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STOCHASTISCHE FINANZMATHEMATIK II Sommersemester 2019

ÜBUNGSBLATT 2

1. Proof Theorem 2.3.5 from the lecture notes.

   Hint: Consider an SDE that starts in X_s and use the representation of the strong solution in terms of Brownian sample paths and initial conditions.

2. Let

$$\mathcal{M}^2 := \{M = (M_t)_{t\in[0,\infty)} : M \text{ is a martingale with right-continuous sample paths, } M_0 = 0,\ \sup_{t\geq 0} E|M_t|^2 < \infty \}$$

be the space of square integrable, right-continuous martingales on $[0,\infty)$ starting in 0. For $M \in \mathcal{M}$ the limit

$$M_\infty := \lim_{t\to\infty} M_t$$

exists alsmost surely and in L^2. The space $\mathcal{M}^2$ is a Hilbert space when equipped with the inner product

$$\langle M, N\rangle := E[M_\infty N_\infty].$$

If $M \in \mathcal{M}^2$, then $\lim_{t\to\infty} E[M_t] = E[M_\infty]$ and $M_t = E[M_\infty|\mathcal{F}_t]$ by uniform integrability. We call $M, N \in \mathcal{M}^2$ weakly orthogonal if $\langle M, N\rangle = 0$ and strongly orthogonal if the product MN is a martingale.

   i) Prove that strong orthogonality implies weak orthogonality. Is the converse also true?

   ii) Let $\mathcal{A} \subset \mathcal{M}^2$ be stable with respect to stopping, i.e. for any $A \in \mathcal{A}$ and any stopping time τ the stopped process $A_t^\tau := A_{\tau\wedge t}$ belongs to $\mathcal{A}$. Prove that the weak orthogonal complement of any stable subset $\mathcal{A}$ of $\mathcal{M}^2$ is strongly orthogonal, i.e. if M is weakly orthogonal to any $N \in \mathcal{A}$, then it is strongly orthogonal.

   iii) Let $M \in \mathcal{M}^2$ and let I be a stochastic integral on $[0,\infty)$. Prove that MI is a martingale.

Hint: In part iii) use Doob's Optional Sample Theorem: For any martingale with (right-) continuous sample paths and any two stopping times $\tau \leq \sigma$,

$$X_\tau = E[X_\sigma|\mathcal{F}_\tau]$$

3. Let C be an observed market price for a call option with strike K and maturity T. The *implied volatility* is the volatility σ^* that makes the Black-Scholes price the observed market price.

   i) Prove that the implied volatility is well defined.

   ii) If the Black-Scholes model were correct, then the implied volatility would be independent of K and T. In reality, however, we often observe a *volatility smile* (or smirk), meaning that the implied volatility is a convex function of the strike and/or time to maturity. Explain this phenomena.