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Interest Rate Modelling and Derivative Pricing, WS 2018/19

Exercise 4

1. **Interest rate sensitivities for bonds and Vanilla swaps**
   Interest rate risk arises from potential changes in present values (of a single instrument or portfolio) induced by changes of the market rates. Financial institutions aim at limiting or reducing interest rate risk by trading swaps.
   Consider Bank A holds a bond receiving 3% coupon (annual payment, 30/360 day count convention) starting today with final maturity in $10y$ (notional payment in $10y$ as well). Market interest rates are given at 3% (flat, annual compounding, 30/360 day count convention). We assume a single curve setting (i.e. Libor projection curves equal discount curves).

   (a) What is the present value of the bond?

   (b) Interest rate *Delta* is the change in present value for a $1bp$ change in interest rates. What is the interest rate Delta for the bond?

   (c) Consider a Vanilla fixed rate payer swap with fixed leg equal to the bond fixed coupon payments (but without notional payment) and floating leg receiving $6m$ Libor rates. What is the present value and interest rate Delta of the swap?

   (d) Show that, in general, the portfolio of a bond and Vanilla swap as constructed above has zero Delta risk.

2. **Affine term-structure models**
   The continuously-compounded spot rate is defined as

$$R(t,T) = -\frac{\ln\left[P(t,T)\right]}{T-t}.$$

   Affine term structure models are characterised by the fact that continuously-compounded spot rates take the form

$$R(t,T) = \alpha(t,T) + \beta(t,T) \cdot r(t)$$

   for deterministic functions α and β.

   (a) Show that for affine models future zero bond prices can be represented as

$$P(t,T) = A(t,T) \cdot e^{-B(t,T)\cdot r(t)}$$

   with deterministic functions A and B.

   (b) Assume a one-factor diffusion for the short rate in the risk-neutral measure via

$$dr(t) = \left[\theta(t) - \chi \cdot r(t)\right] \cdot dt + \sigma\left(t, r(t)\right) \cdot dW(t).$$

   Show that the dynamics of the forward rates $f(t,T)$ become

$$df(t,T) = [\ldots] \cdot dt + \frac{\partial B(t,T)}{\partial T} \cdot \sigma\left(t, r(t)\right) \cdot dW(t)$$

(c) Now consider the case $\sigma(t, r(t)) = \sqrt{a + b \cdot r(t)}$. What are the dynamics of the model in terms of the state variables $x(t)$ and $y(t)$?

3. **Swap rate dynamics in separable HJM model**
Consider the one-factor separable HJM model with $r(t) = f(0,t) + x(t)$ and risk-neutral dynamics

$$dx(t) = [y(t) - \chi \cdot x(t)]\, dt + \sigma_r(t) dW(t),$$
$$dy(t) = \left[\sigma_r(t)^2 - 2\chi \cdot y(t)\right] dt.$$

The forward swap rate (in the single curve setting) is (for $t \le T_0$)

$$S(t) = \frac{P(t, T_0) - P(t, T_N)}{\sum_{i=1}^{N} \tau_i P(t, T_i)}.$$

Recall that zero bonds can be written in terms of state variables via $P(t,T) = P(x(t), y(t); t, T)$.

(a) What are the dynamics of the swap rate in the annuity measure given that HJM model?

(b) Assume the short rate volatility $\sigma_r(t)$ is a deterministic function. Can the swap rate be approximated by a normal, lognormal or shifted log-normal model? What is the corresponding volatility parameters in such an approximated model?