When to Cross the Spread?
- Trading in Two-Sided Limit Order Books -

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Abstract: In this article the problem of optimal trading in illiquid markets is addressed when the deviations from a given stochastic target function describing, for instance, external aggregate client flow are penalised. Using techniques of singular stochastic control, we extend the results of [NW11] to a two-sided limit order market with temporary market impact and resilience, where the bid ask spread is now also controlled. In addition to using market orders, the trader can also submit orders to a dark pool. We first show existence and uniqueness of an optimal control. In a second step, a suitable version of the stochastic maximum principle is derived which yields a characterisation of the optimal trading strategy in terms of a nonstandard coupled FBSDE. We show that the optimal control can be characterised via buy, sell and no-trade regions. The new feature is that we now get a nondegenerate no-trade region, which implies that market orders are only used when the spread is small. This allows to describe precisely when it is optimal to cross the bid ask spread, which is a fundamental problem of algorithmic trading. We also show that the controlled system can be described in terms of a reflected BSDE. As an application, we solve the portfolio liquidation problem with passive orders.

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1. Introduction

In modern financial institutions, due to external regulations, risk management requirements or client preferences, there are often imposed trading targets that need to be followed. These can take the form of a curve giving the desired stock holdings over a given period of time as a function of stochastic market factors. Typical examples include portfolio liquidation with random external client flow and Δ-hedging under market impact. In an idealised setting the trader would simply stay on the target (such as the aggregate exogenous client flow or the Black-Scholes Δ). Preventing this are often associated transaction costs and limited availability of liquidity, thus the trader needs to balance the two conflicting objectives of staying close to the target and minimizing trading costs.

In [NW11] the problem of curve following in an order book model with instantaneous price impact and absolutely continuous market orders is solved. In particular it is assumed that trades have no lasting impact on future prices. However in limit order markets the best bid and best ask prices typically recover only slowly after large discrete trades. In the present work we extend their results to a two-sided limit order market model with temporary market impact and resilience, where the price impact of trading decays only gradually. Trading strategies now include infinitesimally small ("continuous") as well as block ("discrete") trades, so that we are in the framework of singular stochastic control. The singular nature of the market order complicates the analysis. The optimal market order cannot be characterised as the pointwise maximiser of the Hamiltonian as in the absolutely continuous case. Moreover, we now face an optimisation problem with constraints, since passive buy and sell orders are modelled separately and both are nonnegative.

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Methods of singular control have been applied in different fields including the monotone follower problem as in [BSW80], the consumption-investment problem with proportional transaction costs in [DN90] and finite fuel problems as in [KOWZ00]. Most of them rely on the dynamic programming approach or a martingale optimality principle. We shall prove a version of the stochastic maximum principle. In contrast to dynamic programming, this does not require regularity of the value function and provides information on the optimal control directly. Maximum principles for singular stochastic control problems can be found in [CH94], [ØS01] and [BM05], among others. These results cannot directly be applied to the optimisation problem under consideration, since it involves jumps and state dependent singular cost terms. The recent paper [ØS10] provides necessary and sufficient maximum principles for jump diffusions with partial information. Despite being fairly general, their setup does not cover the particular model we consider; instead we give a direct proof based on [CH94] and the ideas developped in [NW11].

We consider an investor who wants to minimise the deviation of his stock holdings from a prespecified target function, which is driven by a vector of uncontrolled stochastic signals. The applications we have in mind are index tracking, portfolio liquidation, hedging and inventory management in the presence of external client flow. Our problem can be seen as an extension of the monotone follower problem to an order book framework with market and passive orders. In our model, the investor can use market orders and simultaneously place passive orders. We think of the passive orders as orders submitted to a dark pool. They generally incur lower trading costs that market orders but their execution is uncertain. The trader thus faces a tradeoff between the penalty for deviating from the target function, the costs of market order submission and the use of passive orders with uncertain execution times. The investor's market orders widen the spread temporarily; the gap then attracts new limit orders from other market participants and the spread recovers. The key decision the trader has to take is the following: If the spread is small, trading is cheap and a market order might be beneficial. For large spreads however it might be better to stop trading and wait until the spread recovers. When to cross the spread is a fundamental question of algorithmic trading in limit order markets. An equivalent question would be when to convert a limit into a market order. To the best of our knowledge, the problem of when to cross the bid ask spread has not been addressed in the mathematical finance literature on limit order markets. [OW05] and [PSS10], for instance, consider portfolio liquidation for a one-sided order book with initial spread zero and without passive orders; in this case it is optimal *never* to stop trading.

Our order book model is inspired by [OW05], a model which has recently been generalised to arbitrary shape functions by [AFS10] as well as [PSS10] and stochastic order book height in [Fru11]. While the mentioned articles focus on portfolio liquidation, we consider here the more general problem of curve following and therefore need a two-sided order book model. In addition, we allow for passive orders. These are orders with random execution which do not induce liquidity costs, such as limit orders or orders placed in a dark venue.

Our first mathematical result is an a priori estimate on the control. For the proof, we reduce the curve following problem to an optimisation problem with quadratic penalty and without target function and then use a scaling argument. This result provides the existence and uniqueness of an optimal control via a Komlós argument. Next we prove a suitable version of the stochastic maximum principle and characterise the optimal trading strategy in terms of a coupled forward backward stochastic differential equation (FBSDE).[1] The proof builds on results from [CH94] and extends them to the present case where we have jumps, state-dependent singular cost terms and general dynamics for the stochastic signal. Next we give a second characterisation of optimality in terms of buy, sell and no-trade regions. It turns out that there is always a region where the costs of trading are larger than the penalty for deviating, so that it is optimal to stop trading when the controlled system is inside this region. This is in contrast to [NW11], where only absolutely continuous trading strategies are allowed and a smoothness condition on the cost function is imposed. It was shown therein that under these conditions the no-trade region

[1] Unlike the classical HJB approach, our proof of existence does not need smoothness of the value function. Moreover, Pontryagins maximum principle provides a more explicit characterisation of the optimal trading strategies than the more traditional dynamic programming approach.

is degenerate, so that the investor always trades. In the present model the no-trade region is defined in terms of a threshold for the bid ask spread. We show that spread crossing is optimal if the spread is smaller than or equal to the threshold. If it is larger, then no market orders should be used. The threshold is given semi-explicitly in terms of the FBSDE and as a result, we can precisely characterise when spread crossing is optimal for a large class of optimisation problems. We will see that market orders are applied such that the controlled system remains inside (the closure of) the no-trade region at all times, and that its trajectory is reflected at the boundary. To make this precise, we show that the adjoint process together with the optimal control provides the solution to a reflected BSDE.

In general it is difficult to solve the coupled forward backward SDE (or the corresponding Hamilton-Jacobi-Bellman quasi variational inequality) explicitly. This is due to the Poisson jumps (leading to nonlocal terms) and the singular nature of the control. For quadratic penalty function and zero target function though the solution can be given in closed form. This corresponds to the portfolio liquidation problem in limit order markets and extends the result of [OW05] to trading strategies with passive orders. The new feature is that the optimal strategy is not deterministic, but adapted to passive order execution, and the trading rate is not constant but increasing in time.

The remainder of this paper is organised as follows: We describe the market environment and the control problem in Section 2 and show in Section 3 that a unique optimal control exists. We then provide two characterisations of optimality, first via the stochastic maximum principle in Section 4 and then via buy, sell and no-trade regions in Section 5. The link to reflected BSDEs is presented in Section 6, and we discuss the application to portfolio liquidation in Section 7.

2. The Model

Let $(\Omega, \mathcal{F}, \{\mathcal{F}(s) : s \in [0,T]\}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions of right continuity and completeness and $T > 0$ be the terminal time.

Assumption 2.1. *The filtration is generated by the following jointly independent processes,*

(i) A d-dimensional Brownian Motion W, $d \geq 1$.
(ii) Two one-dimensional Poisson processes N_i with respective intensities λ_i for $i = 1, 2$.
(iii) A compound Poisson process M on $[0,T] \times \mathbb{R}^k$ with compensator $m(d\theta)ds$, s.t. $m(\mathbb{R}^k) < \infty$.

The compensated (compound) Poisson processes are denoted $\tilde{N}_i$ for $i = 1, 2$ and $\tilde{M}$, respctively.

In our model trading takes place in a two sided limit order market. There are three price processes: a benchmark price process $(D(s))_{t \leq s \leq T}$ which we assume to be a martingale, the *best ask price process* which is above the benchmark, the *best bid price process* which is always below the benchmark. On the buy side of the order book liquidity is available for prices higher than the best ask price, and we assume a block shaped distribution of available liquidity with constant height $\frac{1}{\kappa_1} > 0$. This assumption is also made in [OW05]; it is key for the current approach as it leads to linear dynamics for the bid ask spread. Similarly, liquidity is available on the sell side for prices lower than the best bid. We assume a block shaped distribution of liquidity available on the sell side with constant height $\frac{1}{\kappa_2} > 0$. The investor's trades will have a temporary impact[2] on the best bid and ask prices. The benchmark price is hypothetical and cannot be observed directly in the market. It represents the "fair" price of the underlying or a reference price in the absence of liquidity costs such as the mid-quote. We assume that the benchmark price is uncontrolled. A stylised snapshot of the order book and a typical trajectory of the price processes are plotted in Figure 1.

[2] A fundamental property of illiquid markets is that trades move prices. There is a large body of empirical literature on the price impact of trading, we refer the reader to [KS72], [HLM87], [HLM90], [BHS95] and [ATHL05].

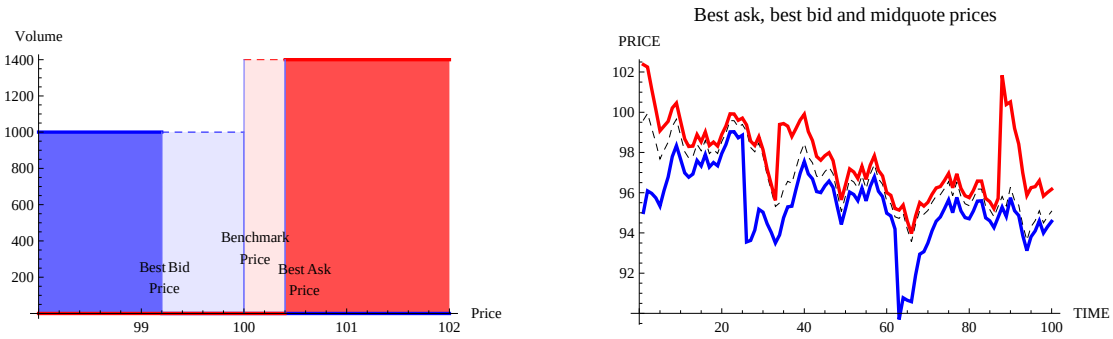

FIG 1. *(a) This stylised snapshot of the order book shows the best bid, benchmark and best ask price as well as liquidity that is available (dark) and consumed (light). (b) Here we see a typical evolution of the price processes over time. The best ask (red) is above the benchmark price (dashed black), which is above the best bid price (blue). Market buy (resp. sell) orders lead to jumps in the best ask (resp. bid) price. In the absence of trading, the best ask and best bid converge to the benchmark.*

2.1. Trading strategies

The investor can apply market buy (sell) orders to consume liquidity on the buy (sell) side of the order book. His cumulated market buy (sell) orders are denoted by η_1 (resp. η_2). These are nondecreasing càdlàg processes, and hence we allow for continuous as well as discrete trades and denote by

$$\Delta\eta_i(s) \triangleq \eta_i(s) - \eta_i(s-) \geq 0$$

for $s \in [0,T]$ and $i = 1,2$ the jumps of η_i. Such control processes are more general than absolutely continuous trading strategies and they seem better suited to describe real world trading strategies, which are purely discrete. In addition, the investor can use passive buy (sell) order volumes u_1 (resp. u_2). We assume that passive orders are fully executed at the benchmark price at random points in time. Thus a passive order always achieves a better price than the corresponding market order, however its execution is uncertain. We think of them as orders placed in a dark pool.

The class of admissible controls is now defined for $t \in [0,T]$ as

$$\begin{aligned}\mathcal{U}_t \triangleq \Bigg\{ (\eta,u) : [t,T]\times\Omega \to \mathbb{R}^2_+ \times \mathbb{R}^2_+ \;\Bigg|\; & \eta_i(t-) = 0,\ \mathbb{E}\left[\eta_i(T)^2 + \int_t^T u_i(r)^2 dr\right] < \infty, \\ & \eta_i \text{ is nondecreasing, càdlàg and progressively measurable and} \\ & u_i \text{ is predictably measurable, for } i = 1,2 \Bigg\}.\end{aligned}$$

Each control consists of the four components η_1, η_2, u_1, u_2, each of them being nonnegative. In particular, we face an optimisation problem with constraints. We note that $\eta_1(s)$ (resp. $\eta_2(s)$) denotes the market buy (resp. sell) orders *accumulated* in $[t,s]$. In contrast, $u_1(s)$ (resp. $u_2(s)$) represents the *volume* placed as a passive buy (resp. sell) order at time $s \in [t,T]$.

2.2. Trading costs

In our model there will be three sources of trading costs associated with the use market orders, passive order placements, and the deviation of the investor's stock holding from some pre-described target function as specified below.

2.2.1. Market orders

Instead of modelling the best bid and best ask price directly, we find it more convenient to work with the buy and sell spreads instead. Specifically, we denote by X_1 the distance of the best ask price to the benchmark price and call this process the buy spread. As in [OW05] and [AFS10] we assume exponential recovery of the buy spread with resilience parameter $\rho_1 > 0$. The dynamics of the *buy spread* are then given for $s \in [t, T]$ by

$$X_1(s) - X_1(t-) = -\int_t^s \rho_1 X_1(r)dr + \int_{[t,s]} \kappa_1 d\eta_1(r), \quad X_1(t-) = x_1 \geq 0.$$

As a convention, we write $\int_{[t,s]}$ for integrals with respect to the singular processes η_i for $i = 1, 2$ to indicate that possible jumps at times s and t are included. Similarly, the *sell spread* X_2 is defined as the distance of the best bid price to the benchmark price and it satisfies

$$X_2(s) - X_2(t-) = -\int_t^s \rho_2 X_2(r)dr + \int_{[t,s]} \kappa_2 d\eta_2(r), \quad X_2(t-) = x_2 \geq 0.$$

An immediate consequence is that the spreads X_1 and X_2 are nonnegative and revert to their lower bound 0. As a consequence, the best ask price is larger than or equal to the best bid price. In our model, the investor's market buy orders have a temporary impact on the best ask price, but not on the best bid (and vice versa). Passive orders do not move prices. Moreover, the price impact of trading decays over time (resilience) and in the absence of trading the price processes converge to the benchmark price.

Remark 2.2.
- In the literature the bid ask spread is defined as the distance of the best ask from the best bid price; in our notation this process is given by $X_1 + X_2$.
- In the seminal paper [Kyl85] three measures of liquidity are defined, all of which are captured in the model we propose. *Depth*, "the size of an order flow innovation required to change prices a given amount", is given by the parameters κ_1 and κ_2 which denote the inverse order book height. *Resiliency*, "the speed with which prices recover from a random, uninformative shock", is captured by the resilience parameters ρ_1 and ρ_2. Finally, *tightness*, "the cost of turning around a position over a short period of time", can be measured in terms of the bid ask spread $X_1 + X_2$.

The costs of market order execution is measured relative to the benchmark price, so the specific nature of the benchmark price process $(D(s))_{s\in[0,T]}$ is not important at this point. An infinitesimal market buy order $d\eta_1(r)$ is executed at the best ask price; the costs of crossing the spread are $X_1(r-)d\eta_1(r)$. A discrete buy order $\Delta\eta_1(r)$ "eats" into the block shaped order book and shifts the spread from $X_1(r-)$ to $X_1(r-) + \kappa_1\Delta\eta_1(r)$. Its liquidity costs are therefore given by

$$\Big(X_1(r-) + \frac{\kappa_1}{2}\Delta\eta_1(r)\Big)\,\Delta\eta_1(r).$$

Altogether, we arrive at the following cost functional for market order execution.

Definition 2.3. The expected trading costs from using market orders is

$$\mathbb{E}_{t,x}\Big[\int_{[t,T]} \Big[X_1(r-) + \frac{\kappa_1}{2}\Delta\eta_1(r)\Big]\, d\eta_1(r) + \int_{[t,T]} \Big[X_2(r-) + \frac{\kappa_2}{2}\Delta\eta_2(r)\Big]\, d\eta_2(r), \tag{2.1}$$

where the jump part is understood as, for $i = 1, 2$

$$\begin{aligned}\mathbb{E}\Big[\int_{[t,T]} \Delta\eta_i(r)d\eta_i(r)\Big] &= \mathbb{E}\Big[\sum_{r\in[t,T]} (\Delta\eta_i(r))^2\Big] \\ &\leq \mathbb{E}\Big[\Big(\sum_{r\in[t,T]} \Delta\eta_i(r)\Big)^2\Big] \leq \mathbb{E}\Big[\eta_i(T)^2\Big] < \infty.\end{aligned}$$

2.2.2. Passive orders

A jump of the Poisson process N_i represents a liquidity event which executes the passive order u_i, for $i = 1, 2$. For simplicity we consider full execution only, i.e. only "fill-or-kill" orders are admissible. This assumption is also made in [Kra11] and in [NW11].[3] As a result, the investor's stock holdings at time $s \in [s, T]$ are given by

$$\begin{aligned} X_3(s) - X_3(t-) &= \int_{[t,s]} d\eta_1(r) - \int_{[t,s]} d\eta_2(r) + \int_t^s u_1(r) N_1(dr) - \int_t^s u_2(r) N_2(dr), \\ X_3(t-) &= x_3 \in \mathbb{R}. \end{aligned}$$

Passive orders are executed at the benchmark price and hence do not incur direct trading costs. However, we allow for *adverse selection costs*. Such costs are commonly observed in practice. They occur if the benchmark price moves in a favorable direction after the passive order is executed, making the execution look less beneficial in hindsight.[4] We postulate the following adverse selection cost structure.

Definition 2.4. Let $\gamma_i \geq 0$ for $i = 1, 2$. The expected adverse selection costs from trading passive buy/sell orders are

$$\mathbb{E}\bigg[\int_0^T \gamma_i u_i(r) dr\bigg] \tag{2.2}$$

A linear penalization of passive orders of the above form has previously been used by [Kra11]. Therein the performance functional in continuous time is not derived from first principles, but taken as the continuous-time analogue of the discrete-time case. The following example illustrates how such a linear cost term can indeed be derived from first principles.

Example 2.5. *Let us assume that every time the passive buy (resp. sell) order is executed, the benchmark prices jumps down (resp. up). The upward and downward jumps can modelled by compound Poisson processes M_i for $i = 1, 2$ whose jump times agree*[5] *with the jump times of the Poisson processes N_i. Denote the compensated Poisson martingales by*

$$\tilde{M}_i([0, s] \times A) \triangleq M_i([0, s] \times A) - \lambda_i m_i(A) s.$$

Let the benchmark price with *adverse selection costs be given by*

$$\bar{D}(s) \triangleq D(s) - \tilde{M}_1(s) + \tilde{M}_2(s).$$

The process $\bar{D}$ is then also a martingale which jumps down (up) if our passive buy (sell) order is executed. The key observation is that the passive buy order is executed before *the jump of the benchmark price, and thus the passive buy orders incur the following costs:*

$$\mathbb{E}\bigg[\sum_{j \geq 1} u_i(\tau_{i,j}) \bar{D}(\tau_{i,j}-)\bigg],$$

where $\tau_{i,j}$ denotes the j-th jump of M_i for $i = 1, 2$ and $j \in \mathbb{N}$. We note that $\bar{D}(\tau_{i,j}) = \bar{D}(\tau_{i,j}-) - \Delta M_i(\tau_{i,j})$, so the above equals

$$\mathbb{E}\bigg[\sum_{j \geq 1; \tau_{i,j} \leq T} u_i(\tau_{i,j}) \big[\bar{D}(\tau_{i,j}) + \Delta M_i(\tau_{i,j})\big]\bigg] = \mathbb{E}\bigg[\int_0^T u_i(r) \big[\bar{D}(r) + \theta\big] M_i(d\theta, dr)\bigg].$$

[3] The assumption of full execution is key to our analysis. While an extension to proportional execution is straightforward, general partial executions are not covered by our method.

[4] See for instance [Kra11] for a detailed analysis in the framework of portfolio liquidation.

[5] In other words, M_i is constructed from N_i by replacing the jumps of size one by a stochastic jump size $\theta > 0$ whose distribution is given by $m_i(\theta)$, for $i = 1, 2$.

It follows that adverse selection leads to an additional loss (relative to the benchmark price $\bar{D}$) of size

$$\mathbb{E}\bigg[\int_0^T u_i(r)\theta M_i(d\theta, dr)\bigg].$$

By [NW11] Lemma A.3, the process $\int_t^{\cdot} u_i(r)\theta\tilde{M}_i(d\theta, dr)$ is a martingale, so the expected losses are of the form (2.2) with

$$\gamma_i \triangleq \lambda_i \int_0^\infty \theta m_i(d\theta).$$

2.2.3. Deviations from the target function

The last and final cost term captures costly deviations from some pre-specified target function. Specifically, we assume that the trader wants to minimise the deviation of his stock holdings to a target function $\alpha : [t,T]\times\mathbb{R}^n \to \mathbb{R}$. This function depends on a vector of uncontrolled stochastic signals Z with dynamics given for $s \in [t,T]$ by

$$\begin{aligned} Z(s) - Z(t-) = &\int_t^s \mu(r, Z(r))dr + \int_t^s \sigma(r, Z(r-))dW(r) \\ &+ \int_t^s \int_{\mathbb{R}^k} \gamma(r, Z(r-), \theta)\tilde{M}(dr, d\theta), \quad Z(t-) = z \in \mathbb{R}^n. \end{aligned}$$

where $\tilde{M}([0,s]\times A) \triangleq M([0,s]\times A) - m(A)s$ denotes the compensated Poisson martingale.

Definition 2.6. The costs of deviating from the target functions are specified in terms of penalty functions $h, f : \mathbb{R} \to \mathbb{R}$. Specifically, the expected cost from curve-following is

$$\mathbb{E}_{t,x,z}\bigg[\int_t^T h\left(X_3(r) - \alpha(r, Z(r))\right) dr + f\left(X_3(T) - \alpha(T, Z(T))\right)\bigg] \tag{2.3}$$

In a model where Z represents exogenous client flow, the trader's net position at time $s \in [t,T]$ is $X_3(s) - Z(s)$. In this case one might choose $\alpha(r,z) = z$ and penalize her aggregate holdings by choosing $h(x) = f(x) = x^2$. Another possible interpretation is that Z is a vector whose entries represent both client flow and the benchmark price and $f\left(X_3(T) - \alpha(T, Z(T))\right)$ represents either the cost of unwinding the terminal position at the market price at time T or the deviation of the actual stock holding from that of a perfect hedge.[6]

2.3. The control problem

Having defined the state processes and their respective dynamics, as well as the various costs of trading, we are now ready to formulate our control problem. The *performance functional* is defined for $(t,x,z) \in [0,T]\times\mathbb{R}^3\times\mathbb{R}^n$ and a control $(\eta, u) \in \mathcal{U}_t$ as

$$\begin{aligned} &J(t,x,z,\eta,u) \\ \triangleq&\mathbb{E}_{t,x,z}\bigg[\int_{[t,T]}\Big[X_1(r-) + \frac{\kappa_1}{2}\Delta\eta_1(r)\Big]d\eta_1(r) + \int_{[t,T]}\Big[X_2(r-) + \frac{\kappa_2}{2}\Delta\eta_2(r)\Big]d\eta_2(r) \\ &\quad + \int_t^T \gamma_1 u_1(r)dr + \int_t^T \gamma_2 u_2(r)dr \\ &\quad + \int_t^T h\left(X_3(r) - \alpha(r, Z(r))\right)dr + f\left(X_3(T) - \alpha(T, Z(T))\right)\bigg] \end{aligned} \tag{2.4}$$

and the optimisation problem under consideration is:

[6]The problem of optimal hedging under market impact and resilience has been studied in a mean-variance setting in [AG12] and in a game-theoretic framework in [HN11].

Problem 2.7.

$$\textit{Minimise} \quad J(\eta, u) \quad \textit{over} \quad (\eta, u) \in \mathcal{U}_t.$$

For $(t, x, z) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^n$ the *value function* is defined as

$$v(t, x, z) = \inf_{(\eta,u)\in\mathcal{U}_t} J(t, x, z, \eta, u).$$

Remark 2.8. Problem 2.7 is a singular stochastic control problem. Maximum principles for singular control are derived for instance in [CH94], [ØS01] and [BM05]. However, the above problem is not covered by their results for several reasons. Firstly, it involves jumps. Secondly, the singular cost terms $\int_{[t,T]} \left[X_i(r-) + \frac{\kappa_i}{2}\Delta\eta_i(r)\right] d\eta_i(r)$ for $i = 1, 2$ depend on the state variable and on the jumps of the control, which is not the case in the "usual" formulation. The standard setup only allows for cost terms of the form $\int_t^T k(s, \omega)d\eta(s)$. A third difficulty in the present model is that the control u (the passive order) does not incur trading costs, so the "standard" characterisation as the pointwise maximiser of the Hamiltonian does not apply. The recent article [ØS10] provides necessary and sufficient maximum principles for the singular control of jump diffusions, where the singular cost term may depend on the state variable. However, they do not allow for terms like $\int_{[t,T]} \Delta\eta_i(r)d\eta_i(r)$ and their sufficient condition is based on a convexity condition on the Hamiltonian which is not satisfied in our specific case. Instead we give a direct proof based on [CH94] and ideas used in [NW11].

To ensure existence and uniqueness of an optimal control, we impose the following assumptions. Here and throughout, we write c for a generic constant, which might be different at each occurrence.

Assumption 2.9. *(i) The penalty functions $f, h : \mathbb{R} \to \mathbb{R}$ are strictly convex, continuously differentiable, nonnegative and normalised in the sense $f(0) = h(0) = 0$.*

(ii) In addition, f and h have at least quadratic growth, i.e. there exists $\varepsilon > 0$ such that $|f(x)|, |h(x)| \geq \varepsilon|x|^2$ for all $x \in \mathbb{R}$.

(iii) The functions μ, σ and γ are Lipschitz continuous, i.e. there exists a constant c such that for all $z, z' \in \mathbb{R}^n$ and $s \in [t, T]$,

$$\|\mu(s, z) - \mu(s, z')\|^2_{\mathbb{R}^n} + \|\sigma(s, z) - \sigma(s, z')\|^2_{\mathbb{R}^{n\times d}} + \int_{\mathbb{R}^k} \|\gamma(s, z, \theta) - \gamma(s, z', \theta)\|^2_{\mathbb{R}^n} m(d\theta) \leq c\|z - z'\|^2_{\mathbb{R}^n}.$$

In addition, they satisfy

$$\sup_{t\leq s\leq T} \left[\|\mu(s, 0)\|^2_{\mathbb{R}^n} + \|\sigma(s, 0)\|^2_{\mathbb{R}^{n\times d}} + \int_{\mathbb{R}^k} \|\gamma(s, 0, \theta)\|^2_{\mathbb{R}^n} m(d\theta)\right] < \infty.$$

(iv) The target function α has at most polynomial growth in the variable z uniformly in s, i.e. there exist constants $c_\alpha, q > 0$ such that for all $z \in \mathbb{R}^n$,

$$\sup_{t\leq s\leq T} |\alpha(s, z)| \leq c_\alpha(1 + \|z\|^q_{\mathbb{R}^n}).$$

(v) The penalty functions f and h have at most polynomial growth.

Remark 2.10. Let us briefly comment on these assumptions. Taking f and h nonnegative is reasonable for penalty functions. Normalisation is no loss of generality, this may always be achieved by a linear shift of f, h and α. Quadratic growth of f and h is only needed in Lemma 3.4 for an a priori L^2-norm bound on the control, which is then used for a Komlós argument. The convexity condition leads naturally to a convex coercive problem which then admits a unique solution.

Once the existence of an optimal control is established, we need one further assumption. It guarantees the existence and uniqueness of the adjoint process.

Assumption 2.11. *The derivatives f' and h' have at most linear growth, i.e. for all $x \in \mathbb{R}$ we have $|f'(x)| + |h'(x)| \leq c(1 + |x|)$.*

3. Existence of a Solution

The aim of the present section is to show that the performance functional is strictly convex and that it is enough to consider controls with a uniform L^2-norm bound. Combining these results with a Komlós argument, we then prove that there is a unique optimal control.

Henceforth we impose Assumption 2.9. For the proof of existence we also assume that there are no adverse selection costs, $\gamma_i = 0$ for $i = 1, 2$. Due to the linear penalization of passive orders the extension to the general case with adverse selection is straightforward. We begin with some growth estimates for the state processes. This result extends [NW11] Lemma 4.1 to the singular control case.

Lemma 3.1. *(i) For every $p \geq 2$ there exists a constant c_p such that for every $(t, x, z) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^n$ we have*

$$\mathbb{E}_{t,x,z}\left[\sup_{t\leq s\leq T} \|Z(s)\|^p_{\mathbb{R}^n}\right] \leq c_p \left(1 + \|z\|^p_{\mathbb{R}^n}\right).$$

(ii) There exists a constant c_x such that for any $(\eta, u) \in \mathcal{U}_t$ we have

$$\mathbb{E}_{t,x,z}\left[\sup_{t\leq s\leq T} \|X^{\eta,u}(s)\|^2_{\mathbb{R}^3}\right] \leq c_x \left(1 + \mathbb{E}_{t,x,z}\left[\|\eta(T)\|^2_{\mathbb{R}^2}\right] + \mathbb{E}\left[\int_t^T \|u(r)\|^2_{\mathbb{R}^2}\, dr\right]\right).$$

In particular, $X^{\eta,u}$ has square integrable supremum for all $(\eta, u) \in \mathcal{U}_t$.

Proof. The argument is as in [NW11] Lemma 4.1. □

A first consequence of the above lemma is that the zero control incurs finite costs.

Corollary 3.2. *The zero control incurs finite costs, i.e. for each $(t, x, z) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^n$ we have*

$$J(t, x, z, 0, 0) < \infty.$$

Proof. This result is a consequence of the polynomial growth of f, h and α together with Lemma 3.1. □

We now show that the performance functional is strictly convex in the control, so that methods of convex analysis can be applied.

Proposition 3.3. *The performance functional $(\eta, u) \mapsto J(t, x, z, \eta, u)$ is strictly convex, for every $(t, x, z) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^n$.*

Proof. From the definition of X_i for $i = 1, 2$ we have $d\eta_i(s) = \frac{dX_i(s)+\rho_i X_i(s)ds}{\kappa_i}$. We use this to rewrite the performance functional as

$$\begin{aligned}
&J(t, x, z, \eta, u)\\
=&\mathbb{E}_{t,x,z}\Bigg[\frac{X_1(T)^2 - x_1^2}{2\kappa_1} + \frac{X_2(T)^2 - x_2^2}{2\kappa_2} + \int_t^T \frac{\rho_1}{\kappa_1} X_1(r)^2 dr + \int_t^T \frac{\rho_2}{\kappa_2} X_2(r)^2 dr\\
&+ \int_t^T h\left(X_3(r) - \alpha(r, Z(r))\right) dr + f\left(X_3(T) - \alpha(T, Z(T))\right)\Bigg]. \qquad (3.1)
\end{aligned}$$

The right hand side is strictly convex in X. Due to the fact that $(\eta, u) \mapsto X^{\eta,u}$ is affine, it follows that $(\eta, u) \mapsto J(t, x, z, \eta, u)$ is strictly convex. □

The aim in this section is to prove existence and uniqueness of an optimal control. For the proof of this result, we need two auxiliary lemmata. We first show a quadratic growth estimate on the value function in Lemma 3.4. This extends Lemma 4.2 from [NW11] to the singular control case.

Lemma 3.4. *There are constants $c_{1,t}, c_2, c_3 > 0$ such that for $(t, x, z) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^n$*

$$v(t, x, z) \geq c_{1,t} x_3^2 - c_2 \left(1 + \|z\|^{c_3}_{\mathbb{R}^n}\right).$$

Proof. The idea is to use the growth conditions on the penalty functions to reduce the optimisation problem to simpler linear-quadratic problem, which can then be estimated in terms of x_3^2. Using the quadratic growth of f and h yields

$$\begin{aligned}
&v(t,x,z)\\
&\geq \inf_{(\eta,u)\in\mathcal{U}_t} \mathbb{E}_{t,x,z}\Bigg[\int_{[t,T]}\Big[X_1(r-)+\frac{\kappa_1}{2}\Delta\eta_1(r)\Big]\,d\eta_1(r)+\int_{[t,T]}\Big[X_2(r-)+\frac{\kappa_2}{2}\Delta\eta_2(r)\Big]\,d\eta_2(r)\\
&\qquad\qquad+\int_t^T \varepsilon\left(X_3(r)-\alpha(r,Z(r))\right)^2 dr+\varepsilon\left(X_3(T)-\alpha(T,Z(T))\right)^2\Bigg].
\end{aligned}$$

Next an application of the inequality $(a-b)^2\geq\frac{1}{2}a^2-b^2$ leads to

$$\begin{aligned}
&v(t,x,z)\\
&\geq \inf_{(\eta,u)\in\mathcal{U}_t} \mathbb{E}_{t,x,z}\Bigg[\int_{[t,T]}\Big[X_1(r-)+\frac{\kappa_1}{2}\Delta\eta_1(r)\Big]\,d\eta_1(r)+\int_{[t,T]}\Big[X_2(r-)+\frac{\kappa_2}{2}\Delta\eta_2(r)\Big]\,d\eta_2(r)\\
&\quad+\int_t^T \frac{\varepsilon}{2}\left|X_3(r)\right|^2 dr+\frac{\varepsilon}{2}\left|X_3(T)\right|^2\Bigg]-\varepsilon\mathbb{E}_{t,x,z}\left[\int_t^T\left|\alpha(r,Z(r))\right|^2 dr+\left|\alpha(T,Z(T))\right|^2\right].
\end{aligned}$$

The polynomial growth of α coupled with Lemma 3.1 provides the existence of constants $c_2,c_3>0$ such that

$$\begin{aligned}
&v(t,x,z) \qquad (3.2)\\
&\geq \inf_{(\eta,u)\in\mathcal{U}_t} \mathbb{E}_{t,x,z}\Bigg[\int_{[t,T]}\Big[X_1(r-)+\frac{\kappa_1}{2}\Delta\eta_1(r)\Big]\,d\eta_1(r)+\int_{[t,T]}\Big[X_2(r-)+\frac{\kappa_2}{2}\Delta\eta_2(r)\Big]\,d\eta_2(r)\\
&\qquad\qquad+\int_t^T \frac{\varepsilon}{2}\left|X_3(r)\right|^2 dr+\frac{\varepsilon}{2}\left|X_3(T)\right|^2\Bigg]-c_2\left(1+\|z\|_{\mathbb{R}^n}^{c_3}\right).
\end{aligned}$$

This provides an estimate of the original value function in terms of an easier optimisation problem with a quadratic penalty function and zero target function. Economically, this may be interpreted as a portfolio liquidation problem. To continue the estimate, we define the following "value" function,

$$\begin{aligned}
&v_1(t,x)\\
&\triangleq \inf_{(\eta,u)\in\mathcal{U}_t} \mathbb{E}_{t,x}\Bigg[\int_{[t,T]}\Big[X_1(r-)+\frac{\kappa_1}{2}\Delta\eta_1(r)\Big]\,d\eta_1(r)+\int_{[t,T]}\Big[X_2(r-)+\frac{\kappa_2}{2}\Delta\eta_2(r)\Big]\,d\eta_2(r)\\
&\qquad\qquad+\int_t^T \frac{\varepsilon}{2}\left|X_3(r)\right|^2 dr+\frac{\varepsilon}{2}\left|X_3(T)\right|^2\Bigg].
\end{aligned}$$

Monotonicity properties of the value function v_1 in the state variable yield

$$v_1(t,x)\geq v_1\big(t,(0,0,x_3)^*\big)\triangleq v_2(t,x_3).$$

Let us denote by J_2 the performance functional associated to the value function v_2. Due to $x_i=0$ for $i=1,2$ the mappings $(\eta,u)\mapsto X_i^{\eta,u}$ are linear and the mapping $(x_3,\eta,u)\mapsto X_3^{\eta,u}-x_3$ is also linear. This can be used to show that J_2 scales quadratically, i.e. for $(t,x_3)\in[0,T]\times\mathbb{R}$ and a scaling factor $\beta>0$ we have

$$J_2(t,\beta x_3,\beta\eta,\beta u)=\beta^2 J_2(t,x_3,\eta,u).$$

As a result, v_2 scales quadratically as well:

$$v_2(t,\beta x_3)=\beta^2 v_2(t,x_3).$$

Moreover, if $x_3 = 0$ then $v_2(t,0) = 0$. Choosing now $\beta = |x_3|$ for $x_3 \neq 0$ we get

$$v_2(t,x_3) = \begin{cases} x_3^2 v_2(t,1), & x_3 > 0 \\ 0, & x_3 = 0 \\ x_3^2 v_2(t,-1), & x_3 < 0, \end{cases}$$

and defining $c_{1,t} \triangleq \min\{v_2(t,1), v_2(t,-1)\}$ leads to

$$v_2(t,x_3) \geq c_{1,t} x_3^2.$$

Plugging this result into (3.2) provides the following estimate

$$v(t,x,z) \geq c_{1,t} x_3^2 - c_2 \left(1 + \|z\|_{\mathbb{R}^n}^{c_3}\right).$$

To prove the assertion of the lemma, it remains to show that the constant $c_{1,t}$ is strictly positive and finite for each $t \in [0,T]$. The proof of this result can be found in [Nau11] Lemma A.2.1. □

We are now ready to prove an a priori estimate on the control, which will be needed in the Komlós argument below. This result extends [NW11] Lemma 4.3 to the singular control case.

Lemma 3.5. *For each $(t,x,z) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^n$ there is a constant $K(t,x,z)$ such that any control with*

$$\mathbb{E}_{t,x,z}\left[\|\eta(T)\|_{\mathbb{R}^2}^2 + \int_t^T \|u(r)\|_{\mathbb{R}^2}^2 \, dr\right] > K(t,x,z)$$

cannot be optimal.

Proof. We first consider the market order η. The dynamics of X_i for $i = 1,2$ imply that for $s \in [t,T]$ we have

$$X_i(s) = e^{-\rho_i(s-t)} x_i + \kappa_i \int_{[t,s]} e^{-\rho_i(r-t)} d\eta_i(r), \tag{3.3}$$

and thus $X_i(T) \geq \kappa_i e^{-\rho_i T} \eta_i(T)$. Combining this with (3.1) yields

$$J(\eta,u) \geq \mathbb{E}_{t,x,z}\left[\frac{X_i(T)^2}{2\kappa_i} - \frac{x_1^2}{2\kappa_1} - \frac{x_2^2}{2\kappa_2}\right] \geq K_1 \mathbb{E}_{t,x,z}[\eta_i(T)^2] - K_{2,x}. \tag{3.4}$$

for constants $K_1, K_{2,x} > 0$. It follows that if $\mathbb{E}_{t,x,z}[\eta_i(T)^2] > \frac{J(0,0)+K_{2,x}+1}{K_1}$ then η cannot be optimal. We have $J(0,0) < \infty$ due to Corollary 3.2.
The estimate in terms of the passive order u is slightly more involved. Let τ_i denote the first jump time of the Poisson process N_i after t for $i = 1,2$, an exponentially distributed random variable with parameter λ_i, and set $\tau \triangleq \tau_1 \wedge \tau_2 \wedge T$. At the jump time τ the state process jumps from $X(\tau-)$ to

$$X(\tau-) + \Delta_N X(\tau) \triangleq X(\tau-) + \begin{pmatrix} 0 \\ 0 \\ u_1(\tau)\mathbb{1}_{\{\tau_1 < \tau_2 \wedge T\}} - u_2(\tau)\mathbb{1}_{\{\tau_2 < \tau_1 \wedge T\}} \end{pmatrix}.$$

We use the definition of the cost functional and the fact that the cost terms are nonnegative to get

$$\begin{aligned} J(\eta,u) =& \mathbb{E}_{t,x,z}\Bigg[\int_{[t,\tau)} \left[X_1(r-) + \frac{\kappa_1}{2}\Delta\eta_1(r)\right] d\eta_1(r) + \int_{[t,\tau)} \left[X_2(r-) + \frac{\kappa_2}{2}\Delta\eta_2(r)\right] d\eta_2(r) \\ &+ \int_t^\tau h\left(X_3(r) - \alpha(r, Z(r))\right) dr + J\left(\tau, X(\tau-) + \Delta_N X(\tau), Z(\tau), \eta, u\right)\Bigg] \qquad (3.5) \\ \geq& \mathbb{E}_{t,x,z}\left[J(\tau, X(\tau-) + \Delta_N X(\tau), Z(\tau), \eta, u)\right] \end{aligned}$$

$$\geq \mathbb{E}_{t,x,z}\left[v(\tau, X(\tau-) + \Delta_N X(\tau), Z(\tau))\right],$$

where J in the above is evaluated at controls on the stochastic interval[7] $[\tau, T]$. Combining this with Lemma 3.4 we get

$$J(\eta, u) \geq \mathbb{E}_{t,x,z}\left[c_{1,t}|X_3(\tau-) + \Delta_N X_3(\tau)|^2 - c_2(1 + \|Z(\tau)\|_{\mathbb{R}^n}^{c_3})\right].$$

In view of Lemma 3.1 we have

$$\mathbb{E}_{t,x,z}\left[\|Z(\tau)\|_{\mathbb{R}^n}^{c_3}\right] \leq \mathbb{E}_{t,x,z}\left[\sup_{s\in[t,T]} \|Z(s)\|_{\mathbb{R}^n}^{c_3}\right] \leq c\left(1 + \|z\|_{\mathbb{R}^n}^{c_3}\right),$$

and thus there is a constant $c_{2,z} \geq 0$ such that

$$J(\eta, u) \geq -c_{2,z} + c_{1,t}\mathbb{E}\left[|X_3(\tau-) + \Delta_N X_3(\tau)|^2\right]. \tag{3.6}$$

By definition, the stock holdings directly after a jump of the Poisson process are given by

$$X_3(\tau-) + \Delta_N X_3(\tau) = x_3 + \eta_1(\tau-) - \eta_2(\tau-) + u_1(\tau)\mathbb{1}_{\{\tau_1<\tau_2\wedge T\}} - u_2(\tau)\mathbb{1}_{\{\tau_2<\tau_1\wedge T\}},$$

and an application of the inequality $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$ leads to

$$\begin{aligned}
&|X_3(\tau-) + \Delta_N X_3(\tau)|^2 \\
\geq&\frac{1}{2}\left(u_1(\tau_1)\mathbb{1}_{\{\tau_1<\tau_2\wedge T\}} - u_2(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\wedge T\}}\right)^2 - \left(x_3 + \eta_1(\tau-) - \eta_2(\tau-)\right)^2 \\
\geq&\frac{1}{2}\left(u_1(\tau_1)\mathbb{1}_{\{\tau_1<\tau_2\wedge T\}} - u_2(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\wedge T\}}\right)^2 - 3\left(|x_3|^2 + |\eta_1(\tau-)|^2 + |\eta_2(\tau-)|^2\right) \\
\geq&\frac{1}{2}\left(u_1(\tau_1)\mathbb{1}_{\{\tau_1<\tau_2\wedge T\}} - u_2(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\wedge T\}}\right)^2 - 3\left(|x_3|^2 + |\eta_1(T)|^2 + |\eta_2(T)|^2\right).
\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7) we get

$$\begin{aligned}
&\frac{1}{2}c_{1,t}\mathbb{E}_{t,x,z}\left[\left(u_1(\tau_1)\mathbb{1}_{\{\tau_1<\tau_2\wedge T\}} - u_2(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\wedge T\}}\right)^2\right] \\
\leq& J(\eta, u) + c_{2,z} + 3c_{1,t}\left(|x_3|^2 + \mathbb{E}_{t,x,z}\left[|\eta_1(T)|^2 + |\eta_2(T)|^2\right]\right).
\end{aligned}$$

Due to equation (3.4) we have for $i = 1, 2$

$$\mathbb{E}_{t,x,z}[|\eta_i(T)|^2] \leq \frac{K_{2,x}}{K_1} + \frac{1}{K_1}J(\eta, u),$$

so combining the last two displays and relabelling constants provides

$$\mathbb{E}_{t,x,z}\left[\left(u_1(\tau_1)\mathbb{1}_{\{\tau_1<\tau_2\wedge T\}} - u_2(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\wedge T\}}\right)^2\right] \leq c_{1,t,x,z} + c_{2,t}J(\eta, u). \tag{3.8}$$

We shall now compute the term on the left hand side of inequality (3.8). The jump times τ_1 and τ_2 are independent and exponentially distributed with parameter λ_1 and λ_2, respectively. We thus have

$$\begin{aligned}
&\mathbb{E}_{t,x,z}\left[\left(u_1(\tau_1)\mathbb{1}_{\{\tau_1<\tau_2\wedge T\}} - u_2(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\wedge T\}}\right)^2\right] \\
=&\int_t^\infty\int_t^\infty \lambda_1 e^{-\lambda_1(r_1-t)}\lambda_2 e^{-\lambda_2(r_2-t)}\left(u_1(r_1)\mathbb{1}_{\{r_1<r_2\wedge T\}} - u_2(r_2)\mathbb{1}_{\{r_2<r_1\wedge T\}}\right)^2 dr_1 dr_2 \\
\geq&\int_T^\infty\int_t^T \lambda_1 e^{-\lambda_1(r_1-t)}\lambda_2 e^{-\lambda_2(r_2-t)}|u_1(r_1)|^2 dr_1 dr_2,
\end{aligned}$$

[7] More precisely, we split the interval $[t, T]$ into the subintervals $[t, \tau]$ and $(\tau, T]$. By definition of the cost functional, the singular order on the second subinterval $(\tau, T]$ includes a possible jump at the left endpoint τ, so this jump must be excluded from the first subinterval $[t, \tau]$. For this reason, the state process directly after the Poisson jump in (3.5) is given by $X(\tau-) + \Delta_N X(\tau)$ and *not* by $X(\tau-) + \Delta_N X(\tau) + \Delta_\eta X(\tau) = X(\tau)$.

where we have used the nonnegativity of the integrand in the last line and restricted integration to $(r_1, r_2) \in [t,T] \times [T,\infty)$. We now compute

$$\begin{aligned}
&\mathbb{E}_{t,x,z}\left[\left(u_1(\tau_1)\mathbb{1}_{\{\tau_1<\tau_2\wedge T\}} - u_2(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\wedge T\}}\right)^2\right] \\
\geq &\int_T^\infty \lambda_2 e^{-\lambda_2(r_2-t)}dr_2 \int_t^T \lambda_1 e^{-\lambda_1(r_1-t)}\mathbb{E}_{t,x,z}\left[|u_1(r_1)|^2\right]dr_1 \\
=&e^{-\lambda_2(T-t)}\int_t^T \lambda_1 e^{-\lambda_1(r_1-t)}\mathbb{E}_{t,x,z}\left[|u_1(r_1)|^2\right]dr_1 \\
\geq&e^{-\lambda_2(T-t)}\lambda_1 e^{-\lambda_1(T-t)}\int_t^T \mathbb{E}_{t,x,z}\left[|u_1(r_1)|^2\right]dr_1.
\end{aligned}$$

Combining this with equation (3.8) and relabelling constants we get

$$\mathbb{E}_{t,x,z}\left[\int_t^T |u_1(r)|^2 dr\right] \leq c_{1,t,x,z} + c_{2,t}J(\eta,u).$$

In particular if

$$\mathbb{E}_{t,x,z}\left[\int_t^T |u_1(r)|^2 dr\right] \geq c_{1,t,x,z} + c_{2,t}J(0,0) + 1,$$

then we see that $J(\eta,u) > J(0,0)$ and the control (η,u) is clearly not optimal. A similar estimate holds for the passive sell order u_2. □

Theorem 3.6. *There is a unique optimal control $(\hat{\eta},\hat{u}) \in \mathcal{U}_t$ for Problem 2.7.*

Proof. Let $(\eta^n,u^n)_{n\in\mathbb{N}} \subset \mathcal{U}_t$ be a minimising sequence, i.e.

$$\lim_{n\to\infty} J(\eta^n,u^n) = \inf_{(\eta,u)\in\mathcal{U}_t} J(\eta,u).$$

Due to the uniform L^2-norm bound of Lemma 3.5 we can then apply the Komlós theorem for singular stochastic control given in [Kab99] Lemma 3.5. It states the existence of a subsequence (also indexed by n) that is Cesaro-convergent along with all its further subsequences. In particular, it provides adapted, non-decreasing and right-continuous processes $\hat{\eta} : [t,T]\times\Omega \to \mathbb{R}^2_+$ such that $\hat{\eta}(T,\cdot) \in L^1$ and the sequence of random measures

$$\bar{\eta}^n \triangleq \frac{1}{n}\sum_{i=1}^n \eta^i$$

converges weakly to $\hat{\eta}$ in the sense that for almost all $\omega\in\Omega$ the measures $\bar{\eta}^n(\omega)$ on $[t,T]$ converge weakly to $\hat{\eta}(\omega)$. Similarly, the same arguments as in the proof of Theorem 3.1 in [NW11] yield a predictable process $\hat{u} : [t,T]\times\Omega \to \mathbb{R}^2_+$ such that the sequence

$$\bar{u}^n \triangleq \frac{1}{n}\sum_{i=1}^n u^i$$

and all its subsequences converges weakly to $\hat{u}$. Having established the appropriate measurability properties, we deduce that $(\hat{\eta},\hat{u})$ is admissible.

We now prove optimality of the limiting process. Weak convergence of the process $(\bar{\eta}^n)_{n\in\mathbb{N}}$ coupled with equation (3.3) implies that for $s\in[t,T]$ such that $\Delta\hat{\eta}(s) = 0$ as well as for $s = T$ we have for $i = 1,2$ that

$$\lim_{n\to\infty} X_i^{\bar{\eta}^n,\bar{u}^n}(s) = \lim_{n\to\infty}\left[e^{-\rho_1(s-t)}x_1 + \kappa_i\int_{[t,s]} e^{-\rho_i(r-t)}d\bar{\eta}_i^n(r)\right]$$

$$=e^{-\rho_1(s-t)}x_i + \kappa_i \int_{[t,s]} e^{-\rho_i(r-t)} d\hat{\eta}_i(r) = X_i^{\hat{\eta},\hat{u}}(s) \quad \mathbb{P}\text{-a.s.}$$

In particular,

$$\lim_{n\to\infty} X_i^{\bar{\eta}^n,\bar{u}^n} = X_i^{\hat{\eta},\hat{u}} \quad \mathbb{P}\otimes\lambda\text{-a.s.}$$

From the definition of $X_i^{\bar{\eta}^n\bar{u}^n}$ for $i=1,2$ we have $d\bar{\eta}_i^n(s) = \frac{dX_i^{\bar{\eta}^n\bar{u}^n}(s)+\rho_i X_i^{\bar{\eta}^n\bar{u}^n}(s)ds}{\kappa_i}$. Using the integration-by-parts formula for Stieltjes integrals we obtain

$$\begin{aligned}
&\mathbb{E}_{t,x,z}\left[\int_{[t,T]}\left[X_i^{\bar{\eta}^n,\bar{u}^n}(r-) + \frac{\kappa_i}{2}\Delta\eta_i(r)\right] d\bar{\eta}_i^n(r)\right] \\
=&\mathbb{E}_{t,x,z}\left[\frac{X_i^{\bar{\eta}^n,\bar{u}^n}(T)^2 - x_i^2}{2\kappa_i} + \int_t^T \frac{\rho_1}{\kappa_i} X_i^{\bar{\eta}^n,\bar{u}^n}(r)^2 dr\right].
\end{aligned}$$

A similar equation holds in the limit with $X_i^{\bar{\eta}^n,\bar{u}^n}$ replaced by $X^{\hat{\eta},\hat{u}}$ and $\bar{\eta}^n$ replaced by $\hat{\eta}$. Hence, by Fatou's lemma

$$\begin{aligned}
&\mathbb{E}_{t,x,z}\left[\int_{[t,T]}\left[X_i^{\hat{\eta},\hat{u}}(r-) + \frac{\kappa_i}{2}\Delta\hat{\eta}_i(r)\right] d\hat{\eta}_i(r)\right] \\
\le&\liminf_{n\to\infty}\mathbb{E}_{t,x,z}\left[\int_{[t,T]}\left[X_i^{\bar{\eta}^n,\bar{u}^n}(r-) + \frac{\kappa_i}{2}\Delta\bar{\eta}_i^n(r)\right] d\bar{\eta}_i^n(r)\right], \quad i=1,2.
\end{aligned}$$

Let us now consider the costs of deviating from the target function. The $d\bar{\eta}_i^n$-integrals in the definition of the process $X_3^{\bar{\eta}^n,\bar{u}^n}$ again converge $\mathbb{P}\otimes\lambda$-a.s. The Poisson integral terms are as in [NW11]. In the proof of their Theorem 3.6 it is shown that

$$\mathbb{E}\left[\sup_{t\le s\le T}\int_t^s |\bar{u}_i^n(r) - \hat{u}(r)|\, dN_i(r)\right] = 0$$

for $i=1,2$. Passing to a subsequence (again denoted $\{(\bar{\eta}^n,\bar{u}^n)\}$) if necessary we may thus assume

$$\lim_{n\to\infty}\sup_{t\le s\le T}\int_t^s |\bar{u}_i^n(r) - \hat{u}(r)|\, dN_i(r) = 0 \quad \mathbb{P}\text{-a.s.}$$

to obtain $\mathbb{P}\otimes\lambda$-almost sure convergence of $X_3^{\bar{\eta}^n,\bar{u}^n}$ to $X_3^{\hat{\eta},\hat{u}}$. We can then again apply Fatou's lemma to obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T h\left(X_3^{\hat{\eta},\hat{u}}(r) - \alpha(r,Z(r))\right) dr + f\left(X_3^{\hat{\eta},\hat{u}}(T) - \alpha(T,Z(T))\right)\right] \\
\le&\liminf_{n\to\infty}\mathbb{E}\left[\int_t^T h\left(X_3^{\bar{\eta}^n,\bar{u}^n}(r) - \alpha(r,Z(r))\right) dr + f\left(X_3^{\bar{\eta}^n,\bar{u}^n}(T) - \alpha(T,Z(T))\right)\right].
\end{aligned}$$

Combining the preceding arguments with the convexity of J gives

$$J(\hat{\eta},\hat{u}) \le \liminf_{n\to\infty} J(\bar{\eta}^n,\bar{u}^n) \le \liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^n J(\eta^i,u^i) = \inf_{(\eta,u)\in\mathcal{U}_t} J(\eta,u).$$

Uniqueness of the optimal strategy is due to the strict convexity of $(\eta,u)\mapsto J(\eta,u)$. □

Throughout, we denote by $(\hat{\eta},\hat{u})$ the optimal control and by $\hat{X} = X^{\hat{\eta},\hat{u}}$ the optimal state trajectory.

4. The Stochastic Maximum Principle

In the preceding section we showed that Problem 2.7 admits a unique solution under Assumption 2.9. We shall now prove a version of the stochastic maximum principle which yields a characterisation of the optimal control in terms of the adjoint equation. In the sequel, we impose Assumption 2.11 and we write $\mathbb{E}$ instead of $\mathbb{E}_{t,x,z}$. The *adjoint equation* is defined as the following BSDE on $[t,T]$,

$$\begin{aligned}\begin{pmatrix}P_1(s)-P_1(t-)\\P_2(s)-P_2(t-)\\P_3(s)-P_3(t-)\end{pmatrix} &= \int_t^s \begin{pmatrix}\rho_1 P_1(r)\\ \rho_2 P_2(r)\\ h'(\hat{X}_3(r)-\alpha(r,Z(r)))\end{pmatrix} dr + \int_t^s \begin{pmatrix}Q_1(r)\\Q_2(r)\\Q_3(r)\end{pmatrix} dW(r)\\ &\quad + \int_t^s \begin{pmatrix}R_{1,1}(r)\\R_{1,2}(r)\\R_{1,3}(r)\end{pmatrix} \tilde{N}_1(dr) + \int_t^s \begin{pmatrix}R_{2,1}(r)\\R_{2,2}(r)\\R_{2,3}(r)\end{pmatrix} \tilde{N}_2(dr)\\ &\quad + \int_t^s \int_{\mathbb{R}^k} \begin{pmatrix}R_{3,1}(r,\theta)\\R_{3,2}(r,\theta)\\R_{3,3}(r,\theta)\end{pmatrix} \tilde{M}(dr,d\theta)\\ &\quad + \int_{[t,s]} \begin{pmatrix}1\\0\\0\end{pmatrix} d\hat{\eta}_1(r) + \int_{[t,s]} \begin{pmatrix}0\\1\\0\end{pmatrix} d\hat{\eta}_2(r), \qquad (4.1)\\ \begin{pmatrix}P_1(T)\\P_2(T)\\P_3(T)\end{pmatrix} &= \begin{pmatrix}0\\0\\-f'(\hat{X}_3(T)-\alpha(T,Z(T)))\end{pmatrix}.\end{aligned}$$

Remark 4.1. Note that the optimal control $\hat{\eta}$ now enters the adjoint equation, which is not the case in the "usual" formulation of singular control problems, see e.g. [CH94]. We will show in Section 6 that the solution to the BSDE defined above provides the solution to a *reflected* BSDE, where the bid ask spread plays the role of the reflecting barrier.

The adjoint process is then a triple of processes (P,Q,R) defined on $[t,T]$ (resp. $[t,T]\times\mathbb{R}^k$) by

$$P(s) \triangleq \begin{pmatrix}P_1(s)\\P_2(s)\\P_3(s)\end{pmatrix}, Q(s) \triangleq \begin{pmatrix}Q_1(s)\\Q_2(s)\\Q_3(s)\end{pmatrix} \text{ and } R(s,\theta) \triangleq \begin{pmatrix}R_{1,1}(s) & R_{1,2}(s) & R_{1,3}(s)\\ R_{2,1}(s) & R_{2,2}(s) & R_{2,3}(s)\\ R_{3,1}(s,\theta) & R_{3,2}(s,\theta) & R_{3,3}(s,\theta)\end{pmatrix},$$

which satisfy for $i=1,2,3$

$$\begin{aligned}P_i &: [t,T]\times\Omega\to\mathbb{R}, \quad Q_i : [t,T]\times\Omega\to\mathbb{R}^d,\\ R_{1,i} &: [t,T]\times\Omega\to\mathbb{R}, \quad R_{2,i} : [t,T]\times\Omega\to\mathbb{R}, \quad R_{3,i} : [t,T]\times\mathbb{R}^k\times\Omega\to\mathbb{R}\end{aligned}$$

and which also satisfy the dynamics (4.1) where P is adapted and Q,R are predictable.

Proposition 4.2. *The BSDE* (4.1) *admits a unique solution which satisfies for* $i=1,2,3$

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\le s\le T}|P_i(s)|^2\right] + \mathbb{E}\left[\int_t^T \|Q_i(r)\|^2_{\mathbb{R}^d} dr\right] + \mathbb{E}\left[\int_t^T |R_{1,i}(r)|^2 dr\right]\\ &+\mathbb{E}\left[\int_t^T |R_{2,i}(r)|^2 dr\right] + \mathbb{E}\left[\int_t^T\int_{\mathbb{R}^k} |R_{3,i}(r,\theta)|^2 m(d\theta)dr\right] < \infty.\end{aligned}$$

It is unique among triples (P,Q,R) *satisfying the above integrability criterion.*

Proof. The backward equation (4.1) is a linear BSDE, so standard arguments imply that its solution can be given in closed form as

$$\begin{cases}P_1(s) = & \mathbb{E}\left[-\int_{(s,T]} e^{-\rho_1(r-s)} d\hat{\eta}_1(r) \,\middle|\, \mathcal{F}_s\right],\\ P_2(s) = & \mathbb{E}\left[-\int_{(s,T]} e^{-\rho_2(r-s)} d\hat{\eta}_2(r) \,\middle|\, \mathcal{F}_s\right],\\ P_3(s) = & \mathbb{E}\left[-\int_s^T h'\left(\hat{X}_3(r)-\alpha(r,Z(r))\right) dr - f'\big(\hat{X}_3(T)-\alpha(T,Z(T))\big) \,\middle|\, \mathcal{F}_s\right].\end{cases} \qquad (4.2)$$

We refer the reader to [Nau11] Proposition 2.4.2 for details. □

The characterisation of the optimal control we shall derive exploits an optimality condition in terms of the Gâteaux derivative of J. Given controls $(\eta,u),(\bar\eta,\bar u)\in\mathcal{U}_t$, it is defined as

$$\langle J'(\bar\eta,\bar u),(\eta,u)\rangle=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[J\left(\bar\eta+\varepsilon\eta,\bar u+\varepsilon u\right)-J(\bar\eta,\bar u)\right].$$

In our particular case, the Gâteaux derivative can be computed explicitly. This is the content of the following lemma.

Lemma 4.3. *The performance functional $J:(\eta,u)\mapsto J(\eta,u)$ is Gâteaux differentiable. Its derivative is given by, for controls $(\eta,u),(\bar\eta,\bar u)\in\mathcal{U}_t$,*

$$\begin{aligned}
&\langle J'(\bar\eta,\bar u),(\eta,u)\rangle\\
=&\mathbb{E}\bigg[\int_{[t,T]}\Big[X_1^{\eta,u}(r-)-e^{-\rho_1(r-t)}x_1+\frac{\kappa_1}{2}\Delta\eta_1(r)\Big]\,d\bar\eta_1(r)\\
&+\int_{[t,T]}\Big[X_2^{\eta,u}(r-)-e^{-\rho_2(r-t)}x_2+\frac{\kappa_2}{2}\Delta\eta_2(r)\Big]\,d\bar\eta_2(r)\\
&\quad+\int_{[t,T]}\Big[X_1^{\bar\eta,\bar u}(r-)+\frac{\kappa_1}{2}\Delta\bar\eta_1(r)\Big]\,d\eta_1(r)+\int_{[t,T]}\Big[X_2^{\bar\eta,\bar u}(r-)+\frac{\kappa_2}{2}\Delta\bar\eta_2(r)\Big]\,d\eta_2(r)\\
&\quad+\int_t^T\gamma_1u_1(r)+\gamma_2u_2(r)+X_3^{\eta,u}(r)h'\left(X_3^{\bar\eta,\bar u}(r)-\alpha(r,Z(r))\right)dr\\
&\qquad+X_3^{\eta,u}(T)f'\left(X_3^{\bar\eta,\bar u}(T)-\alpha(T,Z(T))\right)\bigg].
\end{aligned}$$

Proof. The terms involving γ_i are straightforward. Those involving h and f can be treated exactly as in [NW11] Lemma 5.3, so it is enough to compute the Gâteaux derivative of

$$J_1(\eta,u)\triangleq\mathbb{E}\left[\int_{[t,T]}\Big[X_1(r-)+\frac{\kappa_1}{2}\Delta\eta_1(r)\Big]\,d\eta_1(r)\right].$$

From equation (3.3) it follows that the map $(\eta,u)\mapsto X_1^{\eta,u}$ is affine, so for $s\in[t,T]$, $\varepsilon\in[0,1]$ and $(\eta,u),(\bar\eta,\bar u)\in\mathcal{U}_t$ we have

$$X_1^{\bar\eta+\varepsilon\eta,\bar u+\varepsilon u}(s)=X_1^{\bar\eta,\bar u}(s)+\varepsilon\kappa_1\int_{[t,s]}e^{-\rho_1(r-t)}d\eta_1(r)=X_1^{\bar\eta,\bar u}(s)+\varepsilon\big(X_1^{\eta,u}(s)-e^{-\rho_1(s-t)}x_1\big).$$

We can now compute

$$\begin{aligned}
&\langle J_1'(\bar\eta,\bar u),(\eta,u)\rangle\\
&=\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\left[J_1\left(\bar\eta+\varepsilon\eta,\bar u+\varepsilon u\right)-J_1(\bar\eta,\bar u)\right]\\
&=\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\mathbb{E}\bigg[\int_{[t,T]}\Big[X_1^{\bar\eta+\varepsilon\eta,\bar u+\varepsilon u}(r-)+\frac{\kappa_1}{2}\Delta\bar\eta_1(r)+\varepsilon\frac{\kappa_1}{2}\Delta\eta_1(r)\Big]\,d\left(\bar\eta_1(r)+\varepsilon\eta_1(r)\right)\\
&\qquad-\int_{[t,T]}\Big[X_1^{\bar\eta,\bar u}(r-)+\frac{\kappa_1}{2}\Delta\bar\eta_1(r)\Big]\,d\bar\eta_1(r)\bigg]\\
&=\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\mathbb{E}\bigg[\int_{[t,T]}\Big[X_1^{\bar\eta,\bar u}(r-)+\varepsilon\big(X_1^{\eta,u}(r-)-e^{-\rho_1(r-t)}x_1\big)\\
&\quad+\frac{\kappa_1}{2}\Delta\bar\eta_1(r)+\varepsilon\frac{\kappa_1}{2}\Delta\eta_1(r)\Big]d\bar\eta_1(r)-\int_{[t,T]}\Big[X_1^{\bar\eta,\bar u}(r-)+\frac{\kappa_1}{2}\Delta\bar\eta_1(r)\Big]\,d\bar\eta_1(r)\\
&\quad+\varepsilon\int_{[t,T]}\Big[X_1^{\bar\eta,\bar u}(r-)+\varepsilon\big(X_1^{\eta,u}(r-)-e^{-\rho_1(r-t)}x_1\big)+\frac{\kappa_1}{2}\Delta\bar\eta_1(r)+\varepsilon\frac{\kappa_1}{2}\Delta\eta_1(r)\Big]\,d\eta_1(r)\bigg]
\end{aligned}$$

$$
\begin{aligned}
=&\mathbb{E}\bigg[\int_{[t,T]}\Big[X_1^{\eta,u}(r-)-e^{-\rho_1(r-t)}x_1+\frac{\kappa_1}{2}\Delta\eta_1(r)\Big]\,d\bar\eta_1(r)\\
&+\int_{[t,T]}\Big[X_1^{\bar\eta,\bar u}(r-)+\frac{\kappa_1}{2}\Delta\bar\eta_1(r)\Big]\,d\eta_1(r)\bigg].
\end{aligned}
$$

This completes the proof. □

Our version of the maximum principle is based on an optimality condition on the Gâteaux derivative. As a prerequisite for some algebraic manipulations of the Gâteaux derivative, let us now compute $d(P\cdot X)$ for a fixed control $(\eta,u)\in\mathcal{U}_t$. Using integration by parts, we have for $s\in[t,T]$

$$
\begin{aligned}
&P(s)X(s)-P(t-)X(t-)\\
=&\int_t^s X_3(r-)h'\big(\hat X_3(r)-\alpha(r,Z(r))\big)dr\\
&+\int_t^s\big[\lambda_1u_1(r)\big(P_3(r)+R_{1,3}(r)\big)-\lambda_2u_2(r)\big(P_3(r)+R_{2,3}(r)\big)\big]dr\\
&+\int_t^s\big[X_1(r-)Q_1(r)+X_2(r-)Q_2(r)+X_3(r-)Q_3(r)\big]dW(r)\\
&+\int_t^s\Big[X_1(r-)R_{1,1}(r)+X_2(r-)R_{1,2}(r)+X_3(r-)R_{1,3}(r)+u_1(r)\big(P_3(r-)+R_{1,3}(r)\big)\Big]\tilde N_1(dr)\\
&+\int_t^s\Big[X_1(r-)R_{2,1}(r)+X_2(r-)R_{2,2}(r)+X_3(r-)R_{2,3}(r)-u_2(r)\big(P_3(r-)+R_{2,3}(r)\big)\Big]\tilde N_2(dr)\\
&+\int_t^s\int_{\mathbb{R}^k}\big[X_1(r-)R_{3,1}(r,\theta)+X_2(r-)R_{3,2}(r,\theta)+X_3(r-)R_{3,3}(r,\theta)\big]\tilde M(dr,d\theta)\\
&+\int_{[t,s]}\big[\kappa_1P_1(r)+P_3(r)\big]d\eta_1(r)+\int_{[t,s]}\big[\kappa_2P_2(r)-P_3(r)\big]d\eta_2(r)\\
&+\int_{[t,s]}X_1(r-)d\hat\eta_1(r)+\int_{[t,s]}X_2(r-)d\hat\eta_2(r).
\end{aligned}
$$

This can be written as

$$
Y^{\eta,u}(s)=P(t-)X(t-)+L^{\eta,u}(s),\tag{4.3}
$$

where we define the "local martingale part" $L^{\eta,u}$ for $s\in[t,T]$ by

$$
\begin{aligned}
L^{\eta,u}(s)\triangleq&\int_t^s\big[X_1(r-)Q_1(r)+X_2(r-)Q_2(r)+X_3(r-)Q_3(r)\big]dW(r)\\
&+\int_t^s\Big[X_1(r-)R_{1,1}(r)+X_2(r-)R_{1,2}(r)+X_3(r-)R_{1,3}(r)\\
&\qquad+u_1(r)\big(P_3(r-)+R_{1,3}(r)\big)\Big]\tilde N_1(dr)\\
&+\int_t^s\Big[X_1(r-)R_{2,1}(r)+X_2(r-)R_{2,2}(r)+X_3(r-)R_{2,3}(r)\\
&\qquad-u_2(r)\big(P_3(r-)+R_{2,3}(r)\big)\Big]\tilde N_2(dr)\\
&+\int_t^s\int_{\mathbb{R}^k}\big[X_1(r-)R_{3,1}(r,\theta)+X_2(r-)R_{3,2}(r,\theta)+X_3(r-)R_{3,3}(r,\theta)\big]\tilde M(dr,d\theta),
\end{aligned}
$$

and the "non-martingale part" $Y^{\eta,u}$ for $s\in[t,T]$ by

$$
Y^{\eta,u}(s)\triangleq P(s)X(s)-\int_t^s X_3(r)h'\left(\hat X_3(r)-\alpha(r,Z(r))\right)dr
$$

$$
\begin{aligned}
&- \int_t^s \big[\lambda_1 u_1(r)\big(P_3(r) + R_{1,3}(r)\big) - \lambda_2 u_2(r)\big(P_3(r) + R_{2,3}(r)\big)\big] dr \\
&- \int_{[t,s]} \big[\kappa_1 P_1(r) + P_3(r)\big] d\eta_1(r) - \int_{[t,s]} \big[\kappa_2 P_2(r) - P_3(r)\big] d\eta_2(r) \\
&- \int_{[t,s]} X_1(r-) d\hat{\eta}_1(r) - \int_{[t,s]} X_2(r-) d\hat{\eta}_2(r).
\end{aligned}
$$

Let us now check that L is indeed a martingale.

Lemma 4.4. *For each $(\eta, u) \in \mathcal{U}_t$, the process $L^{\eta,u}$ is a martingale starting in 0.*

Proof. We first consider the process $\int_t^\cdot X_1(r-)Q_1(r)dW(r)$. To prove that it is a true martingale it is enough to check that

$$
\mathbb{E}\left[\sup_{s\in[t,T]} \left| \int_t^s X_1(r-)Q_1(r)dW(r) \right| \right] < \infty.
$$

An application of the Burkholder-Davis-Gundy and Hölder inequalities yields

$$
\begin{aligned}
\mathbb{E}\left[\sup_{s\in[t,T]} \left| \int_t^s X_1(r-)Q_1(r)dW(r) \right| \right] \leq & c\mathbb{E}\left[\left(\int_t^T \|X_1(r-)Q_1(r)\|^2_{\mathbb{R}^d} dr \right)^{\frac{1}{2}} \right] \\
\leq & c\mathbb{E}\left[\sup_{r\in[t,T]} |X_1(r)|^2 \right]^{\frac{1}{2}} \mathbb{E}\left[\int_t^T \|Q_1(r)\|^2_{\mathbb{R}^d} dr \right]^{\frac{1}{2}}.
\end{aligned}
$$

The last expression is finite due to Lemma 3.1 and Proposition 4.2. Now consider the process $\int_t^\cdot \int_{\mathbb{R}^k} X_1(r-)R_1(r,\theta)\tilde{M}(dr, d\theta)$. A Hölder argument as above shows that

$$
\mathbb{E}\left[\int_t^T \int_{\mathbb{R}^k} |X_1(r)R_1(r,\theta)|\, m(d\theta)dr \right] < \infty.
$$

The martingale property now follows from [NW11] Lemma A.3. The remaining terms of $L^{\eta,u}$ can be treated similarly. □

We are now in a position to formulate our second main result, the stochastic maximum principle in integral form.

Theorem 4.5. *A control $(\hat{\eta}, \hat{u}) \in \mathcal{U}_t$ is optimal if and only if for each $(\eta, u) \in \mathcal{U}_t$ we have*

$$
\begin{cases}
\mathbb{E}\left[\int_{[t,T]} \left[\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r)\right] d\left(\eta_1(r) - \hat{\eta}_1(r)\right) \right] \geq 0, \\
\mathbb{E}\left[\int_{[t,T]} \left[\hat{X}_2(r) - \kappa_2 P_2(r) + P_3(r)\right] d\left(\eta_2(r) - \hat{\eta}_2(r)\right) \right] \geq 0, \\
\mathbb{E}\left[\int_t^T \left[u_1(r) - \hat{u}_1(r)\right] \left[R_{1,3}(r) + P_3(r) - \frac{\gamma_1}{\lambda_1}\right] dr \right] \leq 0, \\
\mathbb{E}\left[\int_t^T \left[u_2(r) - \hat{u}_2(r)\right] \left[R_{2,3}(r) + P_3(r) + \frac{\gamma_2}{\lambda_2}\right] dr \right] \geq 0.
\end{cases}
\tag{4.4}
$$

Proof. We proceed as in [CH94] Theorem 4.1. We are minimising the convex functional J over $\mathcal{U}_t$, so by [ET99] Proposition 2.2.1 a necessary and sufficient condition for optimality of $(\hat{\eta}, \hat{u})$ is that

$$
\langle J'(\hat{\eta}, \hat{u}), (\eta - \hat{\eta}, u - \hat{u}) \rangle \geq 0 \text{ for each } (\eta, u) \in \mathcal{U}_t.
$$

Due to Lemma 4.4 we know that $L^{\eta,u}$ is a martingale starting in zero for each $(\eta, u) \in \mathcal{U}_t$. In particular from equation (4.3) we have that $\mathbb{E}[Y^{\eta,u}(T) - Y^{\hat{\eta},\hat{u}}(T)] = 0$. The definition of $Y^{\eta,u}$ together with the terminal condition (4.1) for the adjoint equation allows us to write this as

$$
0 = \mathbb{E}\Big[f'\left(\hat{X}_3(T) - \alpha(T, Z(T))\right) \left[X_3(T) - \hat{X}_3(T)\right]
$$

$$
\begin{aligned}
&+\int_t^T h'\left(\hat{X}_3(r)-\alpha(r,Z(r))\right)\left[X_3(r)-\hat{X}_3(r)\right]dr\\
&+\int_{[t,T]}\left[P_3(r)+\kappa_1 P_1(r)\right]d(\eta_1(r)-\hat{\eta}_1(r))+\int_{[t,T]}\left[-P_3(r)+\kappa_2 P_2(r)\right]d(\eta_2(r)-\hat{\eta}_2(r))\\
&+\int_{[t,T]}\left[X_1(r-)-\hat{X}_1(r-)\right]d\hat{\eta}_1(r)+\int_{[t,T]}\left[X_2(r-)-\hat{X}_2(r-)\right]d\hat{\eta}_2(r)\\
&+\lambda_1\int_t^T\left[u_1(r)-\hat{u}_1(r)\right]\left[P_3(r)+R_{1,3}(r)\right]dr-\lambda_2\int_t^T\left[u_2(r)-\hat{u}_2(r)\right]\left[P_3(r)+R_{2,3}(r)\right]dr\Bigg].
\end{aligned}
$$

Combining this with the explicit formula for the Gâteaux derivative given in Proposition 4.3 yields

$$
\begin{aligned}
&\langle J'(\hat{\eta},\hat{u}),(\eta-\hat{\eta},u-\hat{u})\rangle\\
=&\mathbb{E}\Bigg[\int_{[t,T]}\frac{\kappa_1}{2}\left[\Delta\eta_1(r)-\Delta\hat{\eta}_1(r)\right]d\hat{\eta}_1(r)+\int_{[t,T]}\frac{\kappa_2}{2}\left[\Delta\eta_2(r)-\Delta\hat{\eta}_2(r)\right]d\hat{\eta}_2(r)\\
&+\int_{[t,T]}\left[\hat{X}_1(r-)+\frac{\kappa_1}{2}\Delta\hat{\eta}_1(r)-P_3(r)-\kappa_1 P_1(r)\right]d\left(\eta_1(r)-\hat{\eta}_1(r)\right)\\
&+\int_{[t,T]}\left[\hat{X}_2(r-)+\frac{\kappa_2}{2}\Delta\hat{\eta}_2(r)+P_3(r)-\kappa_2 P_2(r)\right]d\left(\eta_2(r)-\hat{\eta}_2(r)\right)\\
&-\lambda_1\int_t^T\left[u_1(r)-\hat{u}_1(r)\right]\left[P_3(r)+R_{1,3}(r)-\frac{\gamma_1}{\lambda_1}\right]dr\\
&+\lambda_2\int_t^T\left[u_2(r)-\hat{u}_2(r)\right]\left[P_3(r)+R_{2,3}(r)+\frac{\gamma_2}{\lambda_2}\right]dr\Bigg].
\end{aligned}
$$

Note that for $i=1,2$ we have, using the notation from equation (2.2),

$$
\begin{aligned}
&\mathbb{E}\Bigg[\int_{[t,T]}\frac{\kappa_i}{2}\left[\Delta\eta_i(r)-\Delta\hat{\eta}_i(r)\right]d\hat{\eta}_i(r)+\int_{[t,T]}\left[\hat{X}_i(r-)+\frac{\kappa_i}{2}\Delta\hat{\eta}_i(r)\right]d(\eta_i(r)-\hat{\eta}_i(r))\Bigg]\\
=&\mathbb{E}\Bigg[\int_{[t,T]}\hat{X}_i(r-)d(\eta_i(r)-\hat{\eta}_i(r))\\
&\quad+\frac{\kappa_i}{2}\sum_{r\in[t,T]}\left[\Delta\eta_i(r)-\Delta\hat{\eta}_i(r)\right]\Delta\hat{\eta}_i(r)+\Delta\hat{\eta}_i(r)\left[\Delta\eta_i(r)-\Delta\hat{\eta}_i(r)\right]\Bigg]\\
=&\mathbb{E}\Bigg[\int_{[t,T]}\left[\hat{X}_i(r-)+\kappa_i\Delta\hat{\eta}_i(r)\right]d(\eta_i(r)-\hat{\eta}_i(r))\Bigg]\\
=&\mathbb{E}\Bigg[\int_{[t,T]}\hat{X}_i(r)d(\eta_i(r)-\hat{\eta}_i(r))\Bigg].
\end{aligned}
$$

Combining the above two displays leads to

$$
\begin{aligned}
\langle J'(\hat{\eta},\hat{u}),(\eta-\hat{\eta},u-\hat{u})\rangle=\mathbb{E}\Bigg[&\int_{[t,T]}\left[\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)\right]d\left(\eta_1(r)-\hat{\eta}_1(r)\right)\\
&+\int_{[t,T]}\left[\hat{X}_2(r)-\kappa_2 P_2(r)+P_3(r)\right]d\left(\eta_2(r)-\hat{\eta}_2(r)\right)\\
&-\lambda_1\int_t^T\left[u_1(r)-\hat{u}_1(r)\right]\left[P_3(r)+R_{1,3}(r)-\frac{\gamma_1}{\lambda_1}\right]dr\\
&+\lambda_2\int_t^T\left[u_2(r)-\hat{u}_2(r)\right]\left[P_3(r)+R_{2,3}(r)+\frac{\gamma_2}{\lambda_2}\right]dr\Bigg].
\end{aligned}
$$

We conclude that $(\hat{\eta},\hat{u})$ is optimal if and only if for all $(\eta,u)\in\mathcal{U}_t$ equation (4.4) holds. □

5. Buy, Sell and No-Trade Regions

In the preceding section we derived a characterisation of optimality in terms of all admissible controls. This condition is not always easy to verify. Therefore, we derive a further characterisation in the present section, this time in terms of buy, sell and no-trade regions. As a byproduct, this result shows that spread crossing is optimal if and only if the spread is smaller than some threshold.

We start with the main result of this section, which provides a necessary and sufficient condition of optimality in terms of the trajectory of the controlled system

$$\big(s, \hat{X}(s), P(s)\big)_{s\in[t,T]}.$$

The proof builds on arguments from [CH94] Theorem 4.2 and extends them to the present framework where we have jumps and state-dependent singular cost terms.

Theorem 5.1. *A control $(\hat{\eta}, \hat{u}) \in \mathcal{U}_t$ is optimal if and only if it satisfies*

$$\begin{cases} \mathbb{P}\Big(\hat{X}_1(s) - \kappa_1 P_1(s) - P_3(s) \geq 0 \ \forall s \in [t,T]\Big) & = 1, \\ \mathbb{P}\Big(\hat{X}_2(s) - \kappa_2 P_2(s) + P_3(s) \geq 0 \ \forall s \in [t,T]\Big) & = 1, \end{cases} \tag{5.1}$$

as well as

$$\begin{cases} \mathbb{P}\Big(\int_{[t,T]} \mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)>0\}} d\hat{\eta}_1(r) = 0\Big) & = 1, \\ \mathbb{P}\Big(\int_{[t,T]} \mathbb{1}_{\{\hat{X}_2(r)-\kappa_2 P_2(r)+P_3(r)>0\}} d\hat{\eta}_2(r) = 0\Big) & = 1, \end{cases} \tag{5.2}$$

and $ds \times d\mathbb{P}$ a.e. on $[t,T] \times \Omega$

$$\begin{cases} R_{1,3} + P_3 - \frac{\gamma_1}{\lambda_1} \leq 0 \ \text{and} \ \Big(R_{1,3} + P_3 - \frac{\gamma_1}{\lambda_1}\Big)\hat{u}_1 = 0, \\ R_{2,3} + P_3 + \frac{\gamma_2}{\lambda_2} \geq 0 \ \text{and} \ \Big(R_{2,3} + P_3 + \frac{\gamma_2}{\lambda_2}\Big)\hat{u}_2 = 0. \end{cases} \tag{5.3}$$

Proof. First, let $(\hat{\eta}, \hat{u})$ be optimal and define the stopping time

$$\nu(\omega) \triangleq \inf\Big\{s \in [t,T] \,:\, \hat{X}_1(s) - \kappa_1 P_1(s) - P_3(s) < 0\Big\},$$

with the convention $\inf\emptyset \triangleq \infty$. Consider the control defined by $u = \hat{u}, \eta_2 = \hat{\eta}_2$ and

$$\eta_1(s,\omega) \triangleq \hat{\eta}_1(s,\omega) + \mathbb{1}_{[\nu(\omega),T]}(s).$$

Then η_1 is equal to $\hat{\eta}_1$ except for an additional jump of size one at time ν. It also is càdlàg and increasing on $[t,T]$. An application of Theorem 4.5 yields

$$\begin{aligned} 0 \leq& \mathbb{E}\bigg[\int_{[t,T]} \Big[\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r)\Big]\, d\left(\eta_1(r) - \hat{\eta}_1(r)\right)\bigg] \\ =& \mathbb{E}\bigg[\Big(\hat{X}_1(\nu) - \kappa_1 P_1(\nu) - P_3(\nu)\Big)\, \mathbb{1}_{\{\nu\leq T\}}\bigg] \leq 0, \end{aligned}$$

which implies that $\mathbb{P}(\nu = \infty) = 1$. This proves the first line of (5.1), the second line follows by similar arguments. Now consider the control defined by $u = \hat{u}, \eta_2 = \hat{\eta}_2$ and

$$\begin{cases} \eta_1(t-) & = 0, \\ d\eta_1(s,\omega) & \triangleq \mathbb{1}_{\{\hat{X}_1(s,\omega)-\kappa_1 P_1(s,\omega)-P_3(s,\omega)\leq 0\}} d\hat{\eta}_1(s,\omega). \end{cases}$$

Then η_1 is càdlàg and increasing on $[t,T]$, since $\hat{\eta}_1$ is. Due to Theorem 4.5 we have

$$\begin{aligned}0 \leq &\mathbb{E}\bigg[\int_{[t,T]}\Big[\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)\Big]\,d\left(\eta_1(r)-\hat{\eta}_1(r)\right)\bigg]\\ =&\mathbb{E}\bigg[\int_{[t,T]}\Big[\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)\Big]\,\mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)>0\}}d\left(-\hat{\eta}_1(r)\right)\bigg]\leq 0,\end{aligned}$$

and in particular

$$0=\mathbb{E}\bigg[\int_{[t,T]}\mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)>0\}}d\hat{\eta}_1(r)\bigg],$$

which proves the first part of (5.2), the second part follows by similar arguments. It remains to prove (5.3). Again by Theorem 4.5 we have for every control $(\eta,u)\in\mathcal{U}_t$

$$0\geq\mathbb{E}\left[\int_t^T(u_1(r)-\hat{u}_1(r))(R_{1,3}(r)+P_3(r)-\gamma_1/\lambda_1)dr\right].$$

Choosing the control $(\hat{\eta},u)$ with $u_2=\hat{u}_2$ and

$$u_1(r,\omega)=\hat{u}_1(r,\omega)+\mathbb{1}_{\{R_{1,3}(r-,\omega)+P_3(r-,\omega)-\gamma_1/\lambda_1>0\}}$$

we first note that u_1 is predictable and we get

$$0\geq\mathbb{E}\left[\int_t^T\mathbb{1}_{\{R_{1,3}(r)+P_3(r)-\gamma_1/\lambda_1>0\}}(R_{1,3}(r)+P_3(r)-\gamma_1/\lambda_1)dr\right]\geq 0,$$

which shows that $R_{1,3}+P_3-\gamma_1/\lambda_1\leq 0\ ds\times d\mathbb{P}$ a.e. Recall that we also have $\hat{u}_1\geq 0$ by definition. We now want to show at least one of the processes $R_{1,3}+P_3-\gamma_1/\lambda_1$ and $\hat{u}_1$ is zero. To this end, consider the control $(\hat{\eta},u)$ whose passive orders are defined by $u_1=\frac{1}{2}\hat{u}_1$ and $u_2=\hat{u}_2$. We then get

$$\begin{aligned}0&\geq\mathbb{E}\left[\int_t^T(u_1(r)-\hat{u}_1(r))(R_{1,3}(r)+P_3(r)-\gamma_1/\lambda_1)dr\right]\\ &=\mathbb{E}\left[\int_t^T-\frac{1}{2}\hat{u}_1(r)(R_{1,3}(r)+P_3(r)-\gamma_1/\lambda_1)dr\right]\geq 0.\end{aligned}$$

It follows that $ds\times d\mathbb{P}$ a.e. we have $u_1(R_{1,3}+P_3-\gamma_1/\lambda_1)=0$. The argument for the second line in (5.2) is similar. This proves the "only if" part of the assertion.

In order to prove the "if" part, let conditions (5.1), (5.2) and (5.3) be satisfied. We then have for each $(\eta,u)\in\mathcal{U}_t$

$$\begin{aligned}&\mathbb{E}\bigg[\int_{[t,T]}\Big[\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)\Big]\,d\left(\eta_1(r)-\hat{\eta}_1(r)\right)\bigg]\\ =&\mathbb{E}\bigg[\int_{[t,T]}\Big[\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)\Big]\,d\eta_1(r)\bigg] &(5.4)\\ &+\mathbb{E}\bigg[\int_{[t,T]}\Big[\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)\Big]\,\mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)>0\}}d\left(-\hat{\eta}_1(r)\right)\bigg] &(5.5)\\ &+\mathbb{E}\bigg[\int_{[t,T]}\Big[\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)\Big]\,\mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)\leq 0\}}d\left(-\hat{\eta}_1(r)\right)\bigg]. &(5.6)\end{aligned}$$

The integrand of (5.4) is nonnegative due to condition (5.1), so (5.4) is nonnegative. The term (5.5) is zero due to condition (5.2). The term (5.6) has a nonpositive integrand and a decreasing integrator and is therefore also nonnegative. In conclusion, we have

$$\mathbb{E}\bigg[\int_{[t,T]}\Big[\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)\Big]\,d\left(\eta_1(r)-\hat{\eta}_1(r)\right)\bigg]\geq 0,$$

and by a similar argument

$$\mathbb{E}\bigg[\int_{[t,T]}\Big[\hat{X}_2(r)-\kappa_2 P_2(r)+P_3(r)\Big]\,d\left(\eta_2(r)-\hat{\eta}_2(r)\right)\bigg]\geq 0.$$

Still for arbitrary $(\eta,u)\in\mathcal{U}_t$ we have using (5.3) and $u_1\geq 0$

$$\mathbb{E}\bigg[\int_t^T\left[u_1(r)-\hat{u}_1(r)\right]\left[R_{1,3}(r)+P_3(r)-\gamma_1/\lambda_1\right]dr\bigg]=\mathbb{E}\bigg[\int_t^T u_1(r)\left[R_{1,3}(r)+P_3(r)-\gamma_1/\lambda_1\right]dr\bigg]\leq 0.$$

By a similar argument

$$\mathbb{E}\bigg[\int_t^T\left[u_2(r)-\hat{u}_2(r)\right]\left[R_{2,3}(r)-P_3(r)+\gamma_2/\lambda_2\right]dr\bigg]\leq 0.$$

An application of Theorem 4.5 now shows that $(\hat{\eta},\hat{u})$ is indeed optimal. □

The preceding theorem gives an optimality condition in terms of the controlled system $(P,\hat{X})$. We now show how Theorem 5.1 can be used to describe the optimal market order quite explicitly in terms of buy, sell and no-trade regions.

Definition 5.2. We define the buy, sell and no-trade regions (with respect to market orders) by

$$\begin{aligned}
\mathcal{R}_{buy}&\triangleq\left\{(s,x,p)\in[t,T]\times\mathbb{R}^3\times\mathbb{R}^3\,|\,x_1-\kappa_1 p_1-p_3<0\right\},\\
\mathcal{R}_{sell}&\triangleq\left\{(s,x,p)\in[t,T]\times\mathbb{R}^3\times\mathbb{R}^3\,|\,x_2-\kappa_2 p_2+p_3<0\right\},\\
\mathcal{R}_{nt}&\triangleq\left\{(s,x,p)\in[t,T]\times\mathbb{R}^3\times\mathbb{R}^3\,|\,x_1-\kappa_1 p_1-p_3>0\text{ and }x_2-\kappa_2 p_2+p_3>0\right\}.
\end{aligned}$$

Moreover, we define the boundaries of the buy and sell regions by

$$\begin{aligned}
\partial R_{buy}&\triangleq\left\{(s,x,p)\in[t,T]\times\mathbb{R}^3\times\mathbb{R}^3\,|\,x_1-\kappa_1 p_1-p_3=0\right\},\\
\partial R_{sell}&\triangleq\left\{(s,x,p)\in[t,T]\times\mathbb{R}^3\times\mathbb{R}^3\,|\,x_2-\kappa_2 p_2+p_3=0\right\}.
\end{aligned}$$

Let us emphasise that each of the three regions defined above is open. We remark that the time variable s is included into the definition of the buy, sell and no-trade regions such that statements like "the trajectory of the process $\big(s,\hat{X}(s),P(s)\big)$ under the optimal control is inside the no-trade region" make sense. Specifically, we now show that the optimal control remains inside the closure of the no-trade region at all times, i.e. it is either inside the no-trade region or on the boundary of the buy or sell region. Moreover, as long as the controlled system is inside the no-trade region, market orders are not used, i.e. $\hat{\eta}_i$ does not increase for $i=1,2$.

Proposition 5.3. *(i) If $\big(s,\hat{X}(s),P(s)\big)$ is in the no-trade region, it is optimal not to use market orders, i.e. for $i=1,2$*

$$\mathbb{E}\left[\int_{[t,T]}\mathbb{1}_{\{\left(r,\hat{X}(r),P(r)\right)\in\mathcal{R}_{nt}\}}d\hat{\eta}_i(r)\right]=0.$$

(ii) The optimal trajectory remains a.s. inside the closure of the no-trade region,

$$\mathbb{P}\Big(\big(s,\hat{X}(s),P(s)\big)\in\overline{\mathcal{R}_{nt}}\;\forall s\in[t,T]\Big)=1.$$

In particular, it spends no time inside the buy and sell regions.

Proof. Item (1) is a direct consequence of (5.2), while (2) follows from (5.1). □

Example 5.4. *The particular case of portfolio liquidation is solved in Section 7. In this case, the optimal strategy is composed of discrete sell orders at times $t=0,T$ and a trading rate in $(0,T)$. Specifically, these are chosen such that the process $\big(s,\hat{X}(s),P(s)\big)$ remains on the boundary of the sell region until the passive order is executed and all remaining shares are sold.*

The above proposition shows that the controlled system remains inside the closure of the no-trade region and market orders are not used inside the no-trade region. This suggests that markets orders are only used on the boundary, and we shall now make this more precise. To this end, we first note that for $i = 1, 2$ the nondecrasing process $\hat{\eta}_i$ induces a measure on $[t, T] \times \Omega$ by the following map

$$[t, s] \times A \mapsto \mathbb{E} \int_{[t,s]} \mathbb{1}_A d\hat{\eta}_i(r).$$

Proposition 5.5. *(i) We have*

$$\mathbb{P}\left(\hat{\eta}_1(s) = \int_{[t,s]} \mathbb{1}_{\{(r,\hat{X}(r),P(r))\in\partial\mathcal{R}_{buy}\}} d\hat{\eta}_1(r)\, \forall s \in [t, T]\right) = 1.$$

In particular, the support of the measure induced by $\hat{\eta}_1$ is a subset of

$$\Big(r, \hat{X}(r), P(r)\Big) \in \partial\mathcal{R}_{buy},$$

i.e. market buy orders are only used if the controlled system is on the boundary of the buy region.

(ii) Similarly, we have

$$\mathbb{P}\left(\hat{\eta}_2(s) = \int_{[t,s]} \mathbb{1}_{\{(r,\hat{X}(r),P(r))\in\partial\mathcal{R}_{sell}\}} d\hat{\eta}_2(r)\, \forall s \in [t, T]\right) = 1.$$

In particular, the support of the measure induced by $\hat{\eta}_2$ is a subset of

$$\Big(r, \hat{X}(r), P(r)\Big) \in \partial\mathcal{R}_{sell},$$

i.e. market sell orders are only used if the controlled system is on the boundary of the sell region.

Proof. We only show the first assertion. For $s \in [t, T]$ we have using $\hat{\eta}_1(t-) = 0$

$$\begin{aligned}\hat{\eta}_1(s) &= \int_{[t,s]} d\hat{\eta}_1(r)\\ &= \int_{[t,s]} \big[\mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)<0\}} + \mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)>0\}} + \mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)=0\}}\big] d\hat{\eta}_1(r).\end{aligned}$$

We shall show that terms in the second line vanish a.s. By Proposition 5.3 (2) we have

$$\mathbb{P}\left(\big(r, \hat{X}(r), P(r)\big) \notin \mathcal{R}_{buy}\ \forall r \in [t, T]\right) = 1$$

i.e. the optimal trajectory spends no time in the buy region, so that a.s. for each $s \in [t, T]$

$$\int_{[t,s]} \mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)<0\}} d\hat{\eta}_1(r) = \int_{[t,s]} \mathbb{1}_{\{\big(r,\hat{X}(r),P(r)\big)\in\mathcal{R}_{buy}\}} d\hat{\eta}_1(r) = 0.$$

Due to equation (5.2) we have a.s. for each $s \in [t, T]$

$$0 = \int_{[t,T]} \mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)>0\}} d\hat{\eta}_1(r) \geq \int_{[t,s]} \mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)>0\}} d\hat{\eta}_1(r) \geq 0,$$

so that a.s. for each $s \in [t, T]$

$$\int_{[t,s]} \mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)>0\}} d\hat{\eta}_1(r) = 0.$$

This shows that a.s. for each $s \in [t, T]$ we have

$$\hat{\eta}_1(s) = \int_{[t,s]} \mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)=0\}} d\hat{\eta}_1(r) = \int_{[t,s]} \mathbb{1}_{\{\big(r,\hat{X}(r),P(r)\big)\in\partial\mathcal{R}_{buy}\}} d\hat{\eta}_1(r).$$

□

In view of the preceding propositions we have now achieved our main goal, namely to show when spread crossing is optimal. Specifically, there is a threshold $\kappa_1 P_1 + P_3$ for the buy spread. If the buy spread is larger than this threshold, i.e. the controlled system is inside the no-trade region, then the costs of market buy orders are large as compared to the penalty for deviating from the target, and no market orders are used. Note that the threshold can be negative, in this case buying is not optimal at all, irrespective of the spread size. Market orders are only used to prevent a downward crossing of the threshold and as a result the buy spread is never smaller than the threshold. In this sense, the trajectory of the controlled system is reflected at the boundary of the no-trade region. This will be made more precise in Subsection 6 where the link to reflected BSDEs is discussed. A similar interpretation holds for the sell spread, where the threshold is given by $\kappa_2 P_2 - P_3$.

6. Link to Reflected BSDEs

In this section we use the results from the preceding section to show that the adjoint process together with the optimal control is the solution to a reflected BSDE, where the obstacle is the spread. The following definition is taken from [ØS10].

Definition 6.1. Let $F : [t,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ be a measurable function, $L : [t,T] \times \Omega \to \mathbb{R}$ be an adapted càdlàg process and $G \in L^2$. We say that $(\tilde{P}, \tilde{Q}, \tilde{R}, K)$ is a solution to the *reflected BSDE* with driver F, reflecting barrier L and terminal condition G on the time interval $[t,T]$ if the following holds:

(i) $\tilde{P}$ is adapted, $\tilde{Q}$ and $\tilde{R} \triangleq \begin{pmatrix} \tilde{R}_1 \\ \tilde{R}_2 \\ \tilde{R}_3 \end{pmatrix}$ are predictable and they satisfy

$$\begin{aligned} &\tilde{P} : [t,T] \times \Omega \to \mathbb{R}, \quad \tilde{Q} : [t,T] \times \Omega \to \mathbb{R}^d, \\ &\tilde{R}_1 : [t,T] \times \Omega \to \mathbb{R}, \quad \tilde{R}_2 : [t,T] \times \Omega \to \mathbb{R}, \quad \tilde{R}_3 : [t,T] \times \mathbb{R}^k \times \Omega \to \mathbb{R}. \end{aligned}$$

(ii) K is nondecreasing and càdlàg with $K(t-) = 0$.

(iii) For all $s \in [t,T]$ we have

$$\begin{aligned} \tilde{P}(s) - \tilde{P}(t-) &= \int_t^s F(r, \tilde{P}(r))dr + \int_t^s \tilde{Q}(r)dW(r) + \int_t^s \tilde{R}(r,\theta)\tilde{N}(d\theta, dr) - \int_{[t,s]} dK(r), \\ \tilde{P}(T) &= G. \end{aligned}$$

(iv) We have a.s. for all $s \in [t,T]$ that $\tilde{P}(s) \geq L(s)$.

(v) We have a.s. that $\int_{[t,T]} (\tilde{P}(r) - L(r))dK(r) = 0$.

The interpretation is as follows: By item (iv), the process $\tilde{P}$ is never below the barrier L. Item (v) means that the process K increases only if $\tilde{P}$ is at the barrier and is flat otherwise. Let us now define the following linear combinations of the adjoint processes:

$$\begin{aligned} \begin{pmatrix} \bar{P}_1 \\ \bar{P}_2 \end{pmatrix} &\triangleq \begin{pmatrix} -\kappa_1 P_1 - P_3 \\ -\kappa_2 P_2 + P_3 \end{pmatrix}, \quad \begin{pmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{pmatrix} \triangleq \begin{pmatrix} -\kappa_1 Q_1 - Q_3 \\ -\kappa_2 Q_2 + Q_3 \end{pmatrix}, \\ \begin{pmatrix} \bar{R}_{1,1} \bar{R}_{1,2} \\ \bar{R}_{2,1} \bar{R}_{2,2} \\ \bar{R}_{3,1} \bar{R}_{3,2} \end{pmatrix} &\triangleq \begin{pmatrix} -\kappa_1 R_{1,1} - R_{1,3} & -\kappa_2 R_{1,2} + R_{1,3} \\ -\kappa_1 R_{2,1} - R_{2,3} & -\kappa_2 R_{2,2} + R_{2,3} \\ -\kappa_1 R_{3,1} - R_{3,3} & -\kappa_2 R_{3,2} + R_{3,3} \end{pmatrix}. \end{aligned}$$

Proposition 6.2. *The process*

$$\left(\bar{P}_1, \bar{Q}_1, \begin{pmatrix} \bar{R}_{1,1} \\ \bar{R}_{2,1} \\ \bar{R}_{3,1} \end{pmatrix}, \kappa_1 \hat{\eta}_1 \right)$$

is a solution to the reflected BSDE with driver

$$-\kappa_1\rho_1 P_1(r) - h'(\hat{X}_3(r) - \alpha(r, Z(r))),$$

reflecting barrier $-\hat{X}_1$ *and terminal condition* $f'(\hat{X}_3(T) - \alpha(T, Z(T)))$. *Similarly, the process*

$$\left(\bar{P}_2, \bar{Q}_2, \begin{pmatrix}\bar{R}_{1,2}\\ \bar{R}_{2,2}\\ \bar{R}_{3,2}\end{pmatrix}, \kappa_2\hat{\eta}_2\right)$$

is a solution to the reflected BSDE with driver

$$-\kappa_2\rho_2 P_2(r) + h'(\hat{X}_3(r) - \alpha(r, Z(r))),$$

reflecting barrier $-\hat{X}_2$ *and terminal condition* $-f'(\hat{X}_3(T) - \alpha(T, Z(T)))$.

Proof. We only check the first assertion. The first two items of Definition 6.1 are clear. Item (*iii*) follows from the dynamics of the adjoint process by direct computation. Specifically, we have for $s \in [t, T]$

$$\begin{aligned}
\bar{P}_1(s) - \bar{P}_1(t-) =& -\kappa_1(P_1(s) - P_1(t-)) - (P_3(s) - P_3(t-))\\
=& \int_t^s -\kappa_1\rho_1 P_1(r) - h'\left(\hat{X}_3(r) - \alpha(r, Z(r))\right) dr\\
&+ \int_t^s -\kappa_1 Q_1(r) - Q_3(r) dW(r)\\
&+ \int_t^s -\kappa_1 R_{1,1}(r) - R_{1,3}(r)\tilde{N}_1(dr) + \int_t^s -\kappa_1 R_{2,1}(r) - R_{2,3}(r)\tilde{N}_2(dr)\\
&+ \int_t^s \int_{\mathbb{R}^k} -\kappa_1 R_{3,1}(r,\theta) - R_{3,3}(r,\theta)\tilde{M}(dr, d\theta) - \int_{[t,s]} d(\kappa_1\hat{\eta}_1(r)),\\
\bar{P}_1(T) =& -\kappa_1 P_1(T) - P_3(T) = h'\left(\hat{X}_3(T) - \alpha(T, Z(T))\right).
\end{aligned}$$

Item (4) follows from equation (5.1) in Theorem 5.1. In order to verify item (v) we apply Proposition 5.5 to get

$$\begin{aligned}
&\int_{[t,T]} (\bar{P}_1(r) + \hat{X}_1(r)) d(\kappa_1\hat{\eta}_1(r)) \qquad (6.1)\\
=&\kappa_1 \int_{[t,T]} (-\kappa_1 P_1(r) - P_3(r) + \hat{X}_1(r)) \mathbb{1}_{\{\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r) = 0\}} d\hat{\eta}_1(r) = 0.
\end{aligned}$$

The second assertion follows from similar arguments. □

As our main focus is on a solution to the curve following problem and not on reflected BSDEs, we shall not pursue this further and instead refer the interested reader to [ØS10], [EKKP$^+$97] as well as [CM01].

7. Application: Portfolio Liquidation with Singular Control and Passive Orders

In this example section we shall apply the general results on curve following to the portfolio liquidation problem, where an investor wants to unwind a large position of stock shares in a short period of time, with as little adverse price impact as possible. Models and solutions have been proposed among others by [AC01] and [SS08]. Our framework is inspired by [OW05], the new feature here are the passive orders.

The investor starts with stock holdings $X_3(0-) = x_3 > 0$ and wants to sell them such that

$$X_3(T) = 0. \qquad (7.1)$$

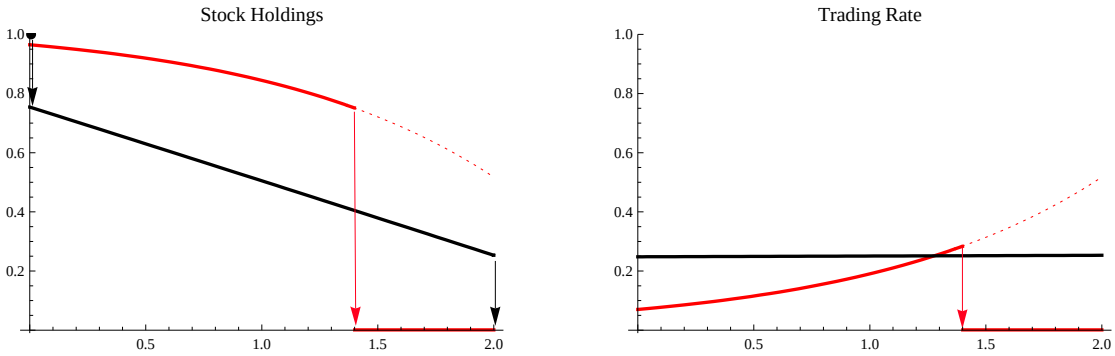

FIG 2. *Stock holdings and trading rate with (red, $\lambda_2 = 1$) and without (black, $\lambda_2 = 0$) passive orders. If there are no passive orders, there are equally sized initial and terminal discrete trades and a constant trading rate in between. If passive orders are allowed, the initial trade is smaller and the trading rate is increasing in time. If the passive order is executed, the stock holdings jump to zero. The parameters in this simulation are $T = 2, x_3 = 1, \rho_2 = 1$ and $\kappa_2 = 0.01$.*

The constraint (7.1) ensures that the portfolio is liquidated by maturity. Thus we do not need to penalise deviation and may choose $h = f = \alpha = 0$. Heuristically, it should be optimal to use only market sell and no buy orders, however we allow for both types of orders and then prove that buying is not optimal. The portfolio liquidation problem with passive orders is

Problem 7.1. *Minimise*

$$J(\eta, u) \triangleq \mathbb{E}\bigg[\int_{[0,T]} \Big[X_1(r-) + \frac{\kappa_1}{2}\Delta\eta_1(r)\Big]\, d\eta_1(r) + \int_{[0,T]} \Big[X_2(r-) + \frac{\kappa_2}{2}\Delta\eta_2(r)\Big]\, d\eta_2(r)\bigg]$$

over controls $(\eta, u) \in \mathcal{U}_0$ such that $\hat{X}_3(T) = 0$.

We introduce a sequence of auxiliary control problems without constraints, but with a penalty for stock holdings at maturity. For $n \in \mathbb{N}$ we define

Problem 7.2. *Minimise*

$$\begin{aligned} J^n(\eta, u) \triangleq \mathbb{E}\bigg[&\int_{[0,T]} \Big[X_1(r-) + \frac{\kappa_1}{2}\Delta\eta_1(r)\Big]\, d\eta_1(r) \\ &+ \int_{[0,T]} \Big[X_2(r-) + \frac{\kappa_2}{2}\Delta\eta_2(r)\Big]\, d\eta_2(r) + nX_3(T)^2\bigg] \end{aligned}$$

over controls $(\eta, u) \in \mathcal{U}_0$.

We first solve the auxiliary control problem.

Proposition 7.3. *The solution to Problem 7.2 is given $ds \times d\mathbb{P}$ a.e. on $[0,T] \times \Omega$ by a passive sell order of size*

$$\hat{u}_2^n(s) = \hat{X}_3(s-),$$

an initial discrete market sell order of size

$$\Delta\hat{\eta}_2^n(0) = \frac{2n\lambda_2\rho_2}{2ne^{\lambda_2 T}(\lambda_2+\rho_2)^2 + e^{\lambda_2 T}\lambda_2\kappa_2(\lambda_2+2\rho_2) - 2n\rho_2^2}x_3,$$

a terminal discrete market sell order of size

$$\Delta\hat{\eta}_2^n(T) = \frac{\lambda_2+\rho_2}{\rho_2}e^{\lambda_2 T}\Delta\hat{\eta}_2^n(0)\mathbb{1}_{\{T<\tau_2\}}$$

and the following rate of market sell orders in $(0,T)$,

$$d\hat{\eta}_2^n(s) = (\lambda_2+\rho_2)e^{\lambda_2 s}\Delta\hat{\eta}_2^n(0)\mathbb{1}_{\{s<\tau_2\}}ds,$$

where τ_2 *denotes the first jump time of the Poisson process* N_2*. Market and passive buy orders are not used, i.e. a.s.* $\hat{\eta}_1^n(s)=0$ *for each* $s\in[0,T]$ *and* $\hat{u}_1^n = 0$ $ds\times d\mathbb{P}$ *a.e. on* $[0,T]\times\Omega$.

Proof. The proof proceeds as follows: Taking the candidate optimal control $(\hat{\eta}^n,\hat{u}^n)$ as given, we first compute the associated state process and then the adjoint equation. This provides a solution to the forward backward system and it then only remains to check the optimality conditions from Theorem 5.1.

The state trajectory associated to the control $(\hat{\eta}^n,\hat{u}^n)$ is given on $[0,T]$ by

$$\begin{aligned}
\hat{X}_1(s) &= x_1 e^{-\rho_1 s},\\
\hat{X}_2(s) &= \begin{cases} \kappa_2 e^{\lambda_2 s}\Delta\hat{\eta}_2^n(0), & \text{if } s\le\tau_2 \text{ and } s<T,\\ \hat{X}_2(\tau_2)e^{-\rho_2(s-\tau_2)}, & \text{if } \tau_2<s,\\ \frac{\kappa_2}{\rho_2}(\lambda_2+2\rho_2)e^{\lambda_2 T}\Delta\hat{\eta}_2^n(0), & \text{if } s=T<\tau_2,\end{cases}\\
\hat{X}_3(s) &= \begin{cases} x_3-\frac{\lambda_2+\rho_2}{\lambda_2}(e^{\lambda_2 s}-1)\Delta\hat{\eta}_2^n(0), & \text{if } s<\tau_2 \text{ and } s<T,\\ \frac{1}{2n}\frac{\kappa_2}{\rho_2}(\lambda_2+2\rho_2)e^{\lambda_2 T}\Delta\hat{\eta}_2^n(0), & \text{if } s=T<\tau_2,\\ 0, & \text{if } s\ge\tau_2.\end{cases}
\end{aligned} \tag{7.2}$$

Note that the stock holdings $\hat{X}_3$ are strictly positive on $[0,\tau_2)$ and jump to zero at τ_2, i.e. if N_2 jumps and the passive order is executed. At this instant, the investor stops trading. Afterwards, the sell spread $\hat{X}_2$ recovers exponentially due to resilience. We will now use the representation (4.2) to construct the adjoint process. First note that $\hat{\eta}_1^n\equiv 0$ implies $P_1=0$ $ds\times d\mathbb{P}$ a.e. We now compute P_3. For $s\in[0,T]$ we have using (4.2)

$$P_3(s) = -\,\mathbb{E}_{s,x}\big[2n\hat{X}_3(T)\big].$$

We know from (7.2) that $\hat{X}_3=0$ on the stochastic interval $[\tau_2,T]$, so that

$$P_3(s)\mathbb{1}_{\{s\ge\tau_2\}} = 0.$$

We also have $P_3(T) = -2n\hat{X}_3(T)$. It remains to consider $s\in[0,\tau_2\wedge T)$ and for such s we compute using the exponential density of τ_2

$$\begin{aligned}
P_3(s) &= -\,\mathbb{E}_{s,x}\left[2n\hat{X}_3(T)\right]\\
&= -\,\mathbb{E}_{s,x}\left[2n\frac{1}{2n}\frac{\kappa_2}{\rho_2}(\lambda_2+2\rho_2)e^{\lambda_2 T}\Delta\hat{\eta}_2^n(0)\mathbb{1}_{\{T<\tau_2\}}\right]\\
&= -\,\frac{\kappa_2}{\rho_2}(\lambda_2+2\rho_2)e^{\lambda_2 T}\Delta\hat{\eta}_2^n(0)\int_T^\infty \lambda_2 e^{-\lambda_2(z-s)}dz\\
&= -\,\frac{\kappa_2}{\rho_2}(\lambda_2+2\rho_2)e^{\lambda_2 T}\Delta\hat{\eta}_2^n(0)e^{-\lambda_2(T-s)}\\
&= -\,\frac{\kappa_2}{\rho_2}(\lambda_2+2\rho_2)e^{\lambda_2 s}\Delta\hat{\eta}_2^n(0).
\end{aligned}$$

We now turn to P_2. A calculation based on the known form of $\hat{\eta}_2^n$, the representation (4.2) and the density of τ_2 shows that

$$\begin{aligned}
P_2(s) =& \mathbb{E}_{s,x}\left[-\int_{(s,T]} e^{-\rho_2(r-s)}d\hat{\eta}_2^n(r)\right]\\
=& -\int_s^T \lambda_2 e^{-\lambda_2(z-s)}\int_s^z e^{\rho_2 s}e^{(\lambda_2-\rho_2)r}(\lambda_2+\rho_2)\Delta\hat{\eta}_2^n(0)drdz
\end{aligned}$$

$$
\begin{aligned}
&- \int_T^\infty \lambda_2 e^{-\lambda_2(z-s)} \Big\{ \int_s^T e^{\rho_2 s} e^{(\lambda_2-\rho_2)r} (\lambda_2+\rho_2) \Delta\hat{\eta}_2^n(0) dr \\
&\qquad\qquad + e^{\rho_2 s} e^{(\lambda_2-\rho_2)T} \frac{1}{\rho_2} (\lambda_2+\rho_2) \Delta\hat{\eta}_2^n(0) \Big\} dz \\
&= - \frac{\lambda_2+\rho_2}{\rho_2} e^{\lambda_2 s} \Delta\hat{\eta}_2^n(0).
\end{aligned}
$$

To sum up, the adjoint process is given explicitly as

$$
\begin{aligned}
P_1(s) &= 0, \\
P_2(s) &= \begin{cases} -\frac{\lambda_2+\rho_2}{\rho_2} e^{\lambda_2 s} \Delta\hat{\eta}_2^n(0), & \text{if } s < \tau_2 \text{ and } s < T, \\ 0, & \text{else,} \end{cases} \\
P_3(s) &= \begin{cases} -\frac{\kappa_2}{\rho_2} (\lambda_2+2\rho_2) e^{\lambda_2 s} \Delta\hat{\eta}_2^n(0), & \text{if } s < \tau_2 \text{ and } s < T, \\ -2n\hat{X}_3(T), & \text{if } s = T < \tau_2 \\ 0, & \text{else.} \end{cases}
\end{aligned}
$$

In particular, P_i is zero on the stochastic interval $[\tau_2, T]$ for $i = 2, 3$.

Having constructed a solution to the forward backward system, we will now use Theorem 5.1 to show that the control $(\hat{u}^n, \hat{\xi}^n)$ is indeed optimal. Using the known form of $\hat{X}_i$ and P_i for $i = 1, 2, 3$, we check the optimality conditions and compute that a.s.

$$
\begin{cases}
\hat{X}_1(s) - P_3(s) - \kappa_1 P_1(s) = -P_3(s) & \geq 0,\ s \in [0, T] \\
\hat{X}_2(s) + P_3(s) - \kappa_2 P_2(s) & = 0,\ s \in [0, \tau_2 \wedge T], \\
\hat{X}_2(s) + P_3(s) - \kappa_2 P_2(s) = \hat{X}_2(s) & \geq 0,\ s \in (\tau_2 \wedge T, T],
\end{cases}
$$

so that condition (5.1) is satisfied. In order to check (5.2), we first note that $\hat{\eta}_1^n(r) = 0$ for each $r \in [0, T]$ a.s. so that

$$
\mathbb{P}\Big(\int_{[0,T]} \mathbb{1}_{\{\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r) > 0\}} d\hat{\eta}_1^n(r) = 0 \Big) = 1.
$$

In addition, we have $\hat{X}_2 - \kappa_2 P_2 + P_3 = 0$ on $[0, \tau_2 \wedge T]$ and $\hat{\eta}_2^n$ is constant on $[\tau_2 \wedge T, T]$ so that

$$
\begin{aligned}
&\int_{[0,T]} \mathbb{1}_{\{\hat{X}_2(r) - \kappa_2 P_2(r) + P_3(r) > 0\}} d\hat{\eta}_2^n(r) \\
&= \int_{[0, \tau_2 \wedge T]} \mathbb{1}_{\{\hat{X}_2(r) - \kappa_2 P_2(r) + P_3(r) > 0\}} d\hat{\eta}_2^n(r) + \int_{(\tau_2 \wedge T, T]} \mathbb{1}_{\{\hat{X}_2(r) - \kappa_2 P_2(r) + P_3(r) > 0\}} d\hat{\eta}_2^n(r) = 0.
\end{aligned}
$$

Finally, let us check condition (5.3). A consequence of $P_1 = 0$ is that $R_{1,3} = 0$ $ds \times d\mathbb{P}$ a.e. and we have

$$
R_{1,3}(s) + P_3(s-) = P_3(s-) \leq 0 \text{ and } \hat{u}_1(s) = 0.
$$

If the Poisson process N_2 jumps, then P_3 jumps to zero, so we have $ds \times d\mathbb{P}$ a.e. on $[0, T] \times \Omega$

$$
R_{2,3}(s) + P_3(s-) = 0.
$$

An application of Theorem 5.1 now yields that $(\hat{u}^n, \hat{\eta}^n)$ is optimal. □

We now proceed to the portfolio liquidation problem with passive orders and terminal constraint.

Proposition 7.4. *The solution to Problem 7.1 is given $ds \times d\mathbb{P}$ a.e. on $[0, T] \times \Omega$ by a passive sell order of size*

$$
\hat{u}_2(s) = \hat{X}_3(s-),
$$

an initial discrete market sell order of size

$$\Delta\hat{\eta}_2(0) = \frac{\lambda_2\rho_2}{e^{\lambda_2 T}(\lambda_2+\rho_2)^2-\rho_2^2}x_3,$$

a terminal discrete market sell order of size

$$\Delta\hat{\eta}_2(T) = \frac{\lambda_2+\rho_2}{\rho_2}e^{\lambda_2 T}\Delta\hat{\eta}_2(0)\mathbb{1}_{\{T<\tau_2\}} = \frac{\lambda_2(\lambda_2+\rho_2)e^{\lambda_2 T}}{e^{\lambda_2 T}(\lambda_2+\rho_2)^2-\rho_2^2}x_3\mathbb{1}_{\{T<\tau_2\}},$$

and the following rate of market sell orders in $(0,T)$*,*

$$d\hat{\eta}_2(s) = (\lambda_2+\rho_2)e^{\lambda_2 s}\Delta\hat{\eta}_2(0)\mathbb{1}_{\{s<\tau_2\}}ds = \frac{\lambda_2\rho_2(\lambda_2+\rho_2)}{e^{\lambda_2 T}(\lambda_2+\rho_2)^2-\rho_2^2}e^{\lambda_2 s}x_3\mathbb{1}_{\{s<\tau_2\}}ds,$$

where τ_2 *denotes the first jump time of the Poisson process* N_2*. Market and passive buy orders are not used, i.e. a.s.* $\hat{\eta}_1(s)=0$ *for each* $s\in[0,T]$ *and* $\hat{u}_1 = 0$ $ds\times d\mathbb{P}$ *a.e. on* $[0,T]\times\Omega$*.*

Proof. We rewrite the performance functional in the following way:

$$\begin{aligned}J(\eta,u) =&\mathbb{E}\bigg[\int_{[0,T]}\Big[X_1(r-)+\frac{\kappa_1}{2}\Delta\eta_1(r)\Big]d\eta_1(r)\\&+\int_{[0,T]}\Big[X_2(r-)+\frac{\kappa_2}{2}\Delta\eta_2(r)\Big]d\eta_2(r)+\delta_{\{\mathbb{R}\setminus\{0\}\}}(X_3(T))\bigg].\end{aligned}$$

where $\delta_{\{\mathbb{R}\setminus\{0\}\}}$ is the indicator function in the sense of convex analysis. We then have for each $(\eta,u)\in\mathcal{U}_0$

$$J^n(\eta,u)\le J(\eta,u). \tag{7.3}$$

Moreover, one can check by direct calculation that the strategy $(\hat{\eta},\hat{u})$ satisfies the liquidation constraint (7.1), i.e. we have $\hat{X}_3(T)=0$ and thus $(\hat{\eta},\hat{u})$ is admissible. Before we prove the optimality, let us establish some convergence results. We first note that the optimal strategies converge in the sense that $\lim_{n\to\infty}\hat{u}^n=\hat{u}$ $ds\times d\mathbb{P}$ a.e. and $\lim_{n\to\infty}\hat{\eta}^n(s)=\hat{\eta}(s)$ for all $s\in[0,T]$ a.s. We now show that the associated trading costs also converge. Indeed, using the known form of $X_3^{\hat{\eta}^n,\hat{u}^n}(T)$ from (7.2) as well as the known form of $\Delta\hat{\eta}_2^n(0)$ implies that the terminal costs satisfy

$$\begin{aligned}&\lim_{n\to\infty}\left\{nX_3^{\hat{\eta}^n,\hat{u}^n}(T)^2\right\}\\=&\lim_{n\to\infty}\left\{n\left[\frac{1}{2n}\frac{\kappa_2}{\rho_2}(\lambda_2+2\rho_2)e^{\lambda_2 T}\Delta\hat{\eta}_2^n(0)\right]^2\mathbb{1}_{\{T<\tau_2\}}\right\}\\=&\lim_{n\to\infty}\left\{n\left[\frac{1}{2n}\frac{\kappa_2}{\rho_2}(\lambda_2+2\rho_2)e^{\lambda_2 T}\frac{2n\lambda_2\rho_2}{2ne^{\lambda_2 T}(\lambda_2+\rho_2)^2+e^{\lambda_2 T}\lambda_2\kappa_2(\lambda_2+2\rho_2)-2n\rho_2^2}x_3\right]^2\mathbb{1}_{\{T<\tau_2\}}\right\}\\=&0.\end{aligned}$$

The integrand of the singular cost term defined in Problem 7.2 converges pointwise in the sense

$$\begin{aligned}&\lim_{n\to\infty}\left\{\Big[X_2^{\hat{\eta}^n,\hat{u}^n}(r-)+\frac{\kappa_2}{2}\Delta\hat{\eta}_2^n(r)\Big]d\hat{\eta}_2^n(r)\right\}\\=&\lim_{n\to\infty}\left\{X_2^{\hat{\eta}^n,\hat{u}^n}(r-)(\lambda_2+\rho_2)e^{\lambda_2 r}\Delta\hat{\eta}_2^n(0)\mathbb{1}_{\{r<\tau_2\}}dr+\frac{\kappa_2}{2}\Delta\hat{\eta}_2^n(0)^2+\frac{\kappa_2}{2}\Delta\hat{\eta}_2^n(T)^2\right\}\\=&\hat{X}_2(r-)(\lambda_2+\rho_2)e^{\lambda_2 r}\Delta\hat{\eta}_2(0)\mathbb{1}_{\{r<\tau_2\}}dr+\frac{\kappa_2}{2}\Delta\hat{\eta}_2(0)^2+\frac{\kappa_2}{2}\Delta\hat{\eta}_2(T)^2\\=&\Big[X_2^{\hat{\eta},\hat{u}}(r-)+\frac{\kappa_2}{2}\Delta\hat{\eta}_2(r)\Big]d\hat{\eta}_2(r).\end{aligned}$$

We now apply Fatou's Lemma together with (7.3) to get for each $(\eta, u) \in \mathcal{U}_0$

$$\begin{aligned} J(\hat{\eta}, \hat{u}) =& \mathbb{E}\bigg[\int_{[0,T]} \Big[\hat{X}_2(r-) + \frac{\kappa_2}{2} \Delta\hat{\eta}_2(r) \Big] d\hat{\eta}_2(r) \bigg] \\ \leq& \liminf_{n\in\mathbb{N}} \mathbb{E}\bigg[\int_{[0,T]} \Big[X_2^{\hat{\eta}^n, \hat{u}^n}(r-) + \frac{\kappa_2}{2} \Delta\hat{\eta}_2^n(r) \Big] d\hat{\eta}_2^n(r) + n X_3^{\hat{\eta}^n, \hat{u}^n}(T)^2 \bigg] \\ =& \liminf_{n\in\mathbb{N}} J^n(\hat{\eta}^n, \hat{u}^n) \leq \liminf_{n\in\mathbb{N}} J^n(\eta, u) \leq J(\eta, u). \end{aligned}$$

This proves that $(\hat{\eta}, \hat{u})$ is indeed the solution to Problem 7.1. □

We conclude with some remarks on the structure of the optimal control.

Remark 7.5.

- It is optimal to offer all outstanding shares as a passive order, and simultaneously trade using market orders.
- Let us compare the solutions with and without passive orders. If no passive orders are allowed, it is shown in [OW05] that the optimal control comprises equally sized initial and terminal discrete trades and a constant trading rate in between. If passive orders are allowed, it follows from Proposition 7.4 that the initial discrete trade is small and the investor starts with a small trading rate, which increases as maturity approaches. The interpretation is that he is reluctant to use market orders and rather waits for passive order execution. See Figure 2 for an illustration.
- While [OW05] work in a one sided model and only consider market sell orders, we consider a larger class of controls and allow for both market buy and sell orders. It is a *consequence* of Proposition 7.4 that market buy orders are never used.
- The sell region is in this case

$$\mathcal{R}_{sell} = \big\{ (s, x, p) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \big| \, x_2 + p_3 - \kappa_2 p_2 < 0 \big\}.$$

  The initial discrete trade is chosen such that the controlled system jumps to the boundary of the sell region. Then a rate of market sell orders is chosen such that the state process remains on this boundary until the passive order is executed.
- The optimal strategy does not depend on the inverse order book height κ_2 and is linear in the initial portfolio size $x_3 = X_3(0-)$.
- The solution to the portfolio liquidation problem with passive orders given in Proposition 7.4 is similar to the one obtained in [KS09] Proposition 4.2; what they call dark pool can be interpreted as a passive order in our setup. Note however that they work in discrete time in a model without spread and resilience. Our solution is also similar to the one obtained in [NW11] Proposition 7.3, where the portfolio liquidation problem is solved in continuous time using passive and market, but no discrete orders and without resilience.
- The solution given above only holds for initial spread zero. If we start with a larger spread, it might be optimal not to use market orders for a certain period of time and wait for the spread to grow back.

Remark 7.6. As the jump intensity λ_2 tends to zero, the solution given in Proposition 7.4 for the model with passive orders converges to the solution given [OW05] for the model without passive orders. Specifically we have for $s \in (0,T)$

$$\begin{aligned} \lim_{\lambda_2\to 0} \Delta\hat{\eta}_2(0) &= \lim_{\lambda_2\to 0} \frac{\lambda_2\rho_2}{e^{\lambda_2 T}(\lambda_2+\rho_2)^2 - \rho_2^2} x_3 = \frac{x_3}{\rho_2 T + 2}, \\ \lim_{\lambda_2\to 0} \Delta\hat{\eta}_2(T) &= \lim_{\lambda_2\to 0} \frac{\lambda_2(\lambda_2+\rho_2)e^{\lambda_2 T}}{e^{\lambda_2 T}(\lambda_2+\rho_2)^2 - \rho_2^2} x_3 = \frac{x_3}{\rho_2 T + 2}, \\ \lim_{\lambda_2\to 0} d\hat{\eta}_2(s) &= \lim_{\lambda_2\to 0} \frac{\lambda_2\rho_2(\lambda_2+\rho_2)}{e^{\lambda_2 T}(\lambda_2+\rho_2)^2 - \rho_2^2} e^{\lambda_2 s} x_3 ds = \frac{\rho_2 x_3}{\rho_2 T + 2} ds. \end{aligned}$$

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