A Constrained Control Problem with Degenerate Coefficients and Degenerate Backward SPDEs with Singular Terminal Condition*

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Abstract

We study a constrained optimal control problem allowing for degenerate coefficients. The coefficients can be random and then the value function is described by a degenerate backward stochastic partial differential equation (BSPDE) with singular terminal condition. For this degenerate BSPDE, we prove the existence and uniqueness of the nonnegative solution.

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1 Introduction

Let $T \in (0,\infty)$ and $(\Omega, \bar{\mathscr{F}}, \mathbb{P})$ be a probability space equipped with a filtration $\{\bar{\mathscr{F}}_t\}_{0\leq t\leq T}$ which satisfies the usual conditions. The probability space carries an m-dimensional Brownian motion W and an independent point process $\tilde{J}$ on a non-empty Borel set $\mathcal{Z} \subset \mathbb{R}^l$ with characteristic measure $\mu(dz)$. We endow the set $\mathcal{Z}$ with its Borel σ-algebra $\mathscr{Z}$ and denote by $\pi(dt, dz)$ the associated Poisson random measure. The filtration generated by W, together with all $\mathbb{P}$ null sets, is denoted by $\{\mathscr{F}_t\}_{t\geq 0}$. The predictable σ-algebra on $\Omega \times [0,+\infty)$ corresponding to $\{\mathscr{F}_t\}_{t\geq 0}$ and $\{\bar{\mathscr{F}}_t\}_{t\geq 0}$ are denoted by $\mathscr{P}$ and $\bar{\mathscr{P}}$, respectively.

In this paper we address the following stochastic optimal control problem with constraints:

$$\min_{\xi,\rho} E\left[\int_0^T\Big(\eta_s(y_s)|\xi_s|^2 + \lambda_s(y_s)|x_s|^2\Big)\,ds + \int_0^T\int_{\mathcal{Z}}\gamma_s(y_s,z)|\rho_s(z)|^2\,\mu(dz)ds\right] \tag{1.1}$$

subject to

$$\begin{cases} x_t = x - \displaystyle\int_0^t \xi_s\,ds - \int_0^t\int_{\mathcal{Z}}\rho_s(z)\,\pi(dz,ds), \quad t\in[0,T] \\ x_T = 0 \\ y_t = y + \displaystyle\int_0^t b_s(y_s)\,ds + \int_0^t \sigma_s(y_s)\,dW_s. \end{cases} \tag{1.2}$$

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The real-valued process $(x_t)_{t\in[0,T]}$ is the *state process*. It is governed by a pair of *controls* (ξ,ρ). The m-dimensional process $(y_t)_{t\in[0,T]}$ is uncontrolled. We sometimes write $x_t^{s,x,\xi,\rho}$ for $0\le s\le t\le T$ to indicate the dependence of the state process on the control (ξ,ρ), the initial time s and initial state $x\in\mathbb{R}$. Likewise, we sometimes write $y_t^{s,y}$ to indicate the dependence on the initial time and state. The set of *admissible controls* consists of all pairs $(\xi,\rho)\in\mathcal{L}^2_{\bar{\mathscr{F}}}(0,T)\times\mathcal{L}^2_{\bar{\mathscr{F}}}(0,T;L^2(\mathcal{Z}))$ s.t. $x_T=0$ a.s.

Control problems of the above form arise in models of optimal portfolio liquidation. In such models x_t denotes the portfolio an investor holds at time $t\in[0,T]$, ξ_t is the rate at which the stock is purchased or sold in a regular exchange at that time, $x_T=0$ is the *liquidation constraint*, ρ_t describes the number of stocks placed in a crossing network, π governs the order execution in the crossing network and y_t is a stochastic factor, often a benchmark price or volume-weighted average price process, against which the cost of a liquidation strategy is benchmarked.

The cost functional is assumed to be of the quadratic form:

$$J_t(x_t,y_t;\xi,\rho)=E\left[\int_t^T\Big(\eta_s(y_s)|\xi_s|^2+\lambda_s(y_s)|x_s|^2\Big)\,ds+\int_t^T\int_{\mathcal{Z}}\gamma_s(y_s,z)|\rho_s(z)|^2\,\mu(dz)ds\Big|\mathscr{F}_t\right]. \tag{1.3}$$

and the value function is given by:

$$V_t(x,y)\triangleq\operatorname*{ess\,inf}_{\xi,\rho}J_t(x_t,y_t;\xi,\rho)\big|_{x_t=x,y_t=y}. \tag{1.4}$$

We refer to [1, 2, 11, 15, 17, 18, 24] and references therein for a detailed discussion of portfolio liquidation problems and an interpretation of the coefficients η,λ and γ.

In a Markovian framework where all coefficients are deterministic functions of the control and state variables, the Hamilton-Jacobi-Bellman (HJB) equation turns out to be a *deterministic* nonlinear parabolic partial differential equation (PDE) with a singularity at the terminal time; see [13] for details. Non-Markovian control problems with pre-specified terminal values have been studied in recent papers by Ankirchner, Jeanblanc and Kruse [2], and Graewe, Horst and Qiu [12]. The former represented the value function in terms of a nonlinear backward stochastic differential equation (BSDE). While BSDEs are tailor-made to study non-Markovian control problems, the optimal control is of an open-loop form and hence it does not give the precise dependence of the optimal control on the factor process $(y_t)_{t\in[0,T]}$.

The BSPDE-approach in [12] is more general. In that paper the authors construct the optimal control in feedback form assuming that there exists another independent n-dimensional Brownian motion B s.t.

$$y_t=y+\int_0^t b_s(y_s)\,ds+\int_0^t\sigma_s(y_s)\,dW_s+\int\bar{\sigma}_s(y_s)\,dB_s, \tag{1.5}$$

where the coefficients b, σ, $\bar{\sigma}$, λ, γ and η are measurable with respect to the filtration $\mathscr{F}$ generated by W, and $\bar{\sigma}$ satisfies the strict non-degeneracy condition. We extend their results to the degenerate case and also allow all coefficients to depend on all sources of randomness. This is important from an application point of view, as the strict separation of the two sources of randomness in (1.5) is not always natural and/or easy to satisfy.

The constrained optimal control problem (1.1) can be formally written as an unconstrained one:

$$\min_{\xi,\rho}E\left[\int_0^T\Big(\eta_s(y_s)|\xi_s|^2+\lambda_s(y_s)|x_s|^2\Big)\,ds+\int_0^T\int_{\mathcal{Z}}\gamma_s(y_s,z)|\rho_s(z)|^2\,\mu(dz)ds+(+\infty)|x_T|^2 1_{\{x_T\neq 0\}}\right] \tag{1.6}$$

subject to

$$\begin{cases} x_t=x-\displaystyle\int_0^t\xi_s\,ds-\int_0^t\int_{\mathcal{Z}}\rho_s(z)\,\pi(dz,ds), & t\in[0,T];\\ y_t=y+\displaystyle\int_0^t b_s(y_s)\,ds+\int_0^t\sigma_s(y_s)\,dW_s. \end{cases}$$

Inspired by Peng's seminal work [21] on non-Markovian stochastic optimal control and in view of the linear-quadratic structure of the cost functional, the dynamic programming principle suggests that the value function is of the form

$$V_t(x,y) = u_t(y)x^2,$$

where u is the first component of the pair (u,ψ) satisfying formally the following backward stochastic partial differential equation (BSPDE) with singular terminal condition:

$$\begin{cases} -du_t(y) = \left[\operatorname{tr}\left(\frac{1}{2}D^2u_t(y) + D\psi_t(y)\sigma_t^*(y)\right) + b_t^*(y)Du_t(y) + F(s,y,u_t(y))\right]dt \\ \qquad\qquad - \psi_t(y)\,dW_t, \quad (t,y)\in[0,T]\times\mathbb{R}^d; \\ u_T(y) = +\infty, \quad y\in\mathbb{R}^d. \end{cases} \tag{1.7}$$

Here

$$F(t,y,r) \triangleq -\int_{\mathcal{Z}} \frac{r^2}{\gamma(t,y,z)+r}\mu(dz) - \frac{r^2}{\eta_t(y)} + \lambda_t(y), \quad (t,y,r)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}. \tag{1.8}$$

BSPDEs were first introduced by Bensoussan [4, 5] as the adjoint equations of *forward* SPDEs. They have since been extensively used to study a wide range of problems in pure and applied probability ranging from stochastic maximum principles to optimal control under partial information as well as from optimal stopping to hedging in incomplete financial markets [5, 6, 9, 10, 14, 20, 25, 26].

To the best of our knowledge degenerate BSPDEs with singular terminal values have never been studied in the literature before, not even in the Markovian case. Due to the degeneracy of the diffusion coefficient, no generalized Itô-Kunita formula for the random filed u satisfying the BSPDE (1.7) in the distributional sense is available. To prove the verification theorem we appeal instead to the link between degenerate BSPDEs and forward-backward stochastic differential equations (FBSDEs). Drawing on recent results on degenerate BSPDEs [8, 9, 16, 19], we first prove that the BSPDE resulting from our control problem but with finite terminal condition has a sufficient regular solution.

Subsequently, we establish a comparison principle for degenerate BSPDEs, from which we deduce that the solutions to BSPDEs with finite terminal values increase with the terminal values. To verify that the limit of a sequence of BSPDEs with increasing terminal values is a solution to our BSPDE (1.7) requires a gradient estimate. Such an estimate is not needed in the non-degenerate case. The non-degenerate case only requires a growth condition on u_t near the terminal time, while the degenerate case requires an additional integrability condition on the gradient Du_t.

The gradient estimate for a solution to a *degenerate* BSPDE generally depends on its gradient at terminal time. In our case, the terminal value of the BSPDE is singular and hence it does in no obvious way characterize the gradient. Instead, we derive our gradient estimate from the gradient estimates of the approximating sequence. Our estimate seems new even in the Markovian case. Along with the gradient estimate, an explicit asymptotic estimate for the solution of our BSPDE near the terminal time is given. Finally, using the Itô formula for the square norm of the positive part of the solutions for BSPDEs, we prove that the obtained solution is the unique nonnegative one.

The remainder of this paper is organized as follows. In Section 2, we introduce some auxiliary notation and state our main result. The verification theorem is proved in Section 3. In Section 4, we show that the BSPDE (1.7) has a solution if the terminal value is finite. As a byproduct, the comparison principle is established for general semi-linear degenerate BSPDEs. Finally, in Section 5, we construct a solution for our BSPDE which is subsequently proved to be the unique nonnegative one.

2 Assumptions and Main Result

Throughout this paper, we use the following notation. D and D^2 denote the first order and second order derivative operators, respectively; partial derivatives are denoted by ∂. For a Banach space U and real number $p \in [1,\infty)$, we denote by $\mathcal{L}^{\infty}_{\bar{\mathscr{F}}}(0,T;U)$ and $\mathcal{L}^{p}_{\bar{\mathscr{F}}}(0,T;U)$ the Banach spaces of all $\bar{\mathscr{P}}$-progressively measurable U-valued processes which are essentially bounded and p-th integrable, respectively. The spaces $\mathcal{L}^{p}_{\mathscr{F}}(0,T;U)$, $p \in [1,\infty]$, are defined analogously with $\bar{\mathscr{P}}$ replaced by $\mathscr{P}$. For $k \in \mathbb{N}$ and $p \in [1,\infty)$, $H^{k,p}$ is the Sobolev space of all real-valued functions ϕ whose first k derivatives belong to $L^p(\mathbb{R}^d)$, equipped with the usual Sobolev norm $\|\phi\|_{H^{k,p}}$. For $k=0$, $H^{0,p} \triangleq L^p(\mathbb{R}^d)$. Moreover,

$$H^{k,p}_{loc} \triangleq \{u;\, u\psi \in H^{k,p},\, \forall\, \psi \in C^{\infty}_c(\mathbb{R}^d)\}$$

with $C^{\infty}_c(\mathbb{R}^d)$ being the set of all the infinitely differentiable functions with compact supports on $\mathbb{R}^d$, and $\mathcal{L}^p(0,T;H^{k,p}_{loc})$ is defined as usual. For simplicity, by $u=(u_1,\ldots,u_l) \in H^{k,p}$, $l \in \mathbb{N}$, we mean $u_1,\ldots,u_l \in H^{k,p}$ and $\|u\|^p_{H^{k,p}} \triangleq \sum_{j=1}^{l} \|u_j\|^p_{H^{k,p}}$. Throughout this paper, we use $\langle\cdot,\,\cdot\rangle$ and $\|\cdot\|$ to denote the inner product and the norm in the usual Hilbert space $L^2(\mathbb{R}^d)$ (L^2 for short), respectively.

We now define what we mean by a solution to a BSPDE whose terminal value may be infinite.

Definition 2.1. A pair of processes (u,ψ) is a solution to the BSPDE

$$\begin{cases} -du_t(y) = f(t,y,Du,D^2u,\psi,D\psi)\,dt - \psi_t(y)\,dW_t, \quad (t,y) \in [0,T]\times\mathbb{R}^d; \\ \quad u_T(y) = G(y), \quad y \in \mathbb{R}^d, \end{cases}$$

if $(u,\psi) \in \mathcal{L}^2_{\mathscr{F}}(0,\tau;H^{1,2}_{loc}) \times \mathcal{L}^2_{\mathscr{F}}(0,\tau;H^{0,2}_{loc})$ for any $\tau \in (0,T)$, and for any $\varphi \in C^{\infty}_c(\mathbb{R}^d)$,

$$\langle\varphi,\,u_t\rangle = \langle\varphi,\,u_\tau\rangle + \int_t^\tau \langle\varphi,\,f(s,y,Du,D^2u,\psi,D\psi)\rangle\,ds - \int_t^\tau \langle\varphi,\,\psi_s dW_s\rangle \text{ a.s.}, \forall\, 0 \le t \le \tau < T,$$

and

$$\lim_{\tau\to T-} u_\tau(y) = G(y) \text{ a.e. in } \mathbb{R}^d \text{ a.s.}$$

We establish existence of a solution to (1.7) under the following regularity conditions on the random coefficients. The first two assumptions are standard and adopted throughout. The third is particular to the degenerate case. It allows for sufficient integrality of the derivative of the value function. It is satisfied if, for instant $\eta_T(y)$ is a constant.

Assumption 2.1. (H.1) The functions $b,\sigma,\eta,\lambda : \Omega\times[0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d\times\mathbb{R}^{d\times m}\times\mathbb{R}^1_+\times\mathbb{R}^1_+$ are $\mathscr{P}\times\mathscr{B}(\mathbb{R}^d)$-measurable and essentially bounded by $\Lambda>0$, $\gamma : \Omega\times[0,T]\times\mathbb{R}^d\times\mathcal{Z} \longrightarrow [0,+\infty]$ is $\mathscr{P}\times\mathscr{B}(\mathbb{R}^d)\times\mathscr{Z}$-measurable. Moreover, there exists a positive constant κ s.t. a.s.

$$\eta_s(y) \ge \kappa, \quad \forall\,(y,s) \in \mathbb{R}^d\times[0,T].$$

(H.2) The first derivatives of b, η, λ and the up to second-order derivatives of σ exist and are bounded by some $L>0$ uniformly for any $(\omega,t) \in \Omega\times[0,T]$.

(H.3) There exists $(T_0,p_0) \in [0,T)\times(2,\infty)$ s.t.

$$\operatorname*{ess\,inf}_{(\omega,t,y)\in\Omega\times[T_0,T]\times\mathbb{R}^d} \eta_t(y) \ge \left(1-\frac{1}{2p_0}\right) \operatorname*{ess\,sup}_{(\omega,t,y)\in\Omega\times[T_0,T]\times\mathbb{R}^d} \eta_t(y).$$

In view of (H.1) the random variable $F(\cdot,\cdot,0)$ belongs to $\mathcal{L}^{\infty}_{\mathscr{F}}(0,T;L^{\infty}(\mathbb{R}^d))$ where F is defined in (1.8). Since the L^p-theory of BSPDEs is much more complete than the L^∞-theory [7, 8, 9] it is more convenient to work with a BSPDE with driver θF where the weight function θ satisfies $\theta F(\cdot,\cdot,0) \in \mathcal{L}^p(0,T;H^{1,p})$ for any $p \in [1,\infty)$. More precisely, for a given $q>d$ we define

$$\theta(y) = (1+|y|^2)^{-q} \quad \text{for} \quad y \in \mathbb{R}^d. \tag{2.1}$$

A direct computation shows that (u, ψ) solves (1.7) if and only if $(v, \zeta) \triangleq (\theta u, \theta\psi)$ solves the BSPDE

$$
\begin{cases}
-dv_t(y) = \Bigg[\mathrm{tr}\left(\dfrac{1}{2}\sigma_t\sigma_t^* D^2 v_t(y) + D\zeta_t\sigma_t^*(y)\right) + \tilde{b}_t^* Dv_t(y) + \beta_t^*\zeta_t(y) + c_t v_t(y) \\
\qquad\qquad + \theta(y)F(t, y, \theta^{-1}(y)v_t(y))\Bigg]\, dt - \zeta_t(y)\, dW_t, \quad (t, y) \in [0, T) \times \mathbb{R}^d; \\
v_T(y) = +\infty, \quad y \in \mathbb{R}^d
\end{cases}
\tag{2.2}
$$

with

$$
\begin{aligned}
\tilde{b}_t^i(y) &\triangleq b_t^i(y) + 2q(1 + |y|^2)^{-1}\sum_{j=1}^d (\sigma_t\sigma_t^*)^{ij}(y)y^j, \quad i = 1, \ldots, d, \\
\beta_t^r(y) &\triangleq 2q(1 + |y|^2)^{-1}\sum_{j=1}^d \sigma_t^{jr}(y)y^j, \quad r = 1, \ldots, m, \\
c_t(y) &\triangleq q(1 + |y|^2)^{-1}\left(\mathrm{tr}(\sigma_t\sigma_t^*(y)) + \sum_{i=1}^d 2y^i b_t^i(y) + 2(q-1)(1 + |y|^2)^{-1}\sum_{i,j=1}^d (\sigma_t\sigma_t^*)^{ij}(y)y^i y^j\right).
\end{aligned}
$$

Let $C_{\mathscr{F}}^w([0, T]; H^{k,p})$ denote the space of all $H^{k,p}$-valued and jointly measurable processes $(X_t)_{t\in[0,T]}$ which are $\mathscr{F}$-adapted, a.s. weakly continuous with respect to t on $[0, T]$[1] and

$$
E\left[\sup_{t\in[0,T]} \|X_t\|_{H^{k,p}}^p\right] < \infty.
$$

In the sequel, we use for any positive integer k

$$
S_{\mathscr{F}}^w([0, T]; H^{k,p}) \triangleq C_{\mathscr{F}}^w([0, T]; H^{k,p}) \cap L^2(\Omega, \mathscr{F}; C([0, T]; H^{k-1,p})), \quad p \in [1, \infty].
$$

We now state our main results. The following theorem is a summary of Theorems 3.3, 5.1 and 5.5.

Theorem 2.1. *Under Conditions (H.1)–(H.3) the BSPDE* (1.7) *admits a unique nonnegative solution* (u, ψ)*, i.e., for any solution* $(\bar{u}, \bar{\psi})$ *to BSPDE* (1.7) *satisfying*

$$
(\theta\bar{u}, \theta\bar{\psi} + \sigma^* D(\theta\bar{\psi})) \in S_{\mathscr{F}}^w([0, t]; H^{1,2}) \times L_{\mathscr{F}}^2(0, t; H^{1,2}), \quad \forall\, t \in (0, T)
$$

and $\bar{u}_t(y) \geq 0$ *a.e. in* $\Omega \times [0, T) \times \mathbb{R}^d$*, we have a.s. for all* $t \in [0, T)$*,* $\bar{u}_t \geq u_t$ *a.e. in* $\mathbb{R}^d$*. If we further have* $p_0 > 2d + 2$ *and* $\theta\tilde{u} \in C_{\mathscr{F}}^w([0, t]; H^{1,p})$ *for some* $p \in (2d + 2, p_0)$*, then a.s. for all* $t \in [0, T)$*,* $\tilde{u}_t = u_t$ *a.e. in* $\mathbb{R}^d$*.*

For this solution, given any $p \in (2, p_0)$ *there exists* $\alpha \in (1, 2)$*, s.t.* $\{(T-t)^\alpha(\theta u_t, \theta\psi_t + \sigma^* D(\theta u_t))(y);\, (t, y) \in [0, T] \times \mathbb{R}^d\}$ *belongs to* $\big(S_{\mathscr{F}}^w([0, T]; H^{1,2}) \cap C_{\mathscr{F}}^w([0, T]; H^{1,p})\big) \times L_{\mathscr{F}}^2(0, T; H^{1,2})$*, and there exist two constants* $c_0 > 0$ *and* $c_1 > 0$ *s.t. a.s.*

$$
\frac{c_0}{T - t} \leq u_t \leq \frac{c_1}{T - t} \qquad \text{a.e. in } \mathbb{R}^d,\ \forall\, t \in [0, T).
$$

Moreover, if the constant p_0 *introduced in (H.3) satisfies* $p_0 > 2d + 2$*, then:*

$$
V(t, y, x) \triangleq u_t(y)x^2, \quad (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d,
$$

coincides with the value function in (1.4)*, and the optimal (feedback) control is given by*

$$
(\xi_t^*,\ \rho_t^*(z)) = \left(\frac{u_t(y_t)x_t}{\eta_t(y_t)},\ \frac{u_t(y_t)x_{t-}}{\gamma_t(z, y_t) + u_t(y_t)}\right). \tag{2.3}
$$

[1]This means that for any $f \in (H^{k,p})^*$, the dual space of $H^{k,p}$, the mapping $t \mapsto f(X_t)$ is a.s. continuous on $[0, T]$.

Remark 2.1. If all the coefficients $b, \sigma, \lambda, \eta, \gamma$ are deterministic, then the optimal control problem is Markovian and the BSPDE (1.7) reduces to the following parabolic PDE:

$$\begin{cases} -\partial_t u_t(y) = \operatorname{tr}\left(\frac{1}{2}\sigma_t\sigma_t^*(y)D^2u_t(y)\right) + b_t^*(y)Du_t(y) - \frac{|u_t(y)|^2}{\eta_t(y)} + \lambda_t(y) \\ \qquad\qquad - \int_{\mathcal{Z}} \frac{|u_t(y)|^2}{\gamma(t,y,z)+u_t(y)}\mu(dz), \quad (t,y) \in [0,T]\times\mathbb{R}^d; \\ u_T(y) = +\infty, \quad y \in \mathbb{R}^d, \end{cases} \tag{2.4}$$

where $\sigma_t\sigma_t^*$ could be degenerate. Under Conditions (H.1)–(H.3), this PDE holds in the distributional (or weak) sense. As such, our results are new even in the Markovian case.

3 The verification theorem

In order to prepare the proof of the verification theorem we now recall selected results on degenerate semi-linear BSPDEs and their connections to FBSDEs from [9]. The link between FBSDEs and BSPDEs is key. It will allow us to compute the dynamics of the process $u_t(y_t)|x_t|^2$.

Theorem 3.1. *Assume that the coefficients b and σ satisfy (H.1) and (H.2) and that $\varrho : \Omega\times[0,T]\times\mathbb{R}^d \to \mathbb{R}^m$ satisfies the same conditions as b. Let $f : \Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}$ satisfy:*

- *the partial derivatives $\partial_y f$ and $\partial_v f$ exist for any quadruple (ω,t,y,v)*
- *$f(\cdot,\cdot,0) \in \mathcal{L}^2_{\mathscr{F}}(0,T;H^{1,2})$*
- *there exists a constant $L_0 > 0$ s.t. for each (ω,t,y),*

$$|f(t,y,v_1) - f(t,y,v_2)| + |\partial_y f(t,y,v_1) - \partial_y f(t,y,v_2)| \le L_0|v_1 - v_2|, \quad \forall\ v_1, v_2 \in \mathbb{R}.$$

Then the following holds:

i) For any $G \in L^2(\Omega,\mathscr{F}_T;H^{1,2})$, the BSPDE

$$\begin{cases} -du_t(y) = \left[\operatorname{tr}\left(\frac{1}{2}\sigma_t\sigma_t^*D^2u_t + D\psi_t\sigma_t^*\right)(y) + b_t^*Du_t(y) + \varrho_t^*\left(\psi_t + \sigma_t^*Du_t\right)(y) + f(t,y,u_t)\right]dt \\ \qquad\qquad - \psi_t(y)\,dW_t, \quad (t,y)\in[0,T]\times\mathbb{R}^d; \\ u_T(y) = G(y), \quad y \in \mathbb{R}^d \end{cases} \tag{3.1}$$

admits a unique solution (u,ψ) s.t.

$$u \in S^w_{\mathscr{F}}([0,T];H^{1,2}) \quad \text{and} \quad \psi + \sigma^*Du \in L^2_{\mathscr{F}}(0,T;H^{1,2}).$$

Moreover, there exists a constant $C_2 = C_2(d,m,\Lambda,L,T,L_0)$ s.t.

$$E\sup_{t\in[0,T]}\|u_t\|^2_{H^{1,2}} + E\int_0^T \|\psi_t + \sigma_t^*Du_t\|^2_{H^{1,2}}\,dt \le C_2\,E\left[\|G\|^2_{H^{1,2}} + \int_0^T \|f(t,\cdot,0)\|^2_{H^{1,2}}\,dt\right]. \tag{3.2}$$

ii) If we further assume that $f(\cdot,\cdot,0)\in\mathcal{L}^p_{\mathscr{F}}(0,T;H^{1,p})$ and $G \in L^p(\Omega,\mathscr{F}_T;H^{1,p})$ for some $p\in[2,\infty)$, then $u \in C^w_{\mathscr{F}}([0,T];H^{1,p})$ and there exists a constant $C_p = C_p(d,m,\Lambda,L,T,L_0,p)$ s.t.

$$E\sup_{t\in[0,T]}\|u_t\|^p_{H^{1,p}} \le C_pE\left[\|G\|^p_{H^{1,p}} + \int_0^T\|f(t,\cdot,0)\|^p_{H^{1,p}}dt\right]. \tag{3.3}$$

iii) If $p > 2d+2$, then $u(t,y)$ is a.s. continuous with respect to (t,y) and it holds a.s. that

$$u(t, y_t^{s,y}) = Y_t^{s,y}, \quad \forall\, (t,y) \in [s,T] \times \mathbb{R}^d, \tag{3.4}$$

where $(y_\cdot^{s,y}, Y_\cdot^{s,y}, Z_\cdot^{s,y})$ is the solution of FBSDE:

$$\begin{cases} dy_t^{s,y} = b_t(y_t^{s,y})\, dt + \sigma_t(y_t^{s,y})\, dW_t, \quad y_s^{s,x} = y; \\ -dY_t^{s,y} = [\varrho_t^*(y_t^{s,y}) Z_t^{s,y} + f(t, y_t^{s,y}, Y_t^{s,y})]\, dt - Z_t^{s,y}\, dW_t, \quad Y_T^{s,y} = G(y_T^{s,y}). \end{cases}$$

Remark 3.1. It is worth noting that assertion (i) of the above theorem extends [9, Theorem 3.1] by replacing $C^w_{\mathscr{F}}([0,T]; H^{1,2})$ therein by $S^w_{\mathscr{F}}([0,T]; H^{1,2}) = C^w_{\mathscr{F}}([0,T]; H^{1,2}) \cap L^2(\Omega, \mathscr{F}; C([0,T]; L^2))$. This follows by applying [22, Theorem 3.2] with the Gelfand triple being realized as $(H^{-1,2}, L^2, H^{1,2})$ therein[2]. In particular, we obtain strong continuity of u in L^2. This allows us to apply the existing Itô formula for BSPDEs in what follows (see Lemma 4.1 (ii) below).

Next, we recall a result from [12]. It states that the optimal control lies in the set of controls $\mathscr{A}$ for which the corresponding state process is monotone.

Lemma 3.2. *For each admissible control pair $(\xi, \rho) \in \mathcal{L}^2_{\bar{\mathscr{F}}}(0,T) \times \mathcal{L}^2_{\bar{\mathscr{F}}}(0,T; L^2(\mathcal{Z}))$, there exists a corresponding admissible control pair $(\hat{\xi}, \hat{\rho}) \in \mathcal{L}^2_{\bar{\mathscr{F}}}(0,T) \times \mathcal{L}^2_{\bar{\mathscr{F}}}(0,T; L^2(\mathcal{Z}))$ whose cost is no more than that of (ξ, ρ) and for which the corresponding state process $x^{0,x;\hat{\xi},\hat{\rho}}$ is a.s. monotone. Moreover, there exists a constant $C > 0$ which is independent of the initial data (t,x) and the control $(\hat{\rho}, \hat{\xi})$, s.t.:*

$$E\left[\sup_{s\in[t,T]} |x_s^{0,x;\hat{\xi},\hat{\rho}}|^2 \middle| \bar{\mathscr{F}}_t \right] = |x_t^{0,x;\hat{\xi},\hat{\rho}}|^2 \leq C(T-t) E\left[\int_t^T |\hat{\xi}_s|^2\, ds \middle| \bar{\mathscr{F}}_t \right] \quad \textit{for each } t \in [0,T]. \tag{3.5}$$

We are now prepared to state and prove the verification theorem. The key assumption is that the gradient of the solution to our BSDE is sufficiently regular so that it can be represented as an FBSDE.

Theorem 3.3. *Assume (H.1)–(H.2). If (u,ψ) is a solution to the BSPDE (1.7) s.t. $\theta u \in C^w(0,t; H^{1,p}) \cap S^w(0,t; H^{1,2})$, $\forall\, t \in (0,T)$, for some $p > 2d+2$, and a.s.*

$$\frac{c_0}{T-t} \leq u_t(y) \leq \frac{c_1}{T-t}, \quad \forall\, (t,y) \in [0,T) \times \mathbb{R}^d \tag{3.6}$$

with two constants $c_0 > 0$ and $c_1 > 0$, then

$$V(t,y,x) \triangleq u_t(y) x^2, \quad \forall\, (t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d, \tag{3.7}$$

coincides with the value function (1.4). Moreover, the optimal feedback control is given by (2.3).

Proof. We first note that $u_t(y)$ is a.s. continuous with respect to $(t,y) \in [0,T) \times \mathbb{R}^d$, due to Sobolev's embedding theorem.

Second, the BSPDE (1.7) is equivalent to the BSPDE (2.2). Thus, if we take $\tau \in (0,T)$ as the terminal time and $\theta u_\tau(y)$ as the terminal condition, then this BSPDE satisfies the assumptions of Theorem 3.1 on $[0,\tau]$ in view of (3.6).

As a result, there exists a unique random field ψ s.t. $\theta\psi + \sigma^* D(\theta u) \in L^2_{\mathscr{F}}(0,\tau; H^{1,2})$ for any $\tau \in (0,T)$ and s.t. $(\theta u,\, \theta\psi)$ is a solution to:

$$\begin{cases} -dv_t(y) = \Big[\operatorname{tr}\left(\frac{1}{2}\sigma_t\sigma_t^* D^2 v_t(y) + D\zeta_t \sigma_t^*(y) \right) + b_t^* D v_t(y) + \theta \sum_{ijr} \sigma_t^{jr} \partial_{y^j} \theta^{-1} \left(\zeta_t^r + \partial_{y^i} v \sigma_t^{ir} \right)(y) \\ \qquad + \theta \left(\frac{1}{2} \sum_{ijr} \partial_{y^i y^j} \theta^{-1} \sigma_t^{ir} \sigma_t^{jr} + b_t^* D\theta^{-1} \right) v_t(y) + \theta(y) F(t, y, \theta^{-1}(y) v_t(y)) \Big]\, dt \\ \qquad - \zeta_t(y)\, dW_t, \quad (t,y) \in [0,\tau) \times \mathbb{R}^d; \\ v_\tau(y) = \theta u_\tau(y), \quad y \in \mathbb{R}^d. \end{cases}$$

[2] $H^{-1,2}$ is the dual space of $H^{1,2}$.

By assumption $p > 2d+2$. Hence, by Theorem 3.1 (iii) we also have the following BSDE representation of θu:

$$-d(\theta u_t)(y_t^{0,y}) = \Bigg[\theta\bigg(\frac{1}{2}\sum_{ijr}\partial_{y^iy^j}\theta^{-1}\sigma_t^{ir}\sigma_t^{jr} + b_t^* D\theta^{-1}\bigg)(\theta u_t)(y_t^{0,y}) + \theta\sum_{ijr}\partial_{y^j}\theta^{-1}\sigma_t^{jr}(y_t^{0,y})\big(Z_t^{0,y}\big)^r$$
$$+ \theta(y_t^{0,y})F(t,y_t^{0,y},u_t(y_t^{0,y}))\Bigg]\,dt - Z_t^{0,y}\,dW_t, \quad t\in[0,T)$$

for some adapted process $Z^{0,y}$ lying in suitable space. Applying the standard Itô formula, we obtain

$$d\theta^{-1}(y_t^{0,y}) = \left[\frac{1}{2}\mathrm{tr}\,\Big(\sigma_t\sigma_t^* D^2\theta^{-1}(y_t^{0,y})\Big) + b_t^* D\theta^{-1}(y_t^{0,y})\right]dt + (D\theta^{-1})^*\sigma_t(y_t^{0,y})\,dW_t,$$

and further,

$$-du_t(y_t^{0,y}) = F(t,y_t^{0,y},u_t(y_t^{0,y}))\,dt - \left[\theta^{-1}(y_t^{0,y})Z_t^{0,y} + \theta u_t(D\theta^{-1})^*\sigma_t(y_t^{0,y})\right]\,dW_t, \quad t\in[0,T).$$

Then the stochastic differential equation for $u_t(y_t^{0,y})|x_t^{0,x;\xi,\rho}|^2$ follows immediately from an application of the standard Itô formula again. Using Lemma 3.2 one can now apply the exact same arguments as in the proof of [12, Theorem 3.1] to deduce that:

$$u_t(y_t^{0,y})|x_t^{0,x;\xi,\rho}|^2 \le J(t,x_t^{0,x;\xi,\rho},y_t^{0,y};\xi,\rho) \quad \text{for any pair} \quad (\xi,\rho)\in\mathscr{A}$$

and that the control (ξ^*,ρ^*) is admissible and satisfies the above inequality with equality. $\square$

4 A Comparison Principle and Solutions of Truncated BSPDEs

In view of the verification theorem it remains to prove that the BSPDE (1.7) has a solution with particular properties. Since Theorem 3.1 does not apply to this equation, we first establish a comparison principle for degenerate BSPDEs, from which we then deduce that a solution to (1.7) can be obtained as the limit of a sequence of solutions to BSPDEs with finite terminal values. The comparison principle is essentially a corollary of the following Itô formula for the square norm of the positive part of the solution to a BSPDE.

Lemma 4.1. *Let $u\in\mathcal{L}^2_{\mathscr{F}}(0,T;H^{1,2})$, $G\in L^2(\Omega,\mathscr{F}_T;L^2(\mathbb{R}^d))$, $\zeta,f,g\in\mathcal{L}^2_{\mathscr{F}}(0,T;L^2(\mathbb{R}^d))$, and assume that for arbitrary $\varphi\in C_c^\infty(\mathbb{R}^d)$,*

$$\langle u_t,\,\varphi\rangle = \langle G,\,\varphi\rangle + \int_t^T (\langle f_s,\,\varphi\rangle - \langle D\varphi,\,g_s\rangle)\,ds - \int_t^T \langle\varphi,\,\zeta_s\,dW_s\rangle, \quad t\in[0,T] \;\; a.s.$$

Then the following holds:

(i) By [22, Theorem 3.2], u admits a version (again denoted by u) lying in $L^2(\Omega,\mathscr{F};C([0,T];H^{0,2}))$. Due to [23, Theorem 3.10 and Corollary 3.11] it also holds that:

$$\|u_t\|^2 + \int_t^T\|\zeta_s\|^2\,ds = \|G\|^2 + \int_t^T 2\left(\langle u_s,\,f_s\rangle - \langle Du_s,\,g_s\rangle\right)ds + \int_t^T 2\langle u_s,\,\zeta_s\,dW_s\rangle, \quad t\in[0,T] \;\; a.s.$$

and

$$\|u_t^+\|^2 + \int_t^T\|\zeta_s 1_{\{u>0\}}\|^2\,ds = \|G^+\|^2 + \int_t^T 2\left(\langle u_s^+,\,f_s\rangle - \langle Du_s^+,\,g_s\rangle\right)ds + \int_t^T 2\langle u_s^+,\,\zeta_s\,dW_s\rangle, \quad t\in[0,T] \;\; a.s.$$

(ii) If we assume $u \in L^2(\Omega, \mathscr{F}; C([0,T]; H^{0,2}))$ and $f = \bar{f} + h$ with $(\bar{f}, h) \in \mathcal{L}^2_{\mathscr{F}}(0,T; L^2(\mathbb{R}^d)) \times \mathcal{L}^1_{\mathscr{F}}(0,T; L^1(\mathbb{R}^d))$ and $h_t(y)u_t^+(y) \le 0$ a.e. in $\Omega \times [0,T] \times \mathbb{R}^d$, then by [12, Lemma A.3] a.s.

$$\|u_t^+\|^2 + \int_t^T \|\zeta_s 1_{u>0}\|^2 \, ds \le \|G^+\|^2 + 2\int_t^T \left\{ \langle u_s^+,\, \bar{f}_s\rangle - \langle Du_s^+,\, g_s\rangle \right\} ds + 2\int_t^T \langle u_s^+,\, \zeta_s \, dW_s\rangle, \quad t \in [0,T]. \quad (4.1)$$

We are now ready to state and prove a comparison principle for nonlinear degenerate BSPDEs.

Proposition 4.2. *Assume that the coefficients b, σ, ϱ and two pairs of terminal value and driver (G,f), (G',f') satisfy the conditions of Theorem 3.1. Let (u,ψ) and (u',ψ') be the solutions of BSPDE (3.1) with (G,f) and (G',f'), respectively, and $(u, \psi + \sigma^* Du)$, $(u', \psi' + \sigma^* Du') \in \mathcal{L}^2(0,T; H^{1,2}) \times \mathcal{L}^2(0,T; H^{1,2})$. If*

$$f(\omega,t,y,u_t(y)) \le f'(\omega,t,y,u_t(y)) \quad \text{and} \quad G(\omega,y) \le G'(\omega,y), \quad \text{a.e. in } \Omega\times[0,T]\times\mathbb{R}^d,$$

then we have a.s.

$$u \le u' \qquad \text{a.e. in } \mathbb{R}^d, \quad \forall t \in [0,T].$$

Proof. We put $(\bar{u}, \bar{\psi}) = (u - u', \psi - \psi')$, $\bar{f} = f' - f$ and $\bar{G} = G' - G$. Then $\bar{G}^+ \equiv 0$, $\bar{f}(t,y,u_t(y)) \ge 0$, and uniform Lipschitz continuity of f' yields a constant $L_1 < \infty$ s.t.

$$|\langle \bar{u}_t^+, f'(u_t) - f'(u_t')\rangle| \le L_1 \langle \bar{u}_t^+, \bar{u}_t^+\rangle, \quad \forall t \in [0,T].$$

Since b, σ and ϱ are bounded with bounded first order derivatives and $\bar{u} \in H^{1,2}$ integration by parts yields a constant $L_2 < \infty$ s.t.

$$|\langle \bar{u}_t^+, (b_t^* + \varrho^*\sigma^*) D\bar{u}_t\rangle| \le L_2 \langle \bar{u}_t^+, \bar{u}_t^+\rangle, \quad \forall t \in [0,T].$$

Let us denote the entries of the diffusion matrix σ_s by σ_s^{jr} and the entries of the vector $\bar{\psi}_s$ by $\bar{\psi}_s^r$. Then integration by parts gives:

$$\sigma_s^{jr}\partial_{y_j}\bar{\psi}_s^r = \partial_{y_j}\left(\sigma_s^{jr}\bar{\psi}_s^r\right) - \partial_{y_j}\sigma_s^{jr}\bar{\psi}_s^r \quad \text{and} \quad \sigma_s^{ir}\sigma_s^{jr}\partial_{y_iy_j}\bar{u}_s = \partial_{y_i}\left(\sigma_s^{ir}\sigma_s^{jr}\partial_{y_j}\bar{u}_s\right) - \partial_{y_i}\left(\sigma_s^{ir}\sigma_s^{jr}\right)\partial_{y_j}\bar{u}_s.$$

Hence, assertion (i) of Lemma 4.1 yields a constant $L_3 < \infty$ s.t. (applying the summation convention):

$$\begin{aligned}
&E\|\bar{u}_t^+\|^2 \\
&\le E\bigg[\int_t^T 2\langle \bar{u}_s^+,\, -\frac{1}{2}\partial_{y_i}\left(\sigma_s^{ir}\sigma_s^{jr}\right)\partial_{y_j}\bar{u}_s^+ - \partial_{y_j}\sigma_s^{jr}\bar{\psi}_s^r + \varrho_s^r\bar{\psi}_s^r + L_3\bar{u}_s^+\rangle\, ds \\
&\quad + \int_t^T \langle \partial_{y_j}\bar{u}_s^+,\, \sigma_s^{ir}\sigma_s^{jr}\partial_{y_i}\bar{u}_s^+ - 2\sigma_s^{jr}\left(\bar{\psi}_s^r + \sigma_s^{ir}\partial_{y_i}\bar{u}_s^+\right)\rangle\, ds - \int_t^T \|\bar{\psi}_s 1_{\{u>u'\}}\|^2\, ds\bigg] \\
&= E\bigg[\int_t^T 2\langle \bar{u}_s^+,\, \frac{1}{4}\partial_{y_iy_j}\left(\sigma_s^{ir}\sigma_s^{jr}\right)\bar{u}_s^+ + \left(\varrho_s^r - \partial_{y_j}\sigma_s^{jr}\right)\left(\bar{\psi}_s^r + \sigma_s^{ir}\partial_{y_i}\bar{u}_s^+\right) + \frac{1}{2}\partial_{y_i}\left(\varrho_s^r\sigma_s^{ir} - \sigma_s^{ir}\partial_{y_j}\sigma_s^{jr}\right)\bar{u}_s^+ \\
&\quad + L_3\bar{u}_s^+\rangle\, ds - \int_t^T \left\|\bar{\psi}_s 1_{\{u>u'\}} + \sigma_s^* D\bar{u}_s^+\right\|^2\, ds\bigg] \\
&\le E\bigg[C\int_t^T \langle \bar{u}_s^+,\, \bar{u}_s^+ + \left|\bar{\psi}_s^r 1_{\{u>u'\}} + \sigma_s^* D\bar{u}_s^+\right|\rangle\, ds - \int_t^T \left\|\bar{\psi}_s 1_{\{u>u'\}} + \sigma_s^* D\bar{u}_s^+\right\|^2\, ds\bigg] \qquad \text{(by (H.1), (H.2))} \\
&\le E\bigg[C\int_t^T \left\|\bar{u}_s^+\right\|^2\, ds - \frac{1}{2}\int_t^T \left\|\bar{\psi}_s 1_{\{u>u'\}} + \sigma_s^* D\bar{u}_s^+\right\|^2\, ds\bigg].
\end{aligned}$$

Thus, an application of Gronwall's inequality leads to:

$$\sup_{t\in[0,T]} E\|\bar{u}_t^+\|^2 + E\left[\int_0^T \|\bar{\psi}_s 1_{\{u>u'\}} + \sigma_s^* D\bar{u}_s^+\|^2\, ds\right] = 0.$$

□

Next, we show that BSPDE (2.2) has a unique solution if the terminal value is finite. To this end, let $N \in \mathbb{N}$ and

$$\hat{F}(t, y, \phi(y)) \triangleq F(t, y, |\phi(y)|)$$

for any $(t, y, \phi) \in \mathbb{R}_+ \times \mathbb{R}^d \times L^0(\mathbb{R}^d)$, and consider then the family of BSPDEs:

$$\begin{cases} -dv_t^N(y) = \Big[\operatorname{tr}\Big(\frac{1}{2}\sigma_t\sigma_t^* D^2 v_t^N(y) + D\zeta_t^N \sigma_t^*(y) \Big) + \tilde{b}_t^* Dv_t^N(y) + \beta_t^* \zeta_t^N(y) + c_t v_t^N(y) \\ \qquad\qquad + \theta(y)\hat{F}(t, y, \theta^{-1}(y) v_t^N(y)) \Big] dt - \zeta_t^N(y)\, dW_t, \quad (t, y) \in [0, T] \times \mathbb{R}^d; \\ v_T^N(y) = N\theta(y), \quad y \in \mathbb{R}^d. \end{cases} \tag{4.2}$$

We note that by introducing the function $\hat{F}$, the bounded solution v^N to (4.2) is nonnegative by the comparison principle and thus $F(t, y, \theta^{1} v_t^N) \equiv \hat{F}(t, y, \theta^{-1} v^N)$.

Proposition 4.3. *Assume (H.1) and (H.2). For each $N \in \mathbb{N}$ and $p \in [2, \infty)$, BSPDE (4.2) has a unique solution (v^N, ζ^N) with*

$$(v^N, \zeta^N + \sigma^* Dv^N) \in \big(S_{\mathscr{F}}^w([0, T]; H^{1,2}) \cap C_{\mathscr{F}}^w([0, T]; H^{1,p}) \big) \times L_{\mathscr{F}}^2(0, T; H^{1,2}),$$

s.t. $\theta^{-1} v^N \in \mathcal{L}_{\mathscr{F}}^\infty(0, T; L^\infty(\mathbb{R}^d))$ and for arbitrary $\varphi \in C_c^\infty(\mathbb{R}^d)$:

$$\begin{aligned} \langle \varphi,\, v_t^N \rangle = \langle \varphi,\, N\theta \rangle + \int_t^T \Big\langle \varphi,\, \operatorname{tr}\Big(\frac{1}{2}\sigma_s\sigma_s^* D^2 v_s^N + D\zeta_s^N \sigma_s^* \Big) + \tilde{b}_s^* Dv_s^N + c_s v_s^N + \beta_s^* \zeta_s^N + \theta \hat{F}(s, \theta^{-1} v_s^N) \Big\rangle\, ds \\ - \int_t^T \langle \varphi,\, \zeta_s^N \rangle\, dW_s \quad a.s., \forall\, 0 \le t \le T. \end{aligned}$$

Proof. To prove existence of a solution, we truncate the quadratic term in $\hat{F}$. More precisely, let

$$h(x) = \begin{cases} \tilde{c} e^{\frac{1}{x^2 - 1}} & \text{if } |x| \le 1; \\ 0 & \text{otherwise,} \end{cases} \quad \text{with} \quad \tilde{c} \triangleq \left(\int_{-1}^1 e^{\frac{1}{x^2-1}}\, dx \right)^{-1}$$

and

$$h_1(x) = \int_{-4}^4 h(x - r)\, dr, \quad h_M(x) = h_1\Big(\frac{x}{M}\Big) \quad \text{for } x \in \mathbb{R},\ M \in \mathbb{N}.$$

For each $M \in \mathbb{N}$, we know from Theorem 3.1 that the BSPDE

$$\begin{cases} -dv_t^{N,M}(y) = \Big[\operatorname{tr}\Big(\frac{1}{2}\sigma_t\sigma_t^* D^2 v_t^{N,M} + D\zeta_t^{N,M}\sigma_t^* \Big)(y) + \tilde{b}_t^* Dv_t^{N,M}(y) + \beta_t^* \zeta_t^{N,M}(y) + c_t v_t^{N,M}(y) + \theta\lambda_t(y) \\ \qquad\qquad - \int_{\mathcal{Z}} \frac{\theta^{-1}|v_t^{N,M}|^2(y)}{\gamma_t(y, z) + |\theta^{-1} v_t^{N,M}(y)|} \mu(dz) - \frac{h_M(\theta^{-1} v_t^{N,M}(y)) |v_t^{N,M}(y)|^2}{\eta_t(y)} \Big] dt \\ \qquad\qquad - \zeta_t^{N,M}(y)\, dW_t, \quad (t, y) \in [0, T] \times \mathbb{R}^d; \\ v_T^{N,M}(y) = N\theta(y), \quad y \in \mathbb{R}^d \end{cases} \tag{4.3}$$

has a unique solution $(v^{N,M}, \zeta^{N,M})$ with

$$(v^{N,M}, \zeta^{N,M} + \sigma^* Dv^{N,M}) \in \big(S_{\mathscr{F}}^w([0, T]; H^{1,2}) \cap C_{\mathscr{F}}^w([0, T]; H^{1,p}) \big) \times L_{\mathscr{F}}^2(0, T; H^{1,2}).$$

Changing the coefficients (λ, γ, M) in the above BSPDE to $(\Lambda, +\infty, 0)$ we get a new equation. For this equation, one readily checks that

$$(\hat{v}_t(y), 0) \triangleq (\theta(y)\,(N + \Lambda(T - t))\,, 0)$$

is a solution, and the comparison principle stated in Proposition 4.2 yields:

$$0 \le v_t^{N,M} \le \hat{v}_t \quad \text{a.e. in } \mathbb{R}^d,\ \forall\, t \in [0, T], \text{ a.s.}$$

Thus, for all sufficiently large $M \in \mathbb{N}$, we see that $v^{N,M}$ and hence the pair $(v^{N,M}, \zeta^{N,M})$ is independent of M and the pair is in fact a solution to (4.2). As for the uniqueness of solutions, it follows from a similar argument. □

Since the solution of (4.3) is bounded, we deduce from Proposition 4.2 a comparison principle for the BSPDE (4.2).

Corollary 4.4. *Assume that the coefficients of the BSPDE* (4.2) *satisfy Conditions (H.1)-(H.2) and denote the solution by* (v^N, ζ^N)*. Let* $(\tilde{\lambda}, \tilde{\gamma}, \tilde{\eta})$ *be another set of coefficients which satisfies the same conditions as* (λ, γ, η)*. Let* $G \in L^2(\Omega, \mathscr{F}_T; H^{1,2})$ *and*

$$(\tilde{v}, \tilde{\zeta}) \in S^w_{\mathscr{F}}([0,T]; H^{1,2}) \times L^2_{\mathscr{F}}(0,T; L^2)$$

with $\theta^{-1}\tilde{v} \in \mathcal{L}^\infty_{\mathscr{F}}(0,T; L^\infty(\mathbb{R}^d))$ *be a solution to the BSPDE:*

$$\begin{cases} -d\tilde{v}_t(y) = \Big[\operatorname{tr}\Big(\frac{1}{2}\sigma_t\sigma_t^* D^2\tilde{v}_t(y) + D\tilde{\zeta}_t\sigma_t^*(y)\Big) + \tilde{b}_t^* D\tilde{v}_t(y) + \beta_t^*\tilde{\zeta}_t(y) + c_t\tilde{v}_t(y) + \theta\tilde{\lambda}_t(y) \\ \qquad\qquad - \displaystyle\int_{\mathcal{Z}} \frac{\theta^{-1}(y)|\tilde{v}_t(y)|^2}{\tilde{\gamma}_t(y,z) + \theta^{-1}|\tilde{v}_t(y)|}\mu(dz) - \frac{\theta^{-1}(y)\,|\tilde{v}_t(y)|^2}{\tilde{\eta}_t(y)}\Big]\,dt - \tilde{\zeta}_t(y)\,dW_t; \\ \tilde{v}_T(y) = G(y), \quad y \in \mathbb{R}^d. \end{cases} \tag{4.4}$$

If $G \geq N\theta$, $\tilde{\lambda} \geq \lambda$, $\tilde{\gamma} \geq \gamma$ *and* $\tilde{\eta} \geq \eta$, *then a.s.*

$$\tilde{v}_t(y) \geq v_t^N(y) \quad a.e. \text{ in } \mathbb{R}^d, \quad \forall t \in [0,T].$$

Moreover, the inequality also holds with all "$\geq$" replaced by "$\leq$" in above statement.

5 Existence and uniqueness of a nonnegative solution to (1.7)

5.1 Existence

In this section we establish the existence of a solution for BSPDE (1.7). More precisely, we prove the following theorem.

Theorem 5.1. *Under Conditions (H.1)-(H.3), for any* $p \in [2, p_0)$ *BSPDE* (1.7) *has a solution* (u, ψ) *s.t. for some* $\alpha \in (1,2)$, $\{(T-t)^\alpha(\theta u_t, \theta\psi_t + \sigma^* D(\theta u_t))(y); (t,y) \in [0,T] \times \mathbb{R}^d\}$ *belongs to*

$$\big(S^w_{\mathscr{F}}([0,T]; H^{1,2}) \cap C^w_{\mathscr{F}}([0,T]; H^{1,p})\big) \times L^2_{\mathscr{F}}(0,T; H^{1,2}),$$

and a.s.

$$\frac{c_0}{T-t} \leq u_t(y) \leq \frac{c_1}{T-t} \quad a.e. \text{ in } \mathbb{R}^d, \quad \forall t \in [0,T). \tag{5.1}$$

with two constants $c_0 > 0$ *and* $c_1 > 0$.

The proof is split into several steps. The idea is to identify a solution as an accumulation point of a convex combination subsequence of the sequence of solutions to the BSPDE (4.2). For $p_1 \in [2, p_0)$ this BSPDE has a unique solution (v^N, ζ^N), due to Proposition 4.3. The sequence $\{v^N\}$ increases in N, due to Corollary 4.4 and hence converges to some limit v. We show below that v satisfies the growth condition (5.1).

The challenge is to establish the existence of a norm-bound and hence an accumulation point of the sequence $\{Dv^N\}$. Due to the singularity of the terminal value such an estimate cannot be obtained from the BSPDE (2.2) directly. Instead we show in Section 5.1.1 that under (H.3) the norm of the gradients essentially depends mainly on η. We then establish our existence of solutions result in Section 5.1.2.

As a first step toward the proof of our existence result, we show that the limiting function v satisfies the growth condition (5.1). To this end, we replace the coefficients (λ, γ, η) by their lower bound $(0, 0, \kappa)$ and upper bound $(\Lambda, +\infty, \Lambda)$, respectively, deduce from Proposition 4.3 that the resulting BSPDEs have unique solutions, verify by direct computation that respective solutions are given by $(\hat{u}^N, 0)$ and $(\tilde{u}^N, 0)$, respectively, where

$$\hat{u}_t^N(y) \triangleq \frac{\kappa\mu(\mathcal{Z})\theta(y)}{\left(1+\frac{\kappa_1\mu(\mathcal{Z})}{N}\right)e^{\mu(\mathcal{Z})(T-t)}-1} \quad \text{and} \quad \tilde{u}_t^N(y) \triangleq \frac{2\Lambda\theta(y)}{1-\frac{N-\Lambda}{N+\Lambda}\cdot e^{-2(T-t)}} - \Lambda\theta(y),$$

and then apply the comparison principle to conclude that a.s.:

$$\hat{u}_t^N(y) \le v_t^N(y) \le \tilde{u}_t^N(y) \quad \text{a.e. in } \mathbb{R}^d, \quad \forall t \in [0,T).$$

Since

$$\tilde{u}_t^N \le \frac{2\Lambda\theta(y)}{1-\frac{N-\Lambda}{N+\Lambda}\cdot e^{-2(T-t)}} = \frac{2\Lambda\theta(y)e^{2(T-t)}}{e^{2(T-t)}-\frac{N-\Lambda}{N+\Lambda}} \le \frac{2\Lambda\theta(y)e^{2(T-t)}}{1+2(T-t)-\frac{N-\Lambda}{N+\Lambda}} \le \frac{\theta(y)e^{2T}}{\frac{1}{N+\Lambda}+\frac{T-t}{\Lambda}}$$

we see that

$$\frac{\kappa\mu(\mathcal{Z})\theta(y)}{\left(1+\frac{\kappa\mu(\mathcal{Z})}{N}\right)e^{\mu(\mathcal{Z})(T-t)}-1} \le v_t^N(y) \le \frac{\theta(y)e^{2T}}{\frac{1}{N+\Lambda}+\frac{T-t}{\Lambda}} \quad \text{a.e. in } \mathbb{R}^d. \tag{5.2}$$

5.1.1 Gradient estimate

Our next goal is to establish a uniform bound for the sequence $\{Dv^N\}$ in $H^{0,p}$. As a byproduct we obtain a bound for the sequence $\{\zeta^N + \sigma^* Dv^N\}$ in $\mathcal{L}^2(0,T;H^{1,2})$. The bound given in Theorem 3.1 (ii) depends on the Lipschitz constant of the driver of the BSPDE. In our case, this means that it depends on the function v^N, due to the quadratic dependence of the driver on v^N. The following corollary provides a better estimate. The estimates in Theorem 3.1 are obtained by applying Itô formulas directly (see [8, 9]); hence we can derive the estimates as well from the monotonicity of the drift for BSPDE (3.1) instead of the Lipschitz condition. The detailed proof is omitted; it is standard but cumbersome.

Corollary 5.2. *Assume the same hypothesis of Theorem 3.1 with*

$$f(\cdot,\cdot,0) \in \mathcal{L}^p_{\mathscr{F}}(0,T;H^{1,p}) \cap \mathcal{L}^2_{\mathscr{F}}(0,T;H^{1,2}) \text{ and } G \in L^p(\Omega,\mathscr{F}_T;H^{1,p}) \cap L^2(\Omega,\mathscr{F}_T;H^{1,2})$$

for some $p \in [2,\infty)$. Let (u,ϕ) be the solution of BSPDE (3.1) in Theorem 3.1. If there exist constant L_1 and function $g \in \mathcal{L}^p_{\mathscr{F}}(0,T;H^{0,p}) \cap \mathcal{L}^2_{\mathscr{F}}(0,T;L^2)$ s.t. a.e. in $\Omega \times [0,T] \times \mathbb{R}^d$,

$$\begin{aligned} &u_s(y)f(s,y,u_s(y)) + \sum_{i=1}^d \partial_{y^i}u_s(y)\left(\partial_{y^i} + \partial_{y^i}u_s(y)\partial_u\right)f(s,y,u_s(y)) \\ &\le |g_s(y)|^2 + L_1\Big(|u_s(y)|^2 + \sum_{i=1}^d |\partial_{y^i}u_s(y)|^2\Big), \end{aligned} \tag{5.3}$$

then we have

$$E\sup_{t\in[0,T]}\|u_t\|^p_{H^{1,p}} \le C'_p E\left[\|G\|^p_{H^{1,p}} + \int_0^T \|g_t\|^p_{H^{0,p}}dt\right],$$

and

$$E\sup_{t\in[0,T]}\|u_t\|^2_{H^{1,2}} + E\int_0^T \|\psi_t + \sigma_t^* Du_t\|^2_{H^{1,2}}\,dt \le C'_2\,E\left[\|G\|^2_{H^{1,2}} + \int_0^T \|g_t\|^2 dt\right]$$

with the constants $C'_2 = C'_2(d,m,\Lambda,L,T,L_1)$ and $C'_p = C'_p(d,m,\Lambda,L,T,L_1,p)$, which are independent of the Lipchitz constant L_0.

We now proceed with the gradient estimate. Since we are mainly interested in the behavior of the gradient near the terminal time, we put
$$\kappa_1 \triangleq \operatorname*{ess\,inf}_{(\omega,t,y)\in\Omega\times[T_0,T]\times\mathbb{R}^d} \eta_t(y)$$
and notice that (5.2) holds with κ replaced by κ_1 on $[T_0,T]$.

Lemma 5.3. *Recall the constant p_0 introduced in (H.3), let $\alpha_0 \triangleq 1-\frac{1}{2p_0}$, and choose $\alpha_1,\alpha_2\in(1,\infty)$ and $p_1\in[2,p_0)$ s.t.*
$$2\alpha_0=\alpha_1\alpha_2 \quad \text{and} \quad (2-\alpha_2)p_1<1.$$
Let $T_1\in[T_0,T)$ and $N_0>2\Lambda+\kappa\mu(\mathcal{Z})$ s.t.
$$\left(1+\frac{\kappa_1\mu(\mathcal{Z})}{N_0}\right)e^{\mu(\mathcal{Z})(T-T_1)}<\alpha_1,$$
and for each $N>N_0$, set
$$\delta^N \triangleq \left(1+\frac{\kappa_1\mu(\mathcal{Z})}{N}\right)e^{\mu(\mathcal{Z})(T-T_1)}.$$
Then the sequence
$$(Q^N_t,\xi^N_t)\triangleq\left(\frac{\kappa_1}{N}+\delta^N(T-t)\right)^{\alpha_2}(v^N_t,\zeta^N_t), \quad t\in[0,T],$$
satisfies
$$\sup_{N>N_0}\left\{E\Big[\sup_{t\in[T_1,T]}\left(\|Q^N_t\|^2_{H^{1,2}}+\|Q^N_t\|^{p_1}_{H^{1,p_1}}\right)\Big]+\|\sigma^*DQ^N+\xi^N\|^2_{\mathcal{L}^2(T_1,T;H^{1,2})}\right\}<\infty.$$

Proof. A direct computation shows that the sequence $\{(Q^N,\xi^N)\}$ is a solution to the BSPDE:
$$\begin{cases} -dQ^N_t(y)=\Big[\operatorname{tr}\left(\frac{1}{2}\sigma_t\sigma^*_tD^2Q^N_t(y)+D\xi^N_t\sigma^*_t(y)\right)+\tilde{b}^*_tDQ^N_t(y)+\beta^*_t\xi^N_t(y)+c_tQ^N_t(y) \\ \qquad +\left(\frac{\kappa_1}{N}+\delta^N(T-t)\right)^{\alpha_2}\left(\theta\lambda_t(y)-\int_{\mathcal{Z}}\frac{\theta^{-1}|v^N_t(y)|^2}{\gamma_t(y,z)+\theta^{-1}|v^N_t(y)|}\mu(dz)-\frac{\theta^{-1}\left|v^N_t(y)\right|^2}{\eta_t(y)}\right) \\ \qquad +\alpha_2\delta^N\left(\frac{\kappa_1}{N}+\delta^N(T-t)\right)^{\alpha_2-1}v^N_t(y)\Big]\,dt-\xi^N_t(y)\,dW_t, \quad (t,y)\in[0,T]\times\mathbb{R}^d; \\ Q^N_T(y)=N^{1-\alpha_2}\theta(y) \quad \text{for } y\in\mathbb{R}^d. \end{cases} \tag{5.4}$$

The assertion follows if we can show that this BSPDE satisfies the the conditions of Corollary 5.2 on $[T_1,T]$ with some constant $L_1<\infty$ independent of N and a function $g^N\in\mathcal{L}^p(T_1,T;H^{0,p})$ which satisfies
$$\sup_{N>N_0}\|g_N\|_{\mathcal{L}^p_{\mathscr{F}}(T_1,T;H^{0,p})}<\infty \quad \text{for } p\in\{2,p_1\}.$$

To obtain the desired result it suffices to estimate $\partial_{y^i}Q^N_t(y)\partial_{y^i}(f^2_N-f^1_N)(s,y)$ where
$$f^1_N(t,y)\triangleq\left(\frac{\kappa_1}{N}+\delta^N(T-t)\right)^{\alpha_2}\frac{\theta^{-1}\left|v^N_t(y)\right|^2}{\eta_t(y)} \quad \text{and} \quad f^2_N(t,y)\triangleq\alpha_2\delta^N\left(\frac{\kappa_1}{N}+\delta^N(T-t)\right)^{\alpha_2-1}v^N_t(y).$$

To this end, notice that $\delta^N<\alpha_1$ and that $e^x\le 1+xe^x$ for any $x\ge 0$. Hence, for each $N>N_0$, each $t\in[T_0,T)$ and almost every $y\in\mathbb{R}^d$ one has:
$$\begin{aligned} &\alpha_2\delta^N\left(\frac{\kappa_1}{N}+\delta^N(T-t)\right)^{-1}-\frac{2\theta^{-1}v^N(y)}{\eta_t(y)} \\ \le{}&\alpha_1\alpha_2\left(\frac{\kappa_1}{N}+\delta^N(T-t)\right)^{-1}-\frac{2\kappa_1\mu(\mathcal{Z})/\eta_t(y)}{\left(1+\frac{\kappa_1\mu(\mathcal{Z})}{N}\right)e^{\mu(\mathcal{Z})(T-t)}-1} \end{aligned}$$

$$
\begin{aligned}
&\le \alpha_1\alpha_2\left(\frac{\kappa_1}{N}+\delta^N(T-t)\right)^{-1} - 2\alpha_0\mu(\mathcal{Z})\left(\left(1+\frac{\kappa_1\mu(\mathcal{Z})}{N}\right)e^{\mu(\mathcal{Z})(T-t)}-1\right)^{-1} \\
&\le 0.
\end{aligned}
$$

Thus, for any $t \in [T_1, T]$, we have

$$
\begin{aligned}
&\partial_{y^i}Q_t^N(y)\partial_{y^i}(f_N^2 - f_N^1)(s,y) \\
&= \partial_{y^i}Q_t^N(y)\alpha_2\delta^N\left(\frac{\kappa_1}{N}+\delta^N(T-t)\right)^{-1}\partial_{y^i}Q_t^N(y) - \partial_{y^i}Q_t^N(y)\frac{2\theta^{-1}v_t^N(y)\partial_{y^i}Q_t^N}{\eta_t(y)} + \partial_{y^i}Q_t^N(y)f_N^3(t,y) \\
&= \left|\partial_{y^i}Q_t^N(y)\right|^2\left(\alpha_2\delta^N\left(\frac{\kappa_1}{N}+\delta^N(T-t)\right)^{-1} - \frac{2\theta^{-1}v_t^N(y)}{\eta_t(y)}\right) + \partial_{y^i}Q_t^N(y)f_N^3(t,y) \\
&\le \left|\partial_{y^i}Q_t^N(y)\right|^2 + \left|f_N^3(t,y)\right|^2 \quad \text{a.e. in } \mathbb{R}^d \text{ a.s.,}
\end{aligned}
$$

where

$$
f_N^3(t,y) \triangleq \left(\frac{\kappa_1}{N}+\delta^N(T-t)\right)^{\alpha_2}\frac{\theta^{-1}\left|v_t^N(y)\right|^2}{\eta_t(y)}\left(\frac{\partial_{y^i}\eta_t(y)}{\eta_t(y)} - \partial_{y^i}\theta^{-1}(y)\theta(y)\right).
$$

In view of the upper bound in (5.2) there exists a constant $C < \infty$ s.t.

$$
\theta^{-1}|v_t^N(y)|^2 \le C\left(\frac{1}{\frac{1}{N}+T-t}\right)^2.
$$

Since $(2-\alpha_2)p_1 < 1$, one therefore has for $p \in \{2, p_1\}$ that

$$
\sup_{N > N_0}\|f_N^3\|_{\mathcal{L}^p_{\mathscr{F}}(T_1,T;H^{0,p})} < \infty.
$$

This proves the assertion. □

Corollary 5.4. *The previous lemma, along with an application of Theorem 3.1 to the time interval $[0, T_1]$, leads to the desired gradient estimate:*

$$
\sup_{N>N_0}\left\{E\Big[\sup_{t\in[0,T]}\left(\|Q_t^N\|^2_{H^{1,2}} + \|Q_t^N\|^{p_1}_{H^{1,p_1}}\right)\Big] + \|\sigma^* DQ^N + \xi^N\|^2_{\mathcal{L}^2(0,T;H^{1,2})}\right\} < \infty. \tag{5.5}
$$

5.1.2 The solution

The estimate (5.5) allows us to extract a subsequence (Q^{N_k}, ξ^{N_k}) s.t. Q^{N_k} converges to Q weakly in $\mathcal{L}^p(0,T;H^{1,p})$ as well as weak-star in $\mathcal{L}^\infty(0,T;H^{1,p})$ for any $p \in \{2, p_1\}$, and $(\xi^{N_k}, \xi^{N_k} + \sigma^* DQ^{N_k})$ converges weakly to $(\xi, \xi + \sigma^* DQ)$ in $\mathcal{L}^2(0,T;L^2) \times \mathcal{L}^2(0,T;H^{1,2})$. Since $\{v^N\}$ increases to v a.e. in $\mathbb{R}^d$ for all $t \in [0,T]$, passing to the limit we get

$$
Q_t(y) = e^{\alpha_2\mu(\mathcal{Z})(T-T_1)}(T-t)^{\alpha_2}v_t(y).
$$

Mazur's Lemma allows us to choose a sequence of convex combinations of $(Q^{N_k}, \xi^{N_k}, \xi^{N_k} + \sigma^* DQ^{N_k})$ which converges strongly in corresponding spaces. Therefore, it is easy to check that (Q, ξ) solves:

$$
\begin{cases}
-dQ_t(y) = \Big[\mathrm{tr}\left(\frac{1}{2}\sigma_t\sigma_t^* D^2 Q_t(y) + D\xi_t^N\sigma_t^*(y)\right) + \tilde{b}_t^* DQ_t(y) + \beta_t^*\xi_t(y) + c_t Q_t(y) \\
\qquad + e^{\alpha_2\mu(\mathcal{Z})(T-T_1)}(T-t)^{\alpha_2}\left(\theta\lambda_t(y) - \int_{\mathcal{Z}}\frac{\theta^{-1}|v_t(y)|^2}{\gamma_t(y,z)+\theta^{-1}|v_t(y)|}\mu(dz) - \frac{\theta^{-1}\left|v_t(y)\right|^2}{\eta_t(y)}\right) \\
\qquad + \alpha_2 e^{\alpha_2\mu(\mathcal{Z})(T-T_1)}(T-t)^{\alpha_2-1}v_t(y)\Big]\,dt - \xi_t(y)\,dW_t, \quad (t,y) \in [T_1,T]\times\mathbb{R}^d; \\
Q_T(y) = 0, \quad y \in \mathbb{R}^d.
\end{cases} \tag{5.6}
$$

By Theorem 3.1 and Proposition 4.2, (Q,ξ) admits a version, still denoted by (Q,ξ), s.t.

$$(Q,\xi+\sigma^*DQ)\in\left(S^w_{\mathscr{F}}([0,T];H^{1,2})\cap C^w_{\mathscr{F}}([0,T];H^{1,p_1})\right)\times L^2_{\mathscr{F}}(0,T;H^{1,2}).$$

Recovering (v,ζ) from (Q,ξ) and setting $(u,\psi)\triangleq\theta^{-1}(v,\zeta)$ we see that (u,ψ) solves (1.7) and that $\{(T-t)^{\alpha_2}(\theta u_t,\theta\psi_t+\sigma^*D(\theta u_t))(y);\,(t,y)\in[0,T]\times\mathbb{R}^d\}$ belongs to $\left(S^w_{\mathscr{F}}([0,T];H^{1,2})\cap C^w_{\mathscr{F}}([0,T];H^{1,p_1})\right)\times L^2_{\mathscr{F}}(0,T;H^{1,2})$. Moreover, (5.1) holds with $c_0=\kappa e^{-\mu(\mathscr{Z})T}$ and $c_1=\Lambda e^{2T}$. Since $p_1\in[2,p_0)$ is arbitrary, this completes the proof of Theorem 5.1.

5.2 Uniqueness

We close our analysis with the following theorem, in which we shall first verify that the solution constructed in the previous section is the minimal one for our BSPDE (1.7) and subsequently, we obtain the uniqueness of the nonnegative solution in a certain class.

Theorem 5.5. *Under Conditions (H.1)–(H.3), for the solution (u,ψ) to* (1.7) *constructed in the proof of Theorem 5.1, if $(\tilde u,\tilde\psi)$ is a solution of* (1.7) *satisfying*

$$(\theta\tilde u,\theta\tilde\psi+\sigma^*D(\theta\tilde\psi))\in S^w_{\mathscr{F}}([0,t];H^{1,2})\times L^2_{\mathscr{F}}(0,t;H^{1,2}),\quad\forall\,t\in(0,T)$$

and if $\tilde u_t(y)\geq 0$ a.e. in $\Omega\times[0,T)\times\mathbb{R}^d$, then a.s. for every $t\in[0,T)$, $\tilde u_t\geq u_t$ a.e. in $\mathbb{R}^d$. Moreover, if we further have $p_0>2d+2$ and $\theta\tilde u\in C^w_{\mathscr{F}}([0,t];H^{1,p})$ for some $p\in(2d+2,p_0)$, then a.s. for all $t\in[0,T)$, $\tilde u_t=u_t$ a.e. in $\mathbb{R}^d$.

Proof. Let (v^N,ζ^N) be the unique solution to BSPDE (4.2) and set $(\tilde v,\tilde\zeta)=\theta\,(\tilde u,\tilde\psi)$. Since v^N increases to v as $N\to\infty$, in order to verify the minimality we only need to prove that a.s.

$$\tilde v_t\geq v^N_t\quad\text{a.e. in }\mathbb{R}^d,\quad\forall t\in[0,T].\tag{5.7}$$

For this, we put $(\bar v,\bar\zeta)=(v^N-\tilde v,\zeta^N-\tilde\zeta)$ and notice that

$$\left(F(t,y,\phi_1(y))-F(t,y,\phi_2(y))\right)(\phi_1(y)-\phi_2(y))^+\leq 0\quad\mathbb{P}\otimes dt\otimes dy\text{-a.e.}\tag{5.8}$$

holds for every non-negative $\phi_1,\phi_2\in L^0(\mathbb{R}^d)$. Taking into account the quadratic growth of F we obtain from Lemma 4.1 (ii) - in a similar way to the proof of Proposition 4.2 - a constant $C<\infty$ s.t. for $0\leq t<\tau<T$:

$$E\|\bar v^+(t)\|^2\leq\ E\bigg[C\int_t^\tau\left\|\bar v^+_s\right\|^2\,ds-\frac12\int_t^\tau\left\|\bar\zeta_s1_{\{v^N>\tilde v\}}+\sigma^*_sD\bar v^+_s\right\|^2ds+\|\bar v^+_\tau\|^2\bigg].$$

Together with Gronwall's inequality, this gives

$$E\left[\|\bar v^+_t\|^2\right]\leq CE\left[\|\bar v^+_\tau\|^2\right],$$

with a constant C independent of τ and t. Since a.s.

$$\bar v^+=(v^N-\tilde v)^+\leq|v^N|\quad\text{a.e. in }\mathbb{R}^d\text{ for all }t\in[0,T],\ \text{a.s.}$$

Proposition 4.3 allows us to integrate $|\bar v^+|^2$ over $\Omega\times[0,T]\times\mathbb{R}^d$. But then an application of Fatou's lemma yields (5.7) because

$$\int_{[0,T]\times\mathbb{R}^d}E\left[|\bar v^+_t(y)|^2\right]\,dydt\leq CT\limsup_{\tau\uparrow T}\int_{\mathbb{R}^d}E\left[|\bar v^+_\tau(y)|^2\right]\,dy\leq CT\int_{\mathbb{R}^d}E\left[\limsup_{\tau\uparrow T}|\bar v^+_\tau(y)|^2\right]\,dy=0.$$

In view of Theorem 3.3, to establish the uniqueness statement it is sufficient to verify that $\tilde{u}$ satisfies the growth condition (3.6). The above minimality arguments have given the lower bound. To establish the upper bound in (3.6) we extend arguments given in [12] and consider the deterministic function:

$$\hat{u}_t \triangleq \Lambda \coth(T-t) = \frac{2\Lambda}{1-e^{-2(T-t)}} - \Lambda \leq \frac{\Lambda e^{2T}}{T-t}.$$

Then, $(\hat{u}, 0)$ is a solution to (1.7) with the triple (λ, γ, η) being replaced by $(\Lambda, +\infty, \Lambda)$. Moreover, $(\hat{u}, 0)$ remains a solution when shifted in time, i.e., for $\delta \in [0, T)$ the pair $(\hat{u}_{\cdot+\delta}, 0)$ is the solution to (1.7) associated with $(\Lambda, +\infty, \Lambda)$, but with a singularity at $t = T - \delta$. Hence, noting that

$$\left(F(t, y, \theta^{-1}(y)\phi_1(y)) - \Lambda + \Lambda^{-1}|\theta^{-1}(y)\phi_2(y)|^2\right)(\phi_1 - \phi_2)^+(y) \leq 0 \quad \mathbb{P} \otimes dt \otimes dy\text{-a.e.},$$

holds for any pair of nonnegative measurable functions ϕ_1 and ϕ_2 on $\mathbb{R}^d$, using arguments similar to those used in the first part of this proof, we conclude

$$\int_{[0,T-\delta]\times\mathbb{R}^d} E|(\theta\tilde{u}_t - \theta\hat{u}_{t+\delta})^+(y)|^2 dy dt \leq C(T-\delta) \int_{\mathbb{R}^d} E \limsup_{\tau \uparrow T-\delta} |(\theta\tilde{u}_\tau - \theta\hat{u}_{\tau+\delta})^+(y)|^2 \, dy = 0.$$

This yields, a.s. for all $t \in [0, T-\delta]$

$$\tilde{u}_t \leq \frac{\Lambda e^{2T}}{T-\delta-t} \qquad \text{a.e. in } \mathbb{R}^d.$$

Finally, letting $\delta \to 0$ we obtain the desired upper bound as well as the uniqueness. □

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