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Stochastische Finanzmathematik I Sommersemester 2019

Übungsblatt 3

1. Let ξ be a bounded adapted process. Show that the density associated with the stochastic exponential of the integral of ξ w.r.t. a Brownian motion belongs to L^p for all $p \in [1, \infty)$.

2. Prove Lemma 2.3.9 from the lecture notes.

3. This exercise discusses the generator of an Itô process.

   (i) Let Y be an $\mathbb{R}^n$ -valued Itô process of the form

$$Y_t^x(\omega) = x + \int_0^t u(s,\omega)\,ds + \int_0^t v(s,\omega)\,dW_s,\ t \in [0,T],$$

   where W is an m-dimensional Brownian motion, and u and v are assumed to be bounded. Let $f \in \mathcal{C}_0^2(\mathbb{R}^n)$ (twice continuously differentiable with compact support). Prove that

$$\mathbb{E}[f(Y_t^x)] = f(x) + \mathbb{E}\left[\int_0^t \sum_{i=1}^n u_i(s)\frac{\partial f(Y_s^x)}{\partial_{y_i}} + \frac{1}{2}\sum_{i,j=1}^n (vv^\top)_{i,j}(s)\frac{\partial^2 f(Y_s^x)}{\partial_{y_i}\partial_{y_j}}\,ds\right].$$

   (ii) Let X be the solution to the Itô diffusion

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = x.$$

   Define $\mathcal{A}f(x) := \lim_{t\searrow 0}\frac{\mathbb{E}[f(X_t)]-f(x)}{t}$. Using (i), prove that $\mathcal{A}$ is the infinitesimal generator of X, i.e., for any f such that the above limit exists, and for all $x \in \mathbb{R}^n$,

$$\mathcal{A}f(x) = \sum_{i=1}^n b_i(x)\frac{\partial f(x)}{\partial_{y_i}} + \frac{1}{2}\sum_{i,j=1}^n (\sigma\sigma^\top)_{i,j}(x)\frac{\partial^2 f(x)}{\partial_{y_i}\partial_{y_j}}.$$

   (iii) (graph of Brownian motion) Let $X = (X_1, X_2)^\top$ be given by,

$$dX_1(t) = dt,\ X_1(0) = t_0; \quad dX_2(t) = dW_t,\ X_2(0) = x_0,$$

   where W is a 1-dimensional Brownian motion, i.e.,

$$dX_t = b\,dt + \sigma\,dW_t,$$

   where $b = (1,0)^\top$ and $\sigma = (0,1)^\top$. In other words, X may be regarded as the graph of the Brownian motion W. Prove the infinitesimal generator of X is the *heat operator*

$$\mathcal{A}f(x) = \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f(x)}{\partial x^2}.$$