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Interest Rate Modelling and Derivative Pricing, SS 2022

Exercise 6

1. **Bermudan options**
   Consider the pricing of a Bermudan option with n exercise times $T_1, \ldots, T_n$. Let $k \in \{0, \ldots n\}$ and set $T_0 = 0$. Denote $U_k(T_k)$ the time-T_k value of the underlying which the option holder receives if she exercises the option at T_k. Further denote $H_k(T_k)$ the time-T_k value of the option where the exercise times are restricted to $T_{k+1}, \ldots T_N$ ($H_k(T_k)$ is the option continuation value). We set $H_k(t) = B(t) \cdot \mathbb{E}_t [H_k(T_k)/B(T_k)]$ for $t \leq T_k$ and expectation $\mathbb{E}_t[\cdot]$ conditional on time-t. The Bermudan option price is derived via the recursion
   $$H_k(T_k) = B(T_k) \cdot \mathbb{E}_{T_k}\left[\frac{H_k(T_{k+1})}{B(T_{k+1})}\right] = B(T_k) \cdot \mathbb{E}_{T_k}\left[\frac{\max\{U_{k+1}, H_{k+1}(T_{k+1})\}}{B(T_{k+1})}\right]$$
   for $k = N-1, \ldots 0$.

   Show that the Bermudan option price $H_0(0)$ can be written as
   $$H_0(0) = B(0) \cdot \sum_{k=1}^{n} \mathbb{E}\left[\frac{[U_k(T_k) - H_k(T_k)]^+}{B(T_k)}\right]$$

2. **Bermudan Swaption Pricing**
   Analyse and explain the pricing of Bermudan swaptions implemented in `BermudanSwaption.ipynb`. Use the Python code and calculate in-the-money, at-the-money and out-of-the-money swaptions. Compare the Bermudan price relative to the co-terminal European bond option prices.

3. **Accuracy of Density Integration Methods**
   Consider the pricing of a $10y$ expiry into $10y$ maturity coupon bond call option (i.e. bond payments in $11y$, $12y$, ...,$20y$). Chose interest rates such that the (forward) bond is at-the-money and calculate the option price in a Hull White model via analytic formula. Also set up the coupon bond option as Bermudan option (with a couple of trivial additional exercises) and use density integration methods to calculate the option price.

   (a) Compare the pricing accuracy of Simpson's method (with and without break-even point calculation) for various choices of number of grid points and grid sizes.

   (b) Compare the pricing accuracy of Hermite integration method for various choices of number of grid points and Hermite polynomial degrees.

   (c) Compare the pricing accuracy of exact cubic spline integration method (with and without break-even point calculation) for various choices of number of grid points and grid sizes.

   (d) Which integration method would you recommend in terms of accuracy and computational effort?

4. **Accuracy of PDE Methods**
   Consider the pricing of a $10y$ expiry into $10y$ maturity coupon bond call option (i.e. bond payments in $11y$, $12y$, ...,$20y$). Chose interest rates such that the (forward) bond is at-the-money and calculate the option price in a Hull White model via analytic formula. Also set up the coupon bond option as Bermudan option (with a couple of trivial additional exercises) and use PDE methods to calculate the option price.

(a) Compare the pricing accuracy of PDE method for various choices of number of grid points, grid sizes and time step sizes.

(b) How should the time step size be chosen in relation to the number of grid points and/or grid size?

(c) How does the choice of θ in the time integration impact the accuracy?

(d) Analyse the impact of the choice of boundary condition ($\lambda_{0,N} = 0$ or via bond-option-approximation) versus the grid size (in terms of standard deviations) on the accuracy of the numerical method.

5. **Convergence of Monte-Carlo Method**
Consider the pricing of a $10y$ expiry into $10y$ maturity coupon bond call option (i.e. bond payments in $11y$, $12y$, ...,$20y$). Chose interest rates such that the (forward) bond is at-the-money and calculate the option price in a Hull White model via analytic formula. Also set up the coupon bond option as Bermudan option with a single exercise and use Monte-Carlo method to calculate the option price.

   (a) Analyse the convergence of the Monte-Carlo method to the analytical option price for increasing number of paths.

   (b) How does the convergence change if you change the Hull White model volatility and mean reversion?

6. **Accuracy of American Monte-Carlo Methods**
Consider the pricing of a $10y$ expiry into $10y$ maturity Bermudan coupon bond call option with annual call right. (i.e. bond payments in $11y$, $12y$, ...,$20y$, exercises in $10y$, $11y$, ...,$19y$). Choose interest rates such that the (forward) bond is at-the-money. Calculate a reference option price in a Hull White model via PDE (or density integration) method with high accuracy. Also calculate the Bermudan option price using American Monte-Carlo method.

   (a) Analyse the convergence of the Monte-Carlo method to the reference option price for increasing number of paths.

   (b) How does the price change if you change the max. polynomial order for regression?

   (c) How does the price change depending on regression on the continuation value compared to regression on the exercise decision only?

   (d) Add a modified American Monte-Carlo method that uses the following basis functions:

      i. a swap rate $S(T_E)$ from exercise to final maturity (i.e. co-terminal swap rate), and

      ii. the term $[S(T_E) - K]^+$, where K is the fixed rate of the underlying coupon bond.

   Analyse the impact on the pricing for increasing max. polynomial order for regression. How does the pricing conpare to the state variable approach?