

1 Galton-Watson processes

Definition 1.1 A Galton-Watson process (GW-process) is a Markov chain $\{Z_n : n = 0, 1, \dots\}$ on \mathbb{N} defined on

$$Z_n = \sum_{k=1}^{Z_{n-1}} \xi_{n,k},$$

where $\{\xi_{n,k} : n, k \geq 1\}$ are i.i.d random variables with probability law $\{p_k\}_{k=0}^{\infty}$ ($p_k \geq 0$ and $\sum_{k=0}^{\infty} p_k = 1$).

Let $f(s)$ be the generating function of ξ defined by

$$f(s) = \sum_{k=0}^{\infty} p_k s^k, \quad s \in [0, 1].$$

Denote by $\{f_n(s)\}_{n \geq 1}$ the iterates of $f(s)$:

$$f_0(s) = s, \quad f_1 = f(s), \quad f_n(s) = f(f_{n-1}(s))$$

Theorem 1.2 Let $m := \mathbb{E}[\xi]$ and q be the smallest root of $f(t) = t$, then $q = 1$ if $m \leq 1$ and $q < 1$ if $m > 1$.

Excise 1 For any $t \in [0, 1]$, we have $f_n(s) \rightarrow q$ as $n \rightarrow \infty$.

Theorem 1.3 As $n \rightarrow \infty$, we have

$$\mathbb{P}\{Z_n = 0\} \rightarrow q \begin{cases} = 1, & \text{if } m \leq 1; \\ < 1, & \text{if } m > 1. \end{cases}$$

We say GW-process Z_n is *supercritical*, *critical* or *subcritical* if $m > 1$, $m = 1$ or $m < 1$.

Theorem 1.4 Suppose $m \in (0, \infty)$. Let $\mathcal{F}_n = \sigma(\{Z_0, \dots, Z_n\})$ and $W_n = Z_n/m^n$. Then $\{W_n : n = 0, 1, \dots\}$ is a (\mathcal{F}_n) -martingale and there exists a nonnegative random variable W such that

$$W_n \xrightarrow{\text{a.s.}} W, \quad \text{as } n \rightarrow \infty.$$

Theorem 1.5 *For the supercritical case $m > 1$, we have*

$$\mathbb{P}\{W = 0\} = q.$$

Moreover, if $\mathbb{E}[|\xi|^2] < \infty$, then

$$\mathbb{E}[|W_n - W|] \rightarrow 0 \quad \text{and} \quad \mathbb{E}[W] = 1.$$

Let τ_0 be the extinction time of GW-process Z_n defined by:

$$\tau_0^- := \inf\{n \geq 0 : Z_n = 0\}.$$

Let Z_n^* be the number of individuals in generation n with an infinite line descent. For the k -th particle in generation n , denote by $\xi_{n+1,k}^*$ the number of its offsprings with an infinite line of descent.

Theorem 1.6 *Assume $Z_0 = 1$,*

(1) *$\{\xi_{n,k}^* : n \geq 1, k \geq 1\}$ are i.i.d. random variables and $\xi^* \leq \xi$. Moreover,*

$$\mathbb{E}[s^{\xi - \xi^*} t^{\xi^*}] = f(qs + (1 - q)t)$$

and

$$\mathbb{E}[s^{Z_n - Z_n^*} t^{Z_n^*}] = f_n(qs + (1 - q)t);$$

(2) *Conditioned on $\{\tau_0 = \infty\}$, Z_n^* is also a GW-process with generating function*

$$f^*(s) = \frac{f(q + (1 - q)s)}{1 - q};$$

(3) *Conditioned on $\{\tau_0 < \infty\}$, Z_n^* is also a GW-process with generating function*

$$f^\circ(s) = \frac{f(qs)}{q};$$