Abstract

An stochastic system is called ergodic if it tends in probability to a limiting form that is independent of the initial conditions. Breakdown of ergodicity gives rise to path dependence. We illustrate the importance of ergodicity and breakdown thereof in economics reviewing some work of non-market interactions including microeconomic models of endogenous preference formation and macroeconomic models of economic growth.

Ergodicity and Non-Ergodicity in Economics

A stochastic system is called ergodic if it tends in probability to a limiting form that is independent of the initial conditions. Breakdown of ergodicity gives rise to path dependence. When path dependence occurs, "history matters". Choices made on the basis of transitory conditions can persist long after those conditions change. Path-dependent features of economics range from small-scale technical standards to large-scale institutions. Prominent path-dependent features in economics include technical standards, such as the "QWERTY" standard typewriter keyboard and the "standard gauge" of railway track. Ergodicity and breakdown thereof plays a major role in models of social interaction. We illustrate this importance summarizing some work on endogenous preference formation, learning dynamics in population games, and models of non-market interaction. As many of these models share a common mathematical basis, we also outline some of the underlying mathematical techniques.

Endogenous Preference Formation

In his pioneering paper on endogenous preference formation, Föllmer (1974) developed an equilibrium analysis of large exchange economies where the preferences of an agent a are subject to random shocks x^a, and where the probabilities governing that randomness have an interactive structure. In an earlier work by Hildenbrand (1971) the *states* x^a were random but independent across agents. In such a situation, and under mild conditions on the agents' excess demand functions $z(x^a, \cdot)$ given their states x^a_t, there exists a price system p such that the per capita excess demand gets small if the number of agents gets large. More precisely,

$$\lim_{n \to \infty} \frac{1}{|\mathbb{A}_n|} \sum_{a \in \mathbb{A}_n} z(x^a, p) = 0 \tag{1}$$

where $\mathbb{A}_n \uparrow \mathbb{A}$ is a sequence of finite sub-populations converging to some infinite set $\mathbb{A}$. In this case all variability in aggregate access demand is eliminated by the law of large numbers.

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The assumption of independence of states can be dropped as long as the probability measure μ governing the joint distribution of the *configuration* $(x^a)_{a \in \mathbb{A}}$ is *ergodic*, i.e., if spatial averages converge to their expected value under μ. However, when preferences are interactive, the probability laws governing *individual* behavior may be insufficient a statistic for the resulting *aggregate* behavior. These probabilities do not necessarily determine the distribution of the entire configuration $(x^a)_{a \in \mathbb{A}}$. This effect can best be illustrated in the context of Föllmer's *Ising economy* where agents $a \in \mathbb{A}$ are indexed by the two dimensional integer lattice $(\mathbb{A} = \mathbb{Z}^2)$ and interact with the agents in their neighborhood $N(a) := \{b \in \mathbb{A} : \|a - b\| = 1\}$. The set of possible states is $\{-1, 1\}$, and the probabilities $(\pi_a)_{a \in \mathbb{A}}$ governing the dependence of an individual agent's preferences on his neighbors' states take the form

$$\pi_a(x^a; x^{-a}) = \frac{\exp\left\{x^a h + x^a \sum_{b \in N(a)} J x^b\right\}}{\exp\left\{x^a h + x^a \sum_{b \in N(a)} J x^b\right\} + \exp\left\{-x^a h - x^a \sum_{b \in N(a)} J x^b\right\}}. \tag{2}$$

Here h specifies to which extend agents are *outer directed* while $J \geq 0$ determines the dependence of an agent's state on the preferences of others. The case $J = 0$ corresponds to independent preferences. Föllmer calls a probability measure μ on the configuration space $S = \{x = (x^a)_{a \in \mathbb{A}} : x^a \in \{-1, +1\}\}$ a *global phase* if its one-dimensional marginal distributions are consistent with the microscopic data given by the local specification (2), i.e., if

$$\mu(x^a = \pm 1 | x^{-a}) = \pi_a(\pm 1; x^{-a}). \tag{3}$$

A global phase μ describes the joint distribution of all the agents' states while the local characteristics $(\pi_a)_{a \in \mathbb{A}}$ describe the conditional dependence of preferences on the states of others. A phase μ can be *equilibrated* if there exists a price system such that (1) holds μ-almost surely. For independent preferences and outer directed $(h \neq 0)$ economies, global phases are always unique and ergodic. However, two ergodic global phases μ_+ and μ_- exist for inner directed $(h = 0)$ economies when the interaction becomes too strong. When J exceeds some critical value the microeconomic data $(\pi_a)_{a \in \mathbb{A}}$ in (2) does not determine equilibrium prices. More precisely, by the ergodic theorem,

$$\lim_{n \to \infty} \frac{1}{|\mathbb{A}_n|} \sum_{a \in \mathbb{A}_n} z(x^a, p) = \int z(x^0, p) \mu_{\pm}(dx^0) \quad \mu_{\pm}\text{-a.s.} \tag{4}$$

and we can find prices p_+ and p_- equilibrating μ_+ and μ_-, respectively. However, there is typically no price system that equilibrates the market *simultaneously* for μ_+ *and* μ_- because $\int z(x^0, p) \mu_+(dx^0) \neq \int z(x^0, p) \mu_-(dx^0)$. When the interaction is too strong, randomness in preferences may thus become a source of uncertainty about equilibrium prices.

Stochastic Strategy Revision in Population Games

Föllmer (1974) beautifully illustrates the interplay between local interactions and the breakdown of ergodicity in large economies. However, his model lacks an element of choice. The pioneering work by Blume (1993) overcomes this shortcoming in a natural way. His work is concerned with the aggregate behavior in population games of bounded rational play looking for "Nash-like play in the aggregate rather than at the level of an individual player". By analogy to Föllmer (1974), Blume's dynamic version of an Ising economy assumes that agents are located on the two-dimensional integer lattice ($\mathbb{A} = \mathbb{Z}^2$), interact only with their nearest neighbors, and choose actions from a binary action set. Choice opportunities arise randomly, and only for one agent at a time. When such an opportunity arises for player $a \in \mathbb{A}$ at time t, his choice $x_t^a \in \{-1, +1\}$ results in an instantaneous payoff $G(x_t^a, x_t^b)$ from his neighbor $b \in N(a)$ and in a total payoff

$$\sum_{b \in N(a)} G(x_t^a, x_t^b). \tag{5}$$

The conditional probability $\pi_a(x_t^a; x_t^{-a})$ with which player $a \in \mathbb{A}$ selects an action x_t^a given that the rest of the population is configured to x_t^{-a} can be chosen to take the form (2) with $h = \beta \hat{h}$ and $J = \beta \hat{J}$. The quantity $\beta \geq 0$ specifies the strength of interaction while $\hat{h}$ and $\hat{J}$ are determined in terms of the 2×2 payoff matrix G. Specifically,

$$\pi_a(x_t^a; x_t^{-a}) = \frac{\exp\left\{\beta \left[x_t^a \hat{h} + x_t^a \sum_{b \in N(a)} \hat{J} x_t^b\right]\right\}}{\exp\left\{\beta \left[x_t^a \hat{h} + x_t^a \sum_{b \in N(a)} \hat{J} x_t^b\right]\right\} + \exp\left\{-\beta \left[x_t^a \hat{h} + x_t^a \sum_{b \in N(a)} \hat{J} x_t^b\right]\right\}}. \tag{6}$$

These *flip rates* correspond to those of the stochastic Ising model of statistical mechanics. They uniquely determine the dynamics of the continuous time Markov process $\{x_t\}_{t \geq 0}$ on S. The process has an *ergodic stationary measure* μ if the distribution of choices does not change over time when the initial state is chosen according to μ and if empirical averages converge to their expected values under μ. The process is called *ergodic* if it has a unique ergodic measure. As it is well known from the theory of interacting particle systems, the set of all ergodic probability measures of the Markov process $\{x_t\}_{t \geq 0}$ is given by the ergodic global phases corresponding to the local specification (6). As a result, Blume's stochastic strategy revision process is ergodic if $\hat{h} \neq 0$ and $\hat{J} \geq 0$. If G describes a two person coordination game, $\{x_t\}_{t \in \mathbb{N}}$ is ergodic and the players eventually coordinate on the risk dominant equilibrium if $\beta \to \infty$. However, ergodicity breaks down for games with symmetric payoff matrices ($\hat{h} = 0$) when the interaction becomes too strong. In this case, initial conditions are persistent. The long run average choice depends on the initial distribution of choices.

Non-ergodic economic Growth

The evolution of individual choices in Blume (1993) is described by a *continuous* time Markov process with *asynchronous* updating. In local interaction models with *synchronous* updating, the dynamics of individual behavior is typically described by a Markov chain whose transition operator takes the product form

$$\Pi(x_t; \cdot) = \prod_{a \in \mathbb{A}} \pi_a \left(\cdot; \{x_t^b\}_{b \in N(a)} \right). \tag{7}$$

While the individual transition probabilities π_a have an interactive structure the actual transition to a new configuration itself is made independently by different agents. The long run dynamics of such Markov chains plays a major role in, e.g., microstructure models of financial markets (Horst (2005a)), and macroeconomic models of economic growth (Durlauf (1993)).

The substantial differences in output levels and growth rates across countries have long become a major focus of macroeconomic research. A hallmark of the stochastic growth model pioneered by Brock and Mirman (1972) is the convergence of economies with identical preferences and production functions to a common level of aggregate output. Yet many analyses of long-run output movements have concluded that per capita production is not equalizing across countries. To explain this divergence, Durlauf (1993) studies a dynamic model of capital accumulation of an economy with an infinite set $\mathbb{A}$ of interacting companies where local technological externalities affect the process of production. Each company $a \in \mathbb{Z}$ chooses a capital stock sequence $\{K_t^a\}_{t \in \mathbb{N}}$ that maximizes the present value of future profits. The technique-specific production functions generate output

$$Y_t^a = f(K_{t-1}^a, x_t^a, F(x_t^a)) \tag{8}$$

where $x_t^a \in \{0, 1\}$. Technique $x_t^a = 1$ is more productive, but comes at a higher fixed cost: $F(1) > F(0)$. Local technological complementarities affect the production as the distribution of x_t^a depends on the techniques implemented by the nearest neighbors $b \in N(a)$ in the previous period. The dynamics of production technologies is then described by an interactive Markov chain of the form (7). Assuming that past choices of technique 1 improve the current relative productivity of the technique and that the high productivity state $x_t^a = 1$ for all $a \in \mathbb{A}$ is an equilibrium, Durlauf (1993) shows that the high productivity state is the only long run outcome if the complementarities are weak enough: there exists $0 < \underline{\theta} < 1$ such that

$$\lim_{t \to \infty} \mathbb{P}\left[x_t^a = 1 \mid x_0^a = 0\right] = 1 \quad \text{if} \quad \pi_a \left(1; \{x_{t-1}^b\}_{b \in N(a)} \right) \geq \underline{\theta}.$$

Even when starting with all low-production industries, an economy eventually coordinates on the high-production-technology when negative feedbacks form low production technologies

are sufficiently weak. Powerful negative complementarities, on the other hand, can generate a non-ergodic growth path. In fact, there exists $0 < \bar{\theta} < \underline{\theta} < 1$ such that

$$\lim_{t \to \infty} \mathbb{P}\left[x_t^a = 1 \mid x_0^a = 0\right] < 1 \quad \text{if} \quad \pi_a\left(1; \{x_{t-1}^b\}_{b \in N(a)}\right) \leq \bar{\theta}.$$

If the complementarities are too strong, industries fail to coordinate on high-productivity equilibria, and economies may get trapped in low-productivity equilibria.

Models of social Interaction - Mean-Field Interaction

Much of the literature on social interactions assumes very special interaction structures such as nearest neighbor interactions as in Blume (1993) or Durlauf (1993) or mean field interaction. If agents care about the average behavior throughout the whole population, the analysis is most naturally done in the context of an infinity of agents as in Brock and Durlauf (2001). These authors analyze aggregate behavioral outcomes when individual utility exhibits social interaction effects. In the simplest setting agents take actions x^a from the binary action set $\{-1, +1\}$ and their utilities consists of three components:

$$U^a(x^a, m^a, \epsilon(x^a)) = u(x^a) + Jx^a m^a + \epsilon(x^a), \tag{9}$$

Here m^a denotes agent a's expectation about the average choice of all the other agents. The second term in the utility function may thus be viewed as a social utility expressing an agent's desire for conformity ($J > 0$). The quantity $u(x^a)$, on the other hand, represents the private utility associated with a choice while $\epsilon(x^a)$ is a random utility term independent of other agents' utilities and extreme-value distributed with parameter $\beta > 0$. As a result, the agents' conditional choice probabilities are of the form (2) if we replace the dependence of *actual* actions by a dependence on *expected actions*. When agents have homogeneous expectations about the behavior of others ($m^a \equiv m$), then

$$\pi_a(x^a; m) = \frac{\exp\{\beta(u(x^a) + Jx^a m)\}}{\exp\{\beta(u(1) + Jm)\} + \exp\{\beta(u(-1) - Jm)\}}. \tag{10}$$

In the limit of an infinite economy all uncertainty about the average action vanishes as in Hildenbrand (1971), and the average action is $\tanh(\beta h + \beta Jm)$ where $h = \frac{1}{2}(u(1) - u(-1))$. If the agents have rational expectations the average satisfies the fixed point condition

$$m = \tanh(\beta h + \beta Jm). \tag{11}$$

This equation has a unique solution if $h \neq 0$ and β is large enough. The uniqueness property breaks down if $h = 0$ in which case there exist three roots to equation (11).

Models of social Interaction - Local & global Interaction

When agents care about both the average action and the choices of neighbors, the equilibrium analysis becomes more involved. Horst and Scheinkman (2003) provide a general framework for analyzing systems of social interactions with an infinite set $\mathbb{A}$ of locally and globally interacting agents where interaction structures and preferences are random. While the distinction between local and global interactions is unnecessary for models with finitely many agents, it plays a major role in the analysis of infinite economies. The continuity of an agent's utility function with respect to the vector of actions requires implicitly, that the dependence of his utility function on another agent's action decays sufficiently fast as the distance to that other agent grows. Thus, if preferences depend on average actions in a nontrivial manner, utility functions are not continuous. To overcome this problem, Horst and Scheinkman (2003) separated the local and global impact of an action profile x on preferences viewing the average action as an additional parameter of the utility functions u^a. Specifically,

$$u^a(x, \varrho, \vartheta^a) \equiv U(x^a, \{x^b\}_{b \in N(a)}, \varrho, \vartheta^a) \tag{12}$$

for a continuous map U. Here ϱ denotes the agents' common "expected" average action and the random vector $(\vartheta^a)_{a \in \mathbb{A}}$ specifies the interaction pattern and the distribution of taste shocks. In equilibrium, the average $\varrho(x^*)$ associated to the configuration x^* is independent of the realized interaction pattern and taste shocks and correctly anticipated: $\varrho = \varrho(x^*)$. If some form of spatial homogeneity prevails, existence or uniqueness of equilibria can be established under a weak interaction condition that restricts the influence of an agent's choice on the optimal decisions of others. In this case one can also prove a spatial ergodicity result: the equilibrium of the infinite system is the limit of equilibria of finite systems when the number of agents grows to infinity (Horst and Scheinkman (2005)).

When dynamic models of social interactions are studied the analysis is often confined to the case of backward looking myopic dynamics, either as a simple explicit dynamic process with random sequential choice, or as an equilibrium selection procedure. Rational expectations equilibria of economies with local interactions are studied in Bisin, Horst, and Özgür (2003), and Horst (2005b). While agents interact locally in these models, they are forward looking. Their choices are optimally based on the past actions in their neighborhood, as well as on their anticipations of the future actions of their neighbors. The resulting population dynamics can be described by an interactive Markov chain of the form (7) but the transition probabilities π_a are *endogenously* specified in terms of the agents' policy functions. Bisin, Horst, and Özgür (2003) also allow for local and global interactions and combine spatial and temporal ergodicity results. The dynamics on the level of aggregate behavior is deterministic (spatial ergodicity) and the distribution of individual choices settles down in the long run (temporal ergodicity) when the interaction is weak enough. The analysis, however, is

confined to one-sided interactions. It is an open problem to fully embed the theory of social interactions into a dynamics analysis of equilibrium.

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