v_n'=\bar{v}'\quad\text{almost surely uniformly on compact sets.}$$

Since $-\theta v_n'(\theta)+v_n(\theta)\geq 0$ it follows from Fatou's lemma that $-I(\bar{v})\leq\liminf_{n\to\infty}-I(v_n)$ so

$$\begin{aligned}&\liminf_{n\to\infty}\left\{\varrho\left(W-\int_\Theta Z_{v_n}(\theta)\mu(d\theta)\right)-I(v_n)\right\}\\ \geq\ &\liminf_{n\to\infty}\varrho\left(W-\int_\Theta Z_{v_n}(\theta)\mu(d\theta)\right)+\liminf_{n\to\infty}-I(v_n)\\ \geq\ &\liminf_{n\to\infty}\varrho\left(W-\int_\Theta Z_{v_n}(\theta)\mu(d\theta)\right)-I(\bar{v})\end{aligned}$$

and the associated risk process remains to be analyzed. For this, we first observe that for $Z_{v_n}\in\partial X_{v_n}$ Fubini's theorem yields

$$\|Z_{v_n}\|_2^2=\int\int Z_{v_n}^2 d\mathbb{P}d\theta=-\int v_n'(\theta)d\theta=v_n(a).\tag{14}$$

Since all the functions in $\mathcal{C}$ are uniformly bounded, we see that the contracts Z_{v_n} are contained in an L^2 bounded, convex set. Hence, by the Banach-Alaoglu theorem, there exists a square integrable random variable Z such that, after passing to a subsequence if necessary,

$$Z_{v_n}\to Z\quad\text{weakly in}\quad L^2(\mathbb{P}\otimes\mu),\tag{15}$$

which implies the convergence of aggregate risks:

$$\int_\Theta Z_{v_n}(\theta,\omega)d\theta\to\int_\Theta Z(\theta,\omega)d\theta\quad\text{weakly in } L^2(\mathbb{P}).$$

By Corollary I.2.2 in Ekeland and Témam (1976) [11], a lower semi-continuous convex function $f : X \to \mathbb{R}$ remains so with respect to the weak topology $\sigma(X, X^*)$. Hence, the Fatou property of the risk measure ϱ guarantees that

$$\liminf_{n\to\infty} \varrho\left(W - \int_\Theta Z_{v_n}(\theta)\mu(d\theta)\right) \ \geq \ \varrho\left(W - \int_\Theta Z(\theta)\mu(d\theta)\right).$$

Let us now denote by $Z_{\bar{v}} \in \mathcal{X}_{\bar{v}}$ an optimal claim associated with the limiting function $\bar{v}$ as constructed above. If the weak limit Z belongs to $\mathcal{X}_{\bar{v}}$, then $(Z_{\bar{v}}, \bar{v})$ satisfies the principal's problem because

$$\varrho\left(W - \int_\Theta Z(\theta)\mu(d\theta)\right) \geq \varrho\left(W - \int_\Theta Z_{\bar{v}}(\theta)\mu(d\theta)\right). \tag{16}$$

In order to see that Z respects the variance constraint, we assume to the contrary that there exists a set $\tilde{\Theta}$ of positive measure such that

$$\mathrm{Var}[Z(\theta)] > -\bar{v}'(\theta) \quad \text{on} \quad \tilde{\Theta}. \tag{17}$$

Since the weak convergence properties remain valid when restricting all random variables to $\tilde{\Theta}$, we have that

$$Z_n \big|_{\tilde{\Theta}} \to Z \big|_{\tilde{\Theta}} \quad \text{weakly in} \quad L^2(\mathbb{P} \otimes \mu).$$

Furthermore, the norm of a weak limit is bounded from above by the limit inferior of the norms of the approximating sequence so

$$\begin{aligned} \int_{\tilde{\Theta}} -\bar{v}'(\theta)d\theta &< \int_{\tilde{\Theta}} \mathrm{Var}[Z(\theta)]d\theta \\ &\leq \liminf_{n\to\infty} \int_{\tilde{\Theta}} \mathrm{Var}[Z_n(\theta)]d\theta \\ &= \liminf_{n\to\infty} \int_{\tilde{\Theta}} -v_n'(\theta)d\theta. \end{aligned}$$

We recall now that the sequence $\{v_n'\}$ converges a.s. uniformly on compact subsets of Θ to $\bar{v}'$. In particular, we have a.s. uniform convergence on all subsets that are contained in a compact set. Since Θ is an interval we may thus assume that

$$-v_n' \to -v' \quad \text{a.s. uniformly on } \tilde{\Theta};$$

otherwise $\tilde{\Theta}$ contains only the endpoints in which case it is a set of measure zero. By dominated convergence this implies $\liminf_{n\to\infty} \int_{\tilde{\Theta}} -v_n'(\theta)d\theta = \int_{\tilde{\Theta}} -\bar{v}'(\theta)d\theta$ and hence the desired contradiction. Therefore expression (16) holds and we conclude that $(Z_{\bar{v}}, \bar{v})$ solves the Principal's problem.

3.3 Uniqueness of optimal solutions

In the preceding section we characterized the solution to the principal's problem in terms of a convex function v and a contract Z_v. Associated with Z_v was a random variable X_v that solved the relaxed optimization problem. The variable X_v is unique if ϱ is strictly convex. However, there are many ways of "pushing X_v to the boundary", i.e., of defining Z_v. As a result, there is no reason to assume that the solution to principal's problem is unique. Instead, we have the following weaker uniqueness result.

Proposition 3.8 *If the risk measure ϱ is strictly convex, then there exists a unique function $\bar{v}$ that minimizes the overall risk. In particular, the principal's optimization problem has is unique solution up to the definition of $Z_{\bar{v}}$.*

PROOF. Let us assume to the contrary that there exist two distinct optimal functions $u, v \in \mathcal{C}$ and consider two corresponding optimal contracts $Z_u \in \partial\mathcal{X}_u$ and $Z_v \in \partial\mathcal{X}_v$ along with a constant $0 < \alpha < 1$. Strict convexity of the risk measure yields

$$\varrho\left(W + \int_\Theta (\alpha Z_u + (1-\alpha)Z_v)\mu(d\theta)\right) < \alpha\varrho\left(W + \int_\Theta Z_u\mu(d\theta)\right) + (1-\alpha)\varrho\left(W + \int_\Theta Z_v\mu(d\theta)\right).$$

The convexity of the variance operator $X \rightarrow \mathrm{Var}[X]$ implies

$$\mathrm{Var}[\alpha Z_u + (1-\alpha)Z_v] \leq \alpha\mathrm{Var}[Z_u] + (1-\alpha)\mathrm{Var}[Z_v].$$

Hence $\alpha Z_u + (1-\alpha)Z_v$ belongs to the set $\mathcal{X}_{\alpha u+(1-\alpha)v}$ so

$$\varrho\left(W + \int_\Theta Z_{\alpha u+(1-\alpha)v}\mu(d\theta)\right) \leq \varrho\left(W + \int_\Theta (\alpha Z_u + (1-\alpha)Z_v)\,\mu(d\theta)\right)$$

Combining the preceding inequalities we obtain that

$$\varrho\left(W + \int_\Theta Z_{\alpha u+(1-\alpha)v}\mu(d\theta)\right) < \alpha\varrho\left(W + \int_\Theta Z_u\mu(d\theta)\right) + (1-\alpha)\varrho\left(W + \int_\Theta Z_v\mu(d\theta)\right)$$

which contradicts the optimality of u and v. □

3.4 A characterization of optimal contracts

In this section we are going to prove the optimal contracts can be chosen to be colinear. To this end, it is enough to fix a function $v \in \mathcal{C}$ and characterize the infimum among all the random variables $X \in \mathcal{X}$ of the operator

$$R(X) := \varrho\left(W - \int_\Theta X(\theta)d\theta\right)$$

subject to the moment conditions

$$\mathbb{E}_{\mathbb{P}}[X(\theta)] = 0, \quad \text{and} \quad \text{Var}[X(\theta)] + v'(\theta) = 0.$$

In order to deal with the constraints let us introduce the operators $V, U : \mathcal{X} \to L^2(\mu)$ defined by

$$U(X)(\theta) := \int_\Omega X(\theta,\omega) d\mathbb{P} \quad \text{and} \quad V(X)(\theta) := \int_\Omega X^2(\theta,\omega) d\mathbb{P} + v'(\theta),$$

respectively, so that our minimization problem can be rewritten as

$$\inf_{X \in \mathcal{X}} R(X) \quad \text{s.t.} \quad U(X) = 0 \text{ and } V(X) = 0. \tag{18}$$

3.4.1 The Lagrangian and the characterization

In order to study the Lagrangian associated with our constrained optimization problem we notice that the Frechét-differentials of V and U at X in the direction $h \in \mathcal{X}$ are given by, respectively,

$$V'(X)h = \int_\Omega 2X(\theta,\omega)h(\theta,\omega)d\mathbb{P} \quad \text{and} \quad U'(X)h = \int_\Omega h(\theta,\omega)d\mathbb{P}.$$

The Frechét differential of R at X in the direction $h \in \mathcal{X}$ is given by

$$R'(X) = \varrho'\left(W - \int_\Theta X(\theta)d\theta\right)\left(-\int_\Theta h(\theta,\omega)d\theta\right).$$

The derivative is well defined because

$$\varrho' \in (L^2(\mathbb{P}))^* \quad \text{and} \quad -\int_\Theta h(\theta,\omega)d\theta \in L^2(\mathbb{P}).$$

Since, for all $H \in L^2(\mathbb{P})$, the map $K \to \varrho'(H)K$ is linear, the Riesz representation theorem yields a random variable $Z_X \in L^2(\mathbb{P})$ such that

$$\varrho'\left(W - \int_\Theta X(\theta)d\theta\right)\left(-\int_\Theta h(\theta,\omega)d\theta\right) = \int_\Omega Z_X(\omega)\left(-\int_\Theta h(\theta,\omega)d\theta\right)d\mathbb{P}.$$

As a result, the operator R has an extremum at Y under our moment constraints, if there exist Lagrange multipliers $\lambda, \eta \in L^2(\mu)$ such that

$$\int_\Omega \int_\Theta h(\theta,\omega)\left(-Z_Y(\omega) + \eta(\theta) + 2\lambda(\theta)Y(\theta,\omega)\right)d\theta d\mathbb{P} = 0.$$

for all $h \in \mathcal{X}$. Since $(\theta,\omega) \to -Z_Y(\omega) + \eta(\theta) + 2\lambda(\theta)Y(\theta,\omega)$ is an integrable function this implies

$$-Z_Y(\omega) + \eta(\theta) + 2\lambda(\theta)Y(\theta,\omega) = 0 \tag{19}$$

due to the DuBois-Reymond lemma. Integrating both sides of this equation with respect to $\mathbb{P}$, and noting that $\mathbb{E}_{\mathbb{P}}[Y(\theta)] = 0$, we obtain

$$\mathbb{E}_{\mathbb{P}}[-Z_Y] + \eta(\theta) = 0.$$

In particular, η does not depend on θ. Thus, with $\bar{Z} := Z_Y - \eta$ and $\alpha(\theta) := (2\lambda(\theta))^{-1}$ we have that

$$Y(\theta, \omega) = \alpha(\theta)\bar{Z}(\omega).$$

The function α is well-defined because we know already that the variance constraint $\text{Var}[Y(\theta)] = -v'(\theta)$ is binding for all types so $\lambda(\theta) \neq 0$ almost surely. Assuming without loss of generality that $\|\bar{Z}\|_2^2 = 1$ the variance constraint implies that

$$\alpha(\theta) = \sqrt{-v'(\theta)}.$$

This proves the following result which also establishes our characterization of optimal contracts.

Proposition 3.9 *Let Z_v be an optimal solution of the problem (18). Then Z_v takes the form*

$$Z_v(\theta, \omega) = \sqrt{-v'(\theta)}\bar{Z}(\omega).$$

3.4.2 An application to entropic risk measures

Our preceding considerations show that the optimization problem (18) can be reduced to finding

$$\inf_{Z \in \Gamma} \varrho \left(W - Z \int_\Theta \sqrt{-v'(\theta)} d\theta \right)$$

where

$$\Gamma := \{Z \in \mathcal{X} \mid \text{E}[Z] = 0, \|Z\|_2^2 = 1\}.$$

This characterization of the optimal contracts given a price schedule induced by $v \in \mathcal{C}$ can be taken further in the particular case of the *entropic* risk measure

$$\varrho(X) = \frac{1}{\beta} \ln \mathbb{E}_{\mathbb{P}} \left[\exp(-\beta X) \right]$$

which is strictly convex. Assuming for simplicity $\beta = 1$, the Frechét-differential of R is given by

$$R'(X)h = \frac{1}{S(X)} \int_\Omega \exp\left(-W + \int_\Theta X(\theta) d\theta \right) \int_\Theta h(\theta, \omega) d\theta d\mathbb{P}$$

where

$$S(X) = \mathbb{E}[\exp(R(X))].$$

Equation (3.4.2) can be rewritten as

$$R'(X)h = \int_\Theta \int_\Omega S(X)^{-1} \exp\left(-W + \int_\Theta X(\theta) d\theta \right) h(\theta, \omega) d\theta d\mathbb{P}$$

Hence $R'(X) \in \mathcal{X}^*$ can be identified with

$$S(X)^{-1} \exp\left(-W + \int_\Theta X(\theta) d\theta \right),$$

and problem (18) is reduced to finding

$$\inf_{Z\in\Gamma} \ln\left(\int_\Omega e^{-W+Z\int_\Theta \sqrt{-v'(\theta)}d\theta} d\mathbb{P}\right).$$

In terms of $a(v) = \int_\Theta \sqrt{-v'(\theta)}d\theta$ we then seek a stationary point of the Lagrangian

$$L(Z,\tau,\kappa) = \ln\left(\int_\Omega e^{-W+Za(v)}d\mathbb{P}\right) + \tau\int_\Omega Z d\mathbb{P} + \kappa\int_\Omega Z^2 d\mathbb{P}$$

where $\tau = \tau(v)$ and $\kappa = \kappa(v)$ belong to $\mathbb{R}$; they are the Lagrange Multipliers associated with the mean and variance constraint, respectively. Equating the Frechét-derivative to the zero operator in $L^2(\mathbb{P})$ yields the following equation:

$$\frac{e^{-W+Za(v)}}{\int_\Omega e^{-W+Za(v)}d\mathbb{P}} + \tau + 2\kappa Z = 0. \tag{20}$$

The values of the multipliers are obtained from the moment constraints. In particular, we have an implicit expression for the solution of (20):

$$Z = -\frac{e^{-W+Za(v)} - \mathbb{E}\left[e^{-W+Za(v)}\right]}{\sqrt{\mathrm{Var}\left(e^{-W+Za(v)}\right)}}.$$

This equation must hold almost everywhere. It is straightforward to see that it has a unique solution for each realization z of Z and w of W, since the line of negative slope

$$l(z) = \mathbb{E}\left[e^{-W+Za(v)}\right] - \sqrt{\mathrm{Var}\left(e^{-W+Za(v)}\right)}z$$

intersects the exponential function $f(z) = e^{-w+za(v)}$ only once.

4 Examples

Our main theorem states that the principal's risk minimization problem has a solution. The solution can be characterized in terms of a convex function that specifies the agents' net utility. The existence result is based on a min-max optimization scheme whose complexity renders a rather involved numerical analysis. In this section we consider some examples where the principal's choice of contracts is restricted to class of numerically more amenable securities. The first example studies a situation where the principal offers type-dependent multiplies of some benchmark claim. This case is motivated by characterization result of Section 3.4 which states that all the optimal contracts are co-linear. In this case the principal's problem can be reduced to a constrained variational problem that can be solved in closed form. A second example studies the case where the principal limits itself to put options with type dependent strikes. Here we provide a numerical algorithm for approximating the optimal solution.

4.1 A single benchmark claim

In this section we study a model where the principal sells a type-dependent multiple of a benchmark claim $f(W) \geq 0$ to the agents. More precisely, the principal offers contracts of the form

$$X(\theta) = \alpha(\theta)f(W). \tag{21}$$

In order to simplify the notation we shall assume that the T-bond's variance is normalized:

$$\mathrm{Var}[f(W)] = 1.$$

4.1.1 The optimization problems

Let (X, π) be a catalogue where the contract X is of the form (21). By analogy to the general case it will be convenient to view the agents' optimization problem as an optimization problem of the set of claims $\{\gamma f(W) \mid \gamma \in \mathbb{R}\}$ so the function $\alpha : \Theta \to \mathbb{R}$ solves

$$\sup_{\gamma \in \mathbb{R}} \{U(\theta, \gamma f(W)) - \pi(\theta)\}.$$

In view of the variance constraint on the agents' claims, the principal's problem can be written as

$$\inf \{\varrho(W - a(v)f(W)) - I(v) \mid v \in \mathcal{C}\} \quad \text{where} \quad a(v) = \int_\Theta \sqrt{-v'(\theta)}d\theta. \tag{22}$$

Note that $E[f(W)] > 0$, so the term $E[f(W)]\sqrt{-v'(\theta)}$ must be included in the income. Before proceeding with the general case let us first consider a situation where in addition to being coherent and law invariant, the risk measure ϱ is also comonotone. In this case each security the principal sells to some agent increases her risk by the amount

$$\varrho\left(-(f(W) - \mathbb{E}[f(W)])\sqrt{-v'(\theta)}\right) + \left(v(\theta) - \theta v'(\theta)\right) \geq 0.$$

This suggests that it is optimal for the principal not to sell a bond whose payoff moves into the same direction as her initial risk exposure.

Proposition 4.1 *Suppose that ϱ is comonotone additive. If $f(W)$ and W are comonotone, then $v = 0$ is a solution to the principal's problem.*

Proof. If W and $f(W)$ are comonotone, then the risk measure in equation (22) is additive and the principal needs to solve

$$\varrho(W) + \inf_{v \in \mathcal{C}} \int_\Theta \left(v(\theta) + \varrho(f(W) - \mathbb{E}[f(W)])\sqrt{-v'(\theta)} - \theta v'(\theta)\right) d\theta.$$

Since $\varrho(f(W) - E[f(W)]) \geq 0$ and $-\theta v'(\theta) \geq 0$ we see that

$$\int_\Theta \left(v(\theta) + \rho(F(W))\sqrt{-v'(\theta)} - \theta v'(\theta)\right) d\theta \geq 0$$

and hence $v \equiv 0$ is a minimizer. $\square$

In view of the preceding proposition the principal needs to design the payoff function f in such a way that W and $f(W)$ are not comonotone. We construct an optimal payoff function in the following subsection.

4.1.2 A solution to the principal's problem

Considering the fact that $\varrho(\cdot)$ is a decreasing function, the principal's goal must be to make the quantity $a(v)$ as small as possible while keeping the income as large as possible. In a first step we therefore solve, for any constant $A \in \mathbb{R}$ the optimization problem

$$\sup_{v \in \mathcal{C}} a(v) \quad \text{subject to} \quad \int_\Theta \left(\mathbb{E}[f(W)]\sqrt{-v'(\theta)} - v(\theta) + \theta v'(\theta) \right) d\theta = A. \tag{23}$$

The constrained variational problem (23) captures the problem of risk minimizing subject to an income constraint. It can be solved in closed form. The associated Euler-Lagrange equation is given by

$$\lambda = \frac{d}{d\theta} \left(-\lambda\theta + \frac{\lambda \mathbb{E}[f] - 1}{2\sqrt{-v'(\theta)}} \right), \tag{24}$$

where λ is the Lagrange multiplier. The income constraint and boundary conditions are:

$$v'(a) = -\frac{\bar{\lambda}^2}{4\lambda^2 a^2} \quad \text{and} \quad v(1) = 0, \quad \text{where} \quad \bar{\lambda} = (\lambda \mathbb{E}[f] - 1).$$

Integrating both sides of equation (24) and taking into account the normalization condition $v(1) = 0$, we obtain

$$v(\theta) = \frac{1}{8} \left(\frac{\bar{\lambda}}{\lambda} \right)^2 \left[\frac{1}{2\theta - a} - \frac{1}{2 - a} \right]. \tag{25}$$

Inserting this equation into the constraint yields

$$A = \mathbb{E}[f] \sqrt{\left(\frac{\bar{\lambda}}{\lambda} \right)^2} \int_a^1 \frac{d\theta}{2\theta - a} - \left(\frac{\bar{\lambda}}{\lambda} \right)^2 \int_a^1 \left\{ \frac{1}{8} \left[\frac{1}{2\theta - a} - \frac{1}{2 - a} \right] + \frac{1}{4} \frac{\theta}{(2\theta - a)^2} \right\} d\theta.$$

In terms of

$$M := \int_a^1 \left\{ \frac{1}{8} \left[\frac{1}{2\theta - a} - \frac{1}{2 - a} \right] + \frac{1}{4} \frac{\theta}{(2\theta - a)^2} \right\} d\theta \quad \text{and} \quad N := \int_a^1 \frac{d\theta}{2\theta - a}$$

we have the quadratic equation

$$-M \left(\frac{\bar{\lambda}}{\lambda} \right)^2 + N \mathbb{E}[f] \sqrt{\left(\frac{\bar{\lambda}}{\lambda} \right)^2} - A = 0,$$

which has the solution

$$\sqrt{\left(\frac{\bar{\lambda}}{\lambda} \right)^2} = \frac{N\mathbb{E}[f] - \sqrt{(N\mathbb{E}[f])^2 - 4AM}}{2M} \tag{26}$$

We have used the root with alternating signs, as we require the problem to reduce to $\varrho(W)$ for $A = 0$.

Remark 4.2 *Notice that the constrained variational problem (23) is independent of the risk measure employed by the principal. This is because we minimized the risk pointwise subject to a constraint on aggregate revenues.*

In view of the preceding considerations the principal's problem reduces to a one-dimensional minimization problems over the Reals:

$$\inf_A \varrho\left(W - f(W)\frac{N^2\mathbb{E}[f]}{2M} + f(W)\frac{N}{2M}\sqrt{(N\mathbb{E}[f])^2 - 4AM}\right) - A.$$

Once the optimal value A^* has been determined, the principal offers the securities

$$\left(\frac{\lambda\mathbb{E}[f] - 1}{4\theta\lambda - 2\lambda a}\right) f(W)$$

at a price

$$\frac{\lambda\mathbb{E}[f] - 1}{2}\left(\frac{3E\lambda(2\theta - a) - a}{(4\theta\lambda - 2\lambda a)^2} + \frac{\lambda E - 1}{\lambda^2}\frac{1}{2 - a}\right).$$

Example 4.3 *Assume that the principal measures her risk exposure using Average Value at Risk at level* 0.05. *Let* $\tilde{W}$ *be a normally distributed random variable with mean* 1/2 *and variance* 1/20. *One can think that* $\tilde{W}$ *represents temperature. Suppose that the principal's initial income is exposed to temperature risk and it is given by* $W = 0.1(\tilde{W} - 1.1)$ *with associated risk*

$$\varrho(W) = 0.0612.$$

Suppose furthermore that the principal sells units of a put option on $\tilde{W}$ *with strike* 0.5, *i.e.,*

$$f(W) = (W - 0.5)^+$$

By proceeding as above we approximated the principal's risk as -0.6731 *and she offers the security*

$$X(\theta) = \frac{0.5459}{2\theta - a} f(W)$$

to the agent of type θ *for a price*

$$\pi(\theta) = \frac{1.1921}{8(2 - a)} - \frac{(1.1921)\theta - (0.22)(2\theta - a)}{\sqrt{2}(2\theta - a)^2}.$$

4.2 Put options with type dependent strikes

In this section we consider the case where the principal underwrites put options on her income with type-dependent strikes. We assume that $W \leq 0$ is a bounded random variable and consider contracts of the form

$$X(\theta) = (K(\theta) - |W|)^+ \quad \text{with} \quad 0 \leq K(\theta) \leq \|W\|_\infty.$$

The boundedness assumption on the strikes is made with no loss of generality as each equilibrium pricing scheme is necessarily non-negative. Note that in this case the risk measure can be defined on $L^\infty(\mathbb{P})$, so we only require convergence in probability to use the Fatou property. We deduce that both the agents' net utilities and the variance of their positions are bounded from above by some constants K_1 and K_2, respectively. Thus, the principal chooses a function v and contract X from the set

$$\{(X, v) \mid v \in \mathcal{C},\, v \leq K_1,\, -\mathrm{Var}[K(\theta) - |W|] = v'(\theta),\, |v'| \leq K_2,\, 0 \leq K(\theta) \leq \|W\|_\infty\}.$$

The variance constraint $v'(\theta) = -\mathrm{Var}[(K(\theta) - W)^+]$ allows us to express the strikes in terms of a continuous function of v', i.e.,

$$K(\theta) = F(v'(\theta)).$$

The Principal's problem can therefore be written as

$$\inf \left\{ \varrho \left(W - \int_\Theta \left\{ (F(v'(\theta)) - |W|)^+ - \mathbb{E}[(F(v'(\theta)) - |W|)^+] \right\} d\theta \right) - I(v) \right\}$$

where the infimum is taken over the set of all functions $v \in \mathcal{C}$ that satisfy $v \leq K_1$ and $|v'| \leq K_2$.

Remark 4.4 *Within our current framework the contracts are expressed in terms of the derivative of the principal's choice of v. This reflects the fact that the principal restricts itself to type-dependent put options and is not always true in the general case.*

4.2.1 An existence result

Let $\{v_n\}$ be a minimizing sequence for the principal's optimization problem. The functions v_n are uniformly bounded and uniformly equicontinuous so we may with no loss of generality assume that $v_n \to \overline{v}$ uniformly. Recall this also implies a.s. convergence of the derivatives. By dominated convergence and the continuity of F, along with the fact that W is bounded yields

$$\int_\Theta (F(v_n'(\theta)) - |W|)^+ d\theta \longrightarrow \int_\Theta (F(\overline{v}'(\theta)) - |W|)^+ d\theta \quad \mathbb{P}\text{-a.s.}$$

and

$$\lim_{n\to\infty} \int_\Theta \mathbb{E}[(F(v_n'(\theta)) - |W|)^+ d\theta = \int_\Theta \mathbb{E}[(F(\overline{v}'(\theta)) - |W|)^+ d\theta$$

This shows that the principal's positions converge almost surely and hence in probability. Since ϱ is lower-semi-continuous with respect to convergence in probability we deduce that $\overline{v}$ solves the principal's problem.

4.2.2 An algorithm for approximating the optimal solution

We close this paper with a numerical approximation scheme for the principal's optimal solution within the put option framework.[4] We assume that the set of states of the World is finite with cardinality m. Each possible state ω_j can occur with probability p_j. The realizations of the principal's wealth are denoted by $W = (W_1, \ldots, W_m)$. Note that p and W are treated as known data. We implement a numerical algorithm to approximate a solution to the principal's problem when she evaluates risk via the risk measure

$$\varrho(X) = -\sup_{q \in Q_\lambda} \sum_{j=1}^{m} X(\omega_j) p_j q_j,$$

where

$$Q_\lambda := \left\{ q \in \mathbb{R}_+^m \mid p \cdot q = 1,\ q_j \leq \lambda^{-1} \right\}.$$

We also assume the set of agent types is finite with cardinality n, i.e. $\theta = (\theta_1, \ldots, \theta_n)$. The density of the types is given by $M := (M_1, \ldots, M_n)$. In order to avoid singular points in the principal's objective function, we approximate the option's payoff function $f(x) = (K - x)^+$ by the differentiable function

$$T(x, K) = \begin{cases} 0, & \text{if} \quad x \leq K - \epsilon, \\ S(x, K), & \text{if} \quad K - \epsilon < x < K + \epsilon, \\ x - K, & \text{if} \quad x \geq K + \epsilon. \end{cases}$$

where

$$S(x, K) = \frac{x^2}{4\epsilon} + \frac{\epsilon - K}{2\epsilon} x + \frac{K^2 - 2A\epsilon + \epsilon^2}{4\epsilon}.$$

The algorithm uses a penalized Quasi-Newton method, based on Zakovic and Pantelides [22], to approximate a minimax point of

$$\begin{aligned} F(v, K, q) \quad = \quad & -\sum_{i=1}^{n} W_i p_i q_i + \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} T(K_j - |W_i|) \right) p_i q_i - \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{j=1}^{n-1} T(K_j - |W_i|) \right) p_i \\ & + \frac{1}{n} \sum_{i=1}^{n} \left(v_i - \theta_i \frac{v_{i+1} - v_i}{\theta_{i+1} - \theta_i} \right) + \frac{1}{n} \left(v_n - \frac{v_n - v_{n-1}}{1 - \theta_{n-1}} \right) \end{aligned}$$

where $v = (v_1, \ldots, v_n)$ stands for the values of a convex, non-increasing function, $K = (K_1, \ldots, K_n)$ denotes the vector of type dependent strikes and the derivatives $v'(\theta_i)$ are approximated by

$$v'(\theta_i) = \frac{v_{i+1} - v_i}{\theta_{i+1} - \theta_i}.$$

[4] Our approximation is based on an existing algorithm. For a more detailed discussion of perhaps more appropriate numerical schemes we refer to an upcoming paper by Ekeland and Moreno [10].

The need for a penalty method arises from the fact that we face the equality constraints $v'(\theta) = -Var[(K(\theta) - |W|)^+]$ and $p \cdot q = 1$. In order to implement a descent method, these constraints are relaxed and a penalty term is added. We denote by ng the total number of constraints. The principal's problem is to find

$$\min_{(v,K)} \max_{q \in Q_\lambda} F(v, K, q) \quad \text{subject to} \quad G(v, K, q) \leq 0$$

where $G : \mathbb{R}^{2n+m} \to \mathbb{R}^{ng}$ determines the constraints that keep (v, K) within the set of feasible contracts and $q \in Q_\lambda$. The Maple code for our procedure is available upon request.

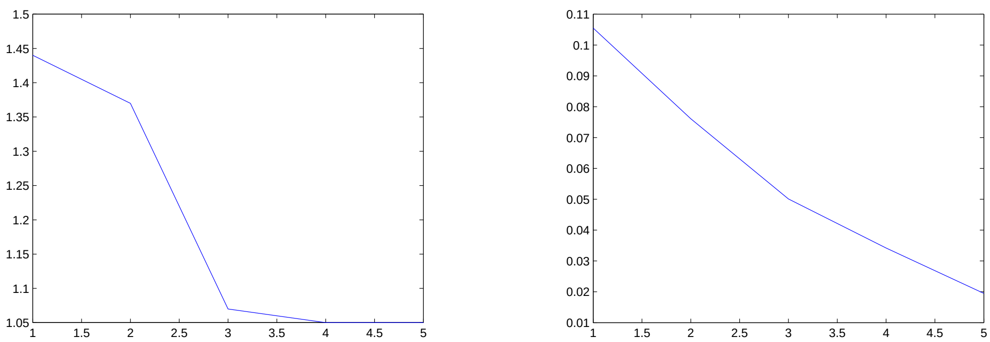
Example 4.5 *Let us illustrate the effects of risk transfer on the principal's position in two models with five agent types and two states of the world. In both cases* $W = (-1, -2)$, $\theta = (1/2, 5/8, 3/4, 7/8, 1)$ *and* $\lambda = 1.1$ *The starting values* v_0, q_0 *and* K_0 *we set are* $(4, 3, 2, 1, 0)$, $(1, 1)$ *and* $(1, 1, 1, 1, 1)$ *respectively.*

i) Let $p = (0.5, 0.5)$ *and the types be uniformly distributed. The principal's initial evaluation of her risk is* 1.52. *The optimal function* v *and strikes are:*

V_1	*0.1055*
V_2	*0.0761*
V_3	*0.0501*
V_4	*0.0342*
V_5	*0.0195*

K_1	*1.44*
K_2	*1.37*
K_3	*1.07*
K_4	*1.05*
K_5	*1.05*

The Principal's valuation of her risk after the exchanges with the agents decreases to 0.2279.



(a) The type-dependent strikes.

(b) The optimal function v.

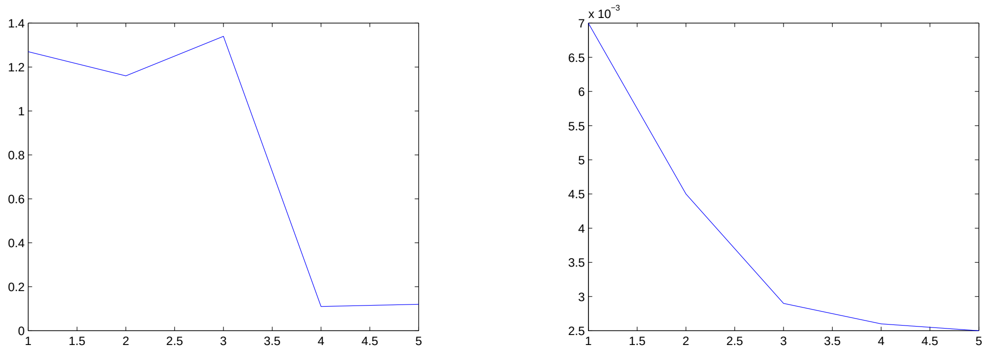
Figure 1: Optimal solution for underwriting put options, Case 1.

ii) In this instance $p = (0.25, 0.75)$ *and* $M = (1/15, 2/15, 3/15, 4/15, 5/15)$*. The principal's initial evaluation of her risk is* 1.825*. The values for the discretized* v *the type-dependent strikes are:*

V_1	*0.0073*
V_2	*0.0045*
V_3	*0.0029*
V_4	*0.0026*
V_5	*0.0025*

K_1	*1.27*
K_2	*1.16*
K_3	*1.34*
K_4	*0.11*
K_5	*0.12*

The Principal's valuation of her risk after the exchanges with the agents is 0.0922*.*



(a) The type-dependent strikes.

(b) The optimal function v.

Figure 2: Optimal solution for underwriting put options, Case 2.

5 Conclusions

In this paper we analyzed a screening problem where the principal's choice space is infinite dimensional. Our motivation was to present a nonlinear pricing scheme for over-the-counter financial products, which she trades with a set of heterogeneous agents with the aim of minimizing the exposure of her income to some non-hedgeable risk. In order to characterize incentive compatible and individually rational catalogues, we have made use of U-convex analysis. To keep the problem tractable we have assumed the agents have mean-variance utilities, but this is not necessary for the characterization of the problem. Considering more general utility functions is an obvious extension to this work. Our main result is a proof of existence of a solution to the principal's risk minimization problem in a general setting. We also characterized the structure of optimal contracts and showed that the optimal solution is - in some sense - unique. The examples we have studied suggest that the methodologies for approaching particular cases are highly dependent on the choice of risk measure, as well as on the kinds of contracts the principal is willing (or able) to offer. In most cases obtaining closed form solutions is not possible and implementations must be done using numerical methods. As a work in progress we are considering agents with heterogenous initial endowments (or risk exposures), as well as a model that contemplates an economy with multiple principals.

A U-Convexity

Our analysis of optimality is based on the notion of U-convex functions. In this appendix we recall the notion of U-convexity and state a characterization result for U-convex functions.

Definition A.1 *Let two spaces A and B and a function $U : A \times B \to \mathbb{R}$ be given.*

(i) The function $f : A \to \mathbb{R}$ is called U-convex if there exists a function $p : B \to \mathbb{R}$ such that

$$f(a) = \sup_{b \in B} \{U(a, b) - p(b)\}.$$

(ii) For a given function $p : B \to \mathbb{R}$ the U-conjugate $p^U(a)$ of p is defined by

$$p^U(a) = \sup_{b \in B} \{U(a, b) - p(b)\}.$$

(iii) The U-subdifferential of p at b is given by the set

$$\partial_U p(b) := \left\{a \in A \mid p^U(a) = U(a, b) - p(b)\right\}.$$

(iv) If $a \in \partial_U p(b)$, then a is called a U-subgradient of $p(b)$.

The following proposition shows that a function is U-convex if and only if it equals the U-conjugate of its U-conjugate.

Proposition A.2 *A function $f : A \to \mathbb{R}$ is U-convex if and only if $\left(f^U\right)^U = f$.*

Proof.

(i) Let us first assume that $\left(f^U\right)^U = f$. Then

$$f(a) = \sup_{b \in B} \{U(a, b) - f^U(b)\},$$

and hence it is U-Convex.

(ii) Conversely, if f is U-convex, then there is a function $p : B \to \mathbb{R}$ such that $f(a) = \sup_{b \in B}\{U(a,b) - p(b)\}$. Since

$$f^U(b) = \sup_{a \in A}\{U(a,b) - \sup_{b \in B}\{U(a,b) - p(b)\}\} \leq \sup_{a \in A}\{U(a,b) - U(a,b) - p(b)\} = p(b)$$

we see that

$$\left(f^U\right)^U(a) = \sup_{b \in B}\left\{U(a,b) - f^U(b)\right\} \geq \sup_{b \in B}\left\{U(a,b) - p^U(b)\right\} = f(a).$$

On the other hand

$$f^U(b) \geq U(a,b) - f(a) \quad \text{for all} \quad a \in A.$$

Thus

$$\left(f^U\right)^U(a) = \sup_{b \in B}\left\{U(a,b) - f^U(b)\right\} \leq \sup_{b \in B}\{U(a,b) - U(a,b) + f(a)\} = f(a).$$

This concludes the proof.

□

B Coherent risk measures on L^2.

In this appendix we recall some well-known properties and representation results for risk measures on L^2 spaces; we refer to the textbook of Föllmer and Schied [13] for a detailed discussion of convex risk measures on L^∞ and to Cheridito and Tianbui [8] for risk measures on rather general state spaces. Bäuerle and Müller [4] establish representation properties of risk law invariant risk measures on L^p spaces for $p \geq 1$. We assume that all random variables are defined on some standard non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition B.1 *(i) A monetary measure of risk on L^2 is a function $\varrho : L^2 \to \mathbb{R} \cup \{\infty\}$ such that for all $X, Y \in L^2$ the following conditions are satisfied:*

- *Monotonicity: if $X \leq Y$ then $\varrho(X) \geq \varrho(Y)$.*
- *Cash Invariance: if $m \in \mathbb{R}$ then $\varrho(X + m) = \varrho(X) - m$.*

(ii) A risk measure is called coherent if it is convex and homogeneous of degree 1, i.e., if the following two conditions hold:

- *Convexity: for all $\lambda \in [0,1]$ and all positions $X, Y \in L^2$:*

$$\varrho(\lambda X + (1-\lambda)Y) \leq \lambda\varrho(X) + (1-\lambda)\varrho(Y)$$

- *Positive Homogeneity: For all* $\lambda \geq 1$

$$\varrho(\lambda X) = \lambda \varrho(X).$$

(iii) The risk measure is called coherent and law invariant, if, in addition,

$$\rho(X) = \rho(Y)$$

for any two random variables X *and* Y *which have the same law.*

(iv) The risk measure ϱ *on* L^2 *has the Fatou property if for any sequence of random variables* $X_1, X_2, \ldots$ *that converges in* L^2 *to a random variable* X *we have*

$$\rho(X) \leq \liminf_{n \to \infty} \rho(X_n).$$

Given $\lambda \in (0, 1]$, the *Average Value at Risk* of level λ of a position Y is defined as

$$AV@R_\lambda(Y) := -\frac{1}{\lambda} \int_0^\lambda q_Y(t)dt,$$

where $q_Y(t)$ is the upper quantile function of Y. If $Y \in L^\infty$, then we have the following characterization

$$AV@R_\lambda(Y) = \sup_{Q \in \mathcal{Q}_\lambda} -\mathbb{E}_Q[Y]$$

where

$$\mathcal{Q}_\lambda = \left\{ Q << P \mid \frac{dQ}{dP} \leq \frac{1}{\lambda} \right\}.$$

Proposition B.2 *For a given financial position* $Y \in L^2$ *the mapping* $\lambda \mapsto AV@R_\lambda(Y)$ *is decreasing in* λ.

It turns out the Average Value of Risk can be viewed as a basis for the space of all law-invariant, coherent risk measures with the Fatou property. More precisely, we have the following result.

Theorem B.3 *The risk measure* $\varrho : L^2 \to \mathbb{R}$ *is law-invariant, coherent and has the Fatou Property if and only if* ϱ *admits a representation of the following form:*

$$\varrho(Y) = \sup_{\mu \in M} \left\{ \int_0^1 AV@R_\lambda(Y)\mu(d\lambda) \right\}$$

where M *is a set of probability measures on the unit interval.*

As a consequence of Proposition B.2 and Theorem B.3 we have the following Corollary:

Corollary B.4 *If $\varrho : L^2 \to \mathbb{R}$ is a law-invariant, coherent risk measure with the Fatou Property then*

$$\varrho(Y) \geq -\mathbb{E}[Y].$$

An important class of risk measures are comonotone risk measures risk. Comonotone risk measures are characterized by the fact that the risk associated with two position whose payoff "moves in the same direction" is additive.

Definition B.5 *A risk measure ϱ is said to be comonotone if*

$$\varrho(X+Y) = \rho(X) + \rho(Y)$$

whenever X and Y are comonotone, i.e., whenever

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0 \quad \mathbb{P}\text{-}a.s.$$

Comonotone, law invariant and coherent risk measures with the Fatou property admit a representation of the form

$$\varrho(Y) = \int_0^1 AV@R_\lambda(Y)\mu(d\lambda).$$

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