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A limit theorem for systems of social interactions☆

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Abstract

In this paper, we establish a convergence result for equilibria in systems of social interactions with many locally and globally interacting players. Assuming spacial homogeneity and that interactions between different agents are not too strong, we show that equilibria of systems with finitely many players converge to the unique equilibrium of a benchmark system with infinitely many agents. We prove convergence of individual actions and of average behavior. Our results also apply to a class of interaction games.
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1. Introduction

Large differences in aggregate social or economic variables are often observed in the absence of corresponding differences in fundamentals. To accommodate such phenomena, a model must generate a multiplier that transforms small changes in exogenous variables into large changes of endogenous variables. Models of social interactions are capable of displaying multiplier effects. In these models an agent's behavior depends, among other things, on the choice of other agents in some reference group and/or the empirical distribution of actions throughout the whole population. In the presence of positive complementarities where the utility of undertaking an action increases with the number of agents undertaking the same action, a change in fundamentals has a direct effect on the behavior of an agent and an indirect effect through the interaction with others that are of the same sign. If these complementarities are powerful enough, small differences in fundamentals are amplified. As a result, significantly different aggregate activity may emerge from slightly distinct fundamentals.

In many examples in the literature an agent's utility is influenced by the average behavior of the population. In this case, the modelling is more naturally done in the context of an infinite number of agents, where one can appeal to a law of large numbers. Horst and Scheinkman (2006) study a general model that allows for local and global interactions and prove existence and uniqueness of equilibria for a class of models with an infinite number of agents. In this paper

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we show that for a class of these models, the equilibrium in a system with an infinite number of agents is the limit of equilibria of large finite systems that naturally approximate the infinite system. Our limit theorem can be viewed as a justification for the analysis of infinite systems. As Horst and Scheinkman (2006) did to establish uniqueness, we assume a form of spacial homogeneity and limits on the strength of social interactions. The results in this paper establish convergence of equilibrium actions of individuals and also the convergence of average actions. These results can be applied also to certain interaction games (see Morris, 1997). Strictly speaking, our results apply to games that satisfy the average action property, as defined by Morris and Shin (2005)—that is when the utility of a player depends on the average action of the other players.[1]

The rest of the paper proceeds as follows. In Sections 2 and 3 we recall the definition of systems of social interactions and existence and uniqueness results for infinite systems, respectively. Section 4 states the main result of this paper: a convergence theorem for equilibria of finite systems. The proof is given in Section 5.

2. Systems of random social interactions

In this section we recall the definition of systems of random social interactions given in Horst and Scheinkman (2006). Each agent is indexed by an $a \in \mathbb{A}$, where $\mathbb{A}$ is a subset of the lattice $\mathbb{Z}^d$ of d-dimensional vectors with integer entries. An agent will choose an action x^a from a common compact and convex set of possible actions X. An *action profile* $x = \{x^a\}_{a \in \mathbb{A}}$ is a list of actions x^b for each $b \in \mathbb{A}$. The *configuration space*

$$S := \{x = \{x^b\}_{b \in \mathbb{A}} : x^b \in X\}$$

of all action profiles is equipped with the product topology, and hence it is compact. The utility function of the agent $a \in \mathbb{A}$ may also depend on the actions chosen by other agents $b \in \mathbb{A}$. In addition, it is random, that is, it also depends on the realization of a random variable ϑ^a defined on the (canonical) probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In short, agent a's utility is of the form

$$\hat{U}^a(x^a, \{x^b\}_{b \neq a}, \vartheta^a).$$

In models of social interactions, the influence of other agents' actions on a player's utility appears in two distinct ways. The first is the impact of the actions of a particular set of *neighbors*. The second is through the distribution of actions throughout the whole population. We call the former the *local* component of social interactions and the latter the *global* component. To describe these distinct influences, and to allow for variable degrees of influence on the utility of an agent by the choices of their neighbors, we write $\vartheta^a = (J^a, \theta^a)$ for each agent $a \in \mathbb{A}$. The random variable θ^a describes a taste shock and assumes values in $\mathbb{R}$. The random variable $J^a = (J^{a,b})_{b \neq a}$, takes values in $\mathbb{R}^{\mathbb{A}}$. The realization of the random variable $J^{a,b}$ defines the effect the choice of the neighbor $b \neq a$ has on the utility of the agent $a \in \mathbb{A}$. To accommodate the global component, Horst and Scheinkman (2006) allowed the utility function to depend also on the distribution of agents' actions. To simplify our exposition and proofs we will assume, as is often done in models of social interactions, that only the average action $\varrho(x)$ associated with the action profile x affects utility. Not all profiles have an empirical average, but we will deal with this problem below.

We assume that the utility function of the agent is of the form:

$$\hat{U}^a(x^a, \{x^b\}_{b \neq a}, J^a, \theta^a) \equiv \hat{U}(x^a, \{J^{a,b}x^b\}_{b \neq a}, \varrho(x), \theta^a). \tag{1}$$

It has for arguments the actions chosen by the different players and the empirical mean of actions. The choice of these arguments reflects the fact that we think of the realizations of the random variable $(J^a, \theta^a)(a \in \mathbb{A})$ as "frozen." The *reference group* of agent a is defined as:

$$N(a) := \{b \in \mathbb{A} : J^{a,b} \neq 0\}.$$

Different realizations of the random variables J^a may yield different peer groups, and so different interaction patterns may emerge. The actions x^b of agents that do not belong to $N(a)$ affect a's optimal action only indirectly through their

[1] In principle, our results could also be extended to deal with the expected utility case. In fact in Horst and Scheinkman (2006) we deal with utility functions with arbitrary dependence on the distribution of actions. Convergence results covering this general case would however require substantial increases in the complexity of the notation and analysis.

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impact on the distribution $\varrho(x)$. The representation of the utility function given by Eq. (1) is not necessarily unique, and at this point it is only useful as an interpretation of the interaction patterns.

The special case $J^{a,b} = 0$ for all $a, b \in \mathbb{A}$ corresponds to a *mean–field* interaction where an agent's utility depends on the actions taken by others only trough the distribution of actions. If interactions are purely global the utility function takes the form:

$$\hat{U}^a(x^a, \{x^b\}_{b \neq a}, J^a, \theta^a) \equiv \hat{U}(x^a, \varrho(x), \theta^a).$$

As argued in Horst and Scheinkman (2006), continuous action versions of the models studied in Brock and Durlauf (2001) and Durlauf (1997) may be viewed as mean–field interactions.

Example 2.1. A utility function of the form

$$\hat{U}^a(x^a, \{x^b\}_{b \neq a}, J^a, \theta^a) = u(x^a) + x^a \sum_{b \in N(a)} J^{a,b} x^b + f(x^a, \varrho(x), \theta^a) \tag{2}$$

captures a situation where an agent's payoff depends on the actual actions taken by his neighbors rather than his expectation about his neighbors actions. Here, the random variables $J^{a,b}$ specify the effect of an increase in the action by a neighbor $b \in N(a)$ on the marginal utility of the agent $a \in \mathbb{A}$.

2.1. Infinite systems of random social interactions

The general definition of a utility function given in Eq. (1) is not convenient for establishing the existence of an equilibrium in models with infinitely many agents. An existence proof typically requires continuity of the utility functions and compactness of the configuration space S. In many interesting examples, the configuration $x \in S$ enters the utility function of an agent $a \in \mathbb{A}$ both *locally* through the actions x^b taken by his neighbors and *globally* via the average behavior throughout the entire population. In case of an infinite number of agents, S is compact in the product topology, but if an agent's utility function depends on x through the empirical average $\varrho(x)$ in a non-trivial manner, it is not a continuous function of x.[2] In addition, a configuration x does not necessarily have an empirical average. To solve these difficulties, Horst and Scheinkman (2006) used a method introduced in Föllmer and Horst (2001) and Horst (2002), that treats the empirical average $\varrho(x)$ of individual actions associated with $x \in S$ as an additional parameter of the utility function. Specifically, in models with infinitely many agents individual utility functions are defined as a *continuous* map defined on the extended state space $S \times X$. Continuity of the utility function then translates into a condition on the random variable J^a. The impact of agents far away must decrease sufficiently fast. In particular, continuity rules out the existence of a "leader" whose actions affect all agents equally.

We are now ready to define an infinite system of random social interactions. This is a slightly simplified version of the definition used in Horst and Scheinkman (2006).

Definition 2.2. A system of random social interactions is a vector

$$\mathcal{E} = (\mathbb{A}, \mathbb{P}, X, (U^a)_{a \in \mathbb{A}})$$

with the following components:

(i) $\mathbb{A} \subset \mathbb{Z}^d$ is the set of agents.
(ii) $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$.
(iii) $X \subset \mathbb{R}^l$ is a common compact, convex, action space.
(iv) $U^a : S \times X \times \mathbb{R}^{\mathbb{A}} \times \mathbb{R} \to \mathbb{R}$ a measurable mapping of the form

$$U^a(x^a, \{x^b\}_{b \neq a}, \varrho, J^a, \theta^a) \equiv U(x^a, \{J^{a,b} x^b\}_{b \neq a}, \varrho, \theta^a),$$

[2] The class of continuous functions $f :\to \mathbb{R}$ which depend only on finitely many coordinates is dense in the space $\mathcal{C}(S)$ of all continuous functions on S equipped with the topology of uniform convergence. Thus, $f \in \mathcal{C}(S)$ depends on an action profile $x \in S$, at least approximately, only through finitely many actions.

and such that for $\mathbb{P}$-a.e. pair (J^a, θ^a), the map

$$(x, \varrho) \mapsto U^a(x^a, \{x^b\}_{b \neq a}, \varrho, J^a, \theta^a)$$

is continuous and strictly concave in x^a; the utility function of agent $a \in \mathbb{A}$.

A priori there is no consistency requirement between the configuration $x \in S$ and the "empirical average" ϱ in the definition of the utility function U. In particular, it is not necessary that we can associate an empirical average with the profile x. However, consistency will be required in equilibrium. An *equilibrium* is a random action profile $g(J, \theta) = \{g_a(J, \theta)\}_{a \in \mathbb{A}}$ with empirical average $\varrho(J, \theta)$ such that all agents play a best reply against their neighbors' actions and the (perceived) average choice $\varrho(J, \theta)$, i.e.,

$$g_a(J, \theta) = \operatorname*{argmax}_{x^a \in X} U(x^a, \{J^{a,b} g_b(J, \theta)\}_{b \neq a}, \varrho(J, \theta), \theta^a) \quad (a \in \mathbb{A}).$$

It should be emphasized that for equilibrium analysis it is equivalent to use the *continuous* utility function U on the extended state space that involves the variables (x, ϱ) or to use the possibly *discontinuous* utility function $\hat{U}$ in (1) that depends only on the actions profile x, since, in equilibrium we will require the "forecast" of the average choice ϱ to coincide with the empirical average of the action profile x. In terms of $\hat{U}$ all agents play a conditional best reply against the actual actions taken by *all* the players, given their private taste shocks and the interaction pattern so

$$g_a(J, \theta) = \operatorname*{argmax}_{x^a \in X} \hat{U}(x^a, \{g_b(J, \theta)\}_{b \neq a}, J^a, \theta^a) \quad (a \in \mathbb{A}).$$

To prove our limit results we will place further restrictions on $\mathcal{E}$.

Assumption 2.3. The system of random social interactions satisfies:

(i) The set of agents is given by the d-dimensional integer lattice, that is, $\mathbb{A} = \mathbb{Z}^d$.
(ii) $\mathbb{P}$ is an ergodic probability measure on $(\Omega, \mathcal{F})$, that is the distribution of the random vector

$$(J, \theta) = (J^a, \theta^a)_{a \in \mathbb{A}}$$

is stationary and satisfies a 0-1-law on the σ-field of all shift invariant events.
(iii) There exists $M \in \mathbb{N}$ such that $\mathbb{P}[J^{a,b} = 0] = 1$ for $|a - b| > M$.

The first two items in this Assumption state that $\mathcal{E}$ is *ergodic*, a property that was used in Horst and Scheinkman (2006) to prove uniqueness of equilibria and to show, in particular, the existence of averages $\varrho(x)$ associated with the candidate equilibrium actions x. The last item in the Assumption simplifies the proofs and guarantees that the dependence of the utility of an agent on the actions of other agents decays "fast enough." In what follows we will always maintain Assumption 2.3.

Our goal is to show that equilibria of infinite systems can be viewed as approximations of equilibria of finite, but large systems. In the following section we propose a way to embed models with finitely many agents into infinite systems. Our approach is analytically convenient for deriving properties of equilibrium actions when the number of agents tends to infinity.

2.2. *Finite systems of social interactions*

There are many ways in which finite systems can be embedded into infinite systems and in which infinite systems can be approximated by finite ones. A natural approach is to consider the increasing sequence $\{\mathbb{A}_n\}_{n \in \mathbb{N}}$, $\mathbb{A}_n := [-n, n]^d \cap \mathbb{A}$, of finite sub-populations and investigate convergence properties of equilibrium action profiles as the number of agents tends to infinity. In models with local interactions where an agent's utility depends on the choices of neighbors, there may be agents in $\mathbb{A}_n$ with neighbors not belonging to $\mathbb{A}_n$. We therefore fix a *boundary condition*

$y \in S$ and assume that all the player $b \notin \mathbb{A}_n$ take the action y^b. In such a situation the utility function of agent $a \in \mathbb{A}_n$ takes the form:

$$U^{n,y}(x^a, \{x^b\}_{b\neq a}, J^a, \theta^a) := U(x^a, \{\hat{x}^b\}_{b\neq a}, \varrho^{n,y}(x), J^a, \theta^a), \tag{3}$$

where the configuration $\hat{x}$ coincides with x on $\mathbb{A}_n$ and with the boundary condition y on the complement $\mathbb{A}_n^c$. Furthermore, $\varrho^{n,y}(x)$ denotes the average choice by the agents in $\mathbb{A}_n$ i.e.,

$$\hat{x}^b := \begin{cases} x^b & \text{if } b \in \mathbb{A}_n \\ y^b & \text{if } b \notin \mathbb{A}_n \end{cases} \quad \text{and} \quad \varrho^{n,y}(x) := \frac{1}{|\mathbb{A}_n|}\sum_{a \in \mathbb{A}_n} x^a. \tag{4}$$

For finite systems $\mathcal{E}^{n,y} = (\mathbb{A}_n, X, \mathbb{P}, U^{n,y})$ the question of existence of an equilibrium $g^{n,y} = \{g_a^{n,y}\}_{a \in \mathbb{A}_n}$ follows from continuity and strict concavity of the utility functions along with compactness and convexity of the action spaces via a standard fixed point argument. Uniqueness can be guaranteed under additional restrictions on the strength of interactions. To prove a convergence result for the sequence of random variables $\{g^{n,y}\}_{n \in \mathbb{N}}$ we also need an existence and uniqueness result for an equilibrium $g = \{g_a\}_{a \in \mathbb{A}}$ in the benchmark model with infinitely many agents. In a second step we prove almost sure convergence of $\{g^{n,y}\}_{n \in \mathbb{N}}$ to g. In this sense, an infinite system $\mathcal{E}$ can be viewed as an approximation of finite, but large systems $\mathcal{E}^{n,y}$.

3. Equilibria in infinite systems of social interactions

This section recalls results on existence and uniqueness of equilibria in infinite systems of random social interactions established by Horst and Scheinkman (2006). If interactions are purely local, existence of equilibria follows from continuity along with strict concavity of the utility functions and compactness and convexity of the actions spaces by a standard fixed point argument. In the case of an infinite number of locally and globally interacting agents we have the additional requirement that, in equilibrium, the agents' forecast of the average choice must equal the actual average of actions, and the question of existence and uniqueness requires additional assumptions. Unless the equilibrium action profiles display a form of spatial homogeneity, there is no reason expect that the these profiles have an average. As a result, we consider only *homogeneous* equilibrium action profiles. To this end, we denote by T^a the a-fold iteration of the canonical shift operator on Ω.

Definition 3.1. A random variable $g(J, \theta) = \{g_a(J, \theta)\}_{a \in \mathbb{A}}$ is a homogeneous equilibrium for the infinite system $\mathcal{E} = (\mathbb{A}, \mathbb{P}, X, U)$ if:

(i) No agent has an incentive to deviate from the proposed strategy. That is, almost surely

$$g_a(J, \theta) = \operatorname*{argmax}_{x^a \in X} U(x^a, \{g_b(J, \theta)\}_{b\neq a}, \varrho(g(J, \theta)), J^a, \theta^a) \quad (a \in \mathbb{A}). \tag{5}$$

(ii) The action profile $g(J, \theta)$ is homogeneous, i.e.,

$$g_a(J, \theta) = g_0 \circ T^a(J, \theta). \tag{6}$$

For a homogeneous equilibrium action profile g the ergodic theorem shows that the associated average action exists and is almost surely independent of the actual interaction pattern and the realized vector of taste shocks because $\mathbb{P}$ is ergodic. That is, there exists ϱ such that

$$\lim_{n\to\infty} \frac{1}{|\mathbb{A}_n|} \sum_{a \in \mathbb{A}_n} g_a(J, \theta) = \varrho \quad \mathbb{P} - \text{a.s.} \tag{7}$$

In particular, the existence of an average action implicitly assumed in (5) is a *property* of homogeneous equilibria, not a condition of equilibrium action profiles. Of course, homogeneous equilibria are unlikely to exist unless, as we assumed, agents' utility functions and the probabilistic structure of interaction patterns and taste shocks are themselves homogeneous.

3.1. Microscopic equilibria of infinite systems

For infinite systems the question of existence and uniqueness of equilibria can conceptually be separated into two parts. The first part consists of taking as given for each (J, θ) an empirical average of actions and asking whether some prescribed profile of actions maximizes utility of each agent when he takes as given the actions of his neighbors and the given empirical average. The second part consists in checking whether the actions assigned to each agent generate the prescribed average choice. To separate the two problems we start by considering *microscopic equilibria*.

Definition 3.2. An action profile $g(\varrho, J, \theta) = \{g_a(\varrho, J, \theta)\}_{a \in \mathbb{A}}$ is a microscopic equilibrium associated with $\varrho \in X$ if

$$g_a(\varrho, J, \theta) = \underset{x^a \in X}{\operatorname{argmax}} U(x^a, \{g_b(\varrho, J, \theta)\}_{b \neq a}, \varrho, J^a, \theta^a) \quad \mathbb{P}-\text{a.s.} \tag{8}$$

for all $a \in \mathbb{A}$. The equilibrium is homogeneous if

$$g_a(\varrho, \cdot) = g_0(\varrho, \cdot) \circ T^a \quad \text{for all} \quad a \in \mathbb{A}.$$

A microscopic equilibrium associated with ϱ is an action profile where each agent maximizes his utility given the actions taken by all the other agents and given the common anticipated average ϱ of actions throughout the entire system. What distinguishes a microscopic equilibrium $g(\varrho, J, \theta)$ from an equilibrium is the fact that the empirical average associated with the configuration $g(\varrho, J, \theta)$ does not necessarily coincide with ϱ. Of course, a microscopic equilibrium associated to some ϱ^* is an equilibrium if

$$\lim_{n \to \infty} \frac{1}{|\mathbb{A}_n|} \sum_{a \in \mathbb{A}_n} g_a(\varrho^*, J, \theta) = \varrho^* \quad \mathbb{P}-\text{a.s.} \tag{9}$$

While the existence of a microscopic equilibrium for infinite systems follows from standard arguments there is no guarantee that this equilibrium is homogenous. However, if $g(\varrho, \cdot)$ is the unique microscopic equilibrium of $\mathcal{E}$ associated with ϱ, Lemma 15 of Horst and Scheinkman (2006) guarantees that $g(\varrho, \cdot)$ is homogeneous.

3.2. Equilibria of infinite systems

The proof of existence and uniqueness of equilibria in infinite systems requires additional assumptions on the strength of interactions between different agents. We need to place a quantitative bound on the dependence of an agent's conditional optimal action on his neighbors' choices and the perceived average behavior. These bounds can be specified in terms of agent 0's best reply function

$$h_0(\{x^a\}_{a \neq 0}, \varrho, J, \theta) := \underset{x^0 \in X}{\operatorname{argmax}} U(x^0, \{x^a\}_{a \neq 0}, \varrho, J^0, \theta^0). \tag{10}$$

Modulo shifts, the same bounds can then be applied to the conditional best reply function h_a of an arbitrary agent $a \in \mathbb{A}$ because the homogeneity property of the utility functions yields

$$h_a(\{x^b\}_{b \neq a}, \varrho, \cdot) = h_0(\{x^{b-a}\}_{b \neq a}, \varrho, \cdot) \circ T^a. \tag{11}$$

The following definition allows us to measure the dependence of an agent's best reply on another agent's action and the anticipated average action.

Definition 3.3. The best reply function h_0 is *Lipschitz continuous* if there exist uniformly bounded random variables $(L^a)_{a \in \mathbb{A}}$ and L^ϱ such that

$$|h_0(\{x^a\}_{a \neq 0}, \hat{\varrho}, J, \theta) - h_0(\{y^a\}_{a \neq 0}, \tilde{\varrho}, J, \theta)| \le \sum_{a \neq 0} L^a(J^0, \theta^0)|x^a - y^a| + L^\varrho(\theta^0)|\hat{\varrho} - \tilde{\varrho}| \quad \mathbb{P}-\text{a.s.}$$

Given an interaction profile J and a configuration of taste shocks θ, the quantity L^a may be viewed as a bound for the influence an action taken by the agent $a \in \mathbb{A}$ has on the optimal choice of agent $0 \in \mathbb{A}$. In a similar manner, the random variable L^{ϱ} measures the dependence of agents 0's best reply on his expectation about the population behavior.[3] Existence and uniqueness of equilibria can be established for systems $\mathcal{E}$ in which the agents' best reply functions satisfy the following *moderate social influence* (MSI) condition in its strong form.

Definition 3.4. Let the agents' best reply functions be Lipschitz continuous and put

$$\alpha_1 := \sup_{J^0,\theta^0} \sum_{a \in \mathbb{A}} L^a(J^0, \theta^0) \quad \text{and} \quad \alpha_2 := \sup_{\theta} L^{\varrho}(\theta^0).$$

We say that $\mathcal{E}$ satisfies the MSI condition, respectively, the MSI condition in its strong form, if the constants L^a and L^{ϱ} can be chosen to satisfy

$$\alpha_1 \le \alpha < 1 \quad \text{respectively} \quad \alpha_1 + \alpha_2 \le \alpha < 1. \tag{12}$$

Let us illustrate how the MSI condition can be derived from restrictions on the utility functions and interaction patterns.

Example 3.5 *(Horst and Scheinkman (2006), Example 27).* Suppose that the agents are located on the two-dimensional lattice, interact only with their four nearest neighbors and that the interaction patterns satisfy $|J^{a,b}| \le 1$. Put $X = [-1, 1]$ and let $m_0(J) := (1/4)\sum_{a \in N(0)} J^{0,a} x^a$ be the weighted average action of agent 0's neighbors. Suppose that there exists a utility function $u : X^3 \times \Theta \to \mathbb{R}$ such that:

$$U^0(x^0, \{x^a\}_{a \neq 0}, \varrho, J^0, \theta^0) = u(x^0, m_0(J), \varrho, \theta^0).$$

We also assume that the map $(x^0, m_0, \varrho) \mapsto u(x^0, m_0, \varrho, \theta^0)$ is twice continuously differentiable with negative second derivative with respect to x^0. Further we assume that an agents' best reply is interior. The first order condition can be written as

$$\frac{\partial u(x^0, m_0(J), \varrho, \theta^0)}{\partial x^0} = 0.$$

Since $(\partial^2 u)/(\partial (x^0)^2) < 0$ it follows that the best reply of agent $0 \in \mathbb{A}$ is uniquely determined. Since $\mathbb{P}[J^{0,a} = 0] = 1$ for $a \notin N(0)$, $L^a = 0$ whenever $a \notin N(0)$. For $a \in N(0)$ it follows from $\mathbb{P}[|J^{0,a}| \le 1] = 1$ and from the implicit function theorem that

$$\left| \frac{\partial h^0(\{x^a\}_{a \neq a0}, \varrho, J, \theta)}{\partial x^a} \right| \le \frac{1}{4} \left| \frac{(\partial^2)/(\partial m_0 \partial x^0) u(x^0, m_0(J), \varrho, \theta^0)}{(\partial^2)/(\partial (x^0)^2) u(x^0, m_0(J), \varrho, \theta^0)} \right| \quad \mathbb{P}-\text{a.s.}$$

In particular, we can choose

$$L^a \le \sup_{x,\varrho} \frac{1}{4} \left| \frac{(\partial^2)/(\partial m_0 \partial x^0) u(x^0, m_0(J), \varrho, \theta^0)}{(\partial^2)/(\partial (x^0)^2) u(x^0, m_0(J), \varrho, \theta^0)} \right| \quad \mathbb{P}-\text{a.s.}$$

Thus, MSI occurs if

$$\sup_{x,\varrho} \left| \frac{(\partial^2)/(\partial m_0 \partial x^0) u(x^0, m_0(J), \varrho, \theta^0)}{(\partial^2)/(\partial (x^0)^2) u(x^0, m_0(J), \varrho, \theta^0)} \right| \le \alpha < 1 \quad \mathbb{P}-\text{a.s.}$$

In this case, the system has an equilibrium. The equilibrium is unique if the following stronger condition holds:

$$\sup_{x,\varrho} \left| \frac{(\partial^2)/(\partial m_0 \partial x^0) u(x^0, m_0(J), \varrho, \theta^0)}{(\partial^2)/(\partial (x^0)^2) u(x^0, m_0(J), \varrho, \theta^0)} \right| + \sup_{x,\varrho} \left| \frac{\partial^2}{\partial \varrho \partial x^0} u(x^0, m_a(J), \varrho, \theta^0) \right| \le \alpha < 1.$$

[3] We may choose $L^a = 0$ for $|a| > M$ because the agents interact locally only with their nearest neighbors, due to Assumption 2.3(iii). For sufficiently smooth utility functions, L^a and L^{ϱ} can be expresses in terms of cross partial derivatives of U. We refer the reader to Section 4.3 in Horst and Scheinkman (2006) for examples.

The following existence and uniqueness result is a consequence of Proposition 18 and Theorem 19 of Horst and Scheinkman (2006), and plays a crucial role in our proofs.

Theorem 3.6. *A system of social interactions $\mathcal{E}$ that satisfies the moderate social influence condition in its strong form has a unique equilibrium. That is, there exists a unique (up to a set of measure zero) spatially homogeneous random vector $g(J, \theta) = \{g_a(J, \theta)\}_{a \in \mathbb{A}}$ whose empirical average ϱ^* is almost surely independent of J and θ,*

$$\lim_{n\to\infty} \frac{1}{|\mathbb{A}_n|} \sum_{a \in \mathbb{A}_n} g_a(J, \theta) = \varrho^* \quad \mathbb{P} - \text{a.s.}$$

such that each agent plays a best reply against his neighbors actions and the perceived average of actions ϱ^*:

$$g_a(J, \theta) = h_a(\{g_b(J, \theta)\}_{b\neq a}, \varrho^*, J, \theta) \quad \mathbb{P} - \text{a.s.}$$

For finite systems equilibria always exist, and the associated average actions typically depend on the realization of the random variables (J, θ). Our convergence result stated below shows that the sequence of average actions converges to the average action of the unique equilibrium of the infinite system if the interaction between different agents is not too strong. In finite but large systems the average choice throughout the whole population can thus be approximated by a deterministic quantity.

4. Approximation of large systems

In this section we will investigate convergence of the equilibria of a sequence $\{\mathcal{E}^{n,y}\}_{n \in \mathbb{N}}$ of finite systems. Given the boundary condition y on $\mathbb{A}_n^c$, we can always choose $L^a = 0$ for $a \notin \mathbb{A}_n$, and hence the strong MSI condition guarantees existence of a unique equilibrium $g^{n,y}(J, \theta) = \{g_a^{n,y}(J, \theta)\}_{a \in \mathbb{A}_n}$ for the system $\mathcal{E}^{n,y}$.

It turns out that under the uniqueness condition for equilibrium of infinite systems stated in Horst and Scheinkman (2006) each individual action and the average action in the equilibrium profiles $\{g^{n,y}\}_{n \in \mathbb{N}}$ converge to the individual and average actions in the unique equilibrium of the infinite system. More precisely, we have the following result. The proof requires some preliminaries and will be given in Section 5.

Theorem 4.1. *Let $\mathcal{E}^{n,y} = (\mathbb{A}_n, X, \mathbb{P}, U^{n,y})$ be finite systems of social interactions that satisfy the moderate social influence condition in its strong form.*

(i) *For any agent $a \in \mathbb{A}$ the sequence of equilibrium actions $\{g_a^{n,y}\}_{n \in \mathbb{N}}$ converges almost surely to the corresponding equilibrium action g_a in the infinite system.*
(ii) *The sequence of empirical averages $\{\varrho^{n,y}\}_{n \in \mathbb{N}}$ converges almost surely to the empirical average ϱ associated with the equilibrium action g.*

That is, the impact of a boundary on both the individual and average behavior vanishes when the number of agents tends to infinity.

For infinite systems with local and global interactions any approximation must take a stand on what happens to agents at the "boundary" of the set $\mathbb{A}_n$. However, for mean field system this care is not necessary. The approximating finite systems have exactly the same utility functions as the infinite system except that the argument is the empirical average in the finite systems. Hence our Theorem 4.1 is a generalization of Proposition 5 in Glaeser and Scheinkman (2002) for models in which an agent's utility only depends on the mean of other agents action and shocks are iid. We state this result as a corollary.

Corollary 4.2. *Let $\mathcal{E}$ be a mean field system. If $L^\varrho \leq \alpha < 1$ almost surely, then the sequence of unique equilibria in $\mathcal{E}^n$ converges almost surely to the unique equilibrium of $\mathcal{E}$.*

5. Proof of the approximation result

In this section we prove our approximation result for large, but finite systems of social interactions. The key is to establish macroscopic convergence, i.e., to prove that the sequence of average actions converges almost surely to

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the average action associated with the unique equilibrium of the infinite system. To this end, we need to show that the impact of the boundary condition on the equilibrium action $g_a^{n,y}$ of the agent $a \in \mathbb{A}_n$ decreases to zero *uniformly* in the agent's distance to the boundary of $\mathbb{A}_n$ as $n \to \infty$. The idea is to view $g_a^{n,y}$ as one component of the limit of a deterministic dynamic process starting in $g^{n,x}$ where the agents myopically optimize their behavior in reaction to the choices of others in the previous period, given the boundary condition y on $\mathbb{A}_n^c$. In general such a process is not guaranteed to converge. However, under our MSI condition convergence follows from, e.g., Proposition 4 in Glaeser and Scheinkman (2002). In a second step we apply the quantitative bounds on the impact of boundary conditions on individual equilibrium actions to prove that, in the limit of an infinite set of agents, the average behavior is independent of boundary conditions. A third step consists of showing that convergence of averages implies convergence of individual actions. Finally, we show that in large systems the average choice is almost surely independent of the actual realization of taste shocks and interaction patterns.

5.1. *Continuous dependence of microscopic equilibria on average actions*

Before establishing convergence of equilibria, it is convenient to prove continuous dependence of microscopic equilibria on the perceived average behavior.

Lemma 5.1. *Under the assumptions of* Theorem 4.1 *the homogeneous microscopic equilibria* $g(\varrho, \cdot)$ *depend continuously on* ϱ. *That is, for all* $a \in \mathbb{A}$ *and almost surely*

$$\lim_{n\to\infty} g_a(\varrho_n, \cdot) = g_a(\varrho, \cdot) \quad if \quad \lim_{n\to\infty} \varrho_n = \varrho.$$

Proof. Since $g(\varrho, \cdot)$ is a homogeneous microscopic equilibrium

$$g_0(\varrho_n, \cdot) = h_0(\{g_a(\varrho_n, \cdot)\}, \varrho_n, \cdot) \quad \text{and} \quad g_a(\varrho_n, \cdot) = g_0(\varrho_n, \cdot) \circ T^a,$$

where T^a denotes the a-fold iteration of the canonical shift operator. Thus, Lipschitz continuity of the best reply function yields

$$|g_0(\varrho_n, \cdot) - g_0(\varrho, \cdot)| \le |h_0(\{g_a(\varrho_n, \cdot)\}_{a\neq 0}, \varrho_n, \cdot) - h_0(\{g_a(\varrho_n, \cdot)\}_{a\neq 0}, \varrho, \cdot)| + |h_0(\{g_a(\varrho_n, \cdot)\}_{a\neq 0}, \varrho, \cdot)$$
$$- h_0(\{g_a(\varrho, \cdot)\}_{a\neq 0}, \varrho, \cdot)| \le L^{\varrho}(\cdot)|\varrho - \varrho^n| + \sum_{a\in\mathbb{A}} L^a(\cdot)|g_0(\varrho_n, \cdot) \circ T^a - g_0(\varrho, \cdot) \circ T^a|$$

and so our weak interaction condition shows that

$$|g_0(\varrho_n, \cdot) - g_0(\varrho, \cdot)| \le \frac{1}{1-\alpha}|\varrho_n - \varrho| \quad \mathbb{P}-\text{a.s.} \qquad \square$$

Proposition 18 of Horst and Scheinkman (2006) guarantees that for a homogeneous microscopic equilibrium action profile $g(\varrho, \cdot)$ the associated average action is almost surely independent of the realization of the random variable (J, θ). The ergodic theorem yields a $\mathbb{P}$-null set $\mathcal{N}_\varrho$ which may depend on ϱ such that

$$\lim_{n\to\infty} \frac{1}{|\mathbb{A}_n|} \sum_{a\in\mathbb{A}_n} g_a(\varrho, J, \theta) = \mu[\varrho] \quad \text{for all} \quad (J, \theta) \in \mathcal{N}_\varrho^c.$$

Continuous dependence of microscopic equilibria on average choices allows us to show that the set $\mathcal{N}_\varrho$ can actually be chosen independently of ϱ.

Lemma 5.2. *There exists a set* $\mathcal{N}$ *of* $\mathbb{P}$*-measure zero such that*

$$\lim_{n\to\infty} \frac{1}{|\mathbb{A}_n|} \sum_{a\in\mathbb{A}_n} g_a(\varrho, J, \theta) = \mu[\varrho] \quad \textit{for all} \quad (J, \theta) \in \mathcal{N}^c \quad \textit{and each} \quad \varrho \in X.$$

Proof. By Lemma 35 in Horst and Scheinkman (2006) that map $\varrho \mapsto \mu[\varrho]$ is continuous and hence uniformly continuous because X is compact. Thus, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\mu[\varrho] - \mu[\hat{\varrho}]| \leq \frac{\epsilon}{2} \quad \text{if} \quad |\varrho - \hat{\varrho}| < \delta.$$

Furthermore, there exists $\varrho_1, \ldots, \varrho_{n(\epsilon)} \in X$ such that X is contained in the union of all the δ-balls $B_1, \ldots, B_{n(\epsilon)}$ centered at $\varrho_1, \ldots, \varrho_{n(\epsilon)}$. For any such ϱ_i there exists a set $\mathcal{N}_{\varrho_i}$ of measure zero such that

$$\lim_{n\to\infty} \frac{1}{|\mathbb{A}_n|} \sum_{a \in \mathbb{A}_n} g_a(\varrho_i, J, \theta) = \mu[\varrho_i] \quad \text{for all} \quad (J, \theta) \notin \mathcal{N}_{\varrho_i}.$$

Let us then fix $\varrho \in X$ and $(J, \theta) \notin \mathcal{N}_{\varrho_1} \cup \cdots \cup \mathcal{N}_{\varrho_{n(\delta)}}$. Clearly, ϱ belongs to some ball B_i so Lemma 5.1 yields

$$\mathit{limsup}_{n\to\infty} \left| \frac{1}{|\mathbb{A}_n|} \sum_{a \in \mathbb{A}_n} g_a(\varrho, J, \theta) - \mu[\varrho] \right| \leq \mathit{limsup}_{n\to\infty} \left| \frac{1}{|\mathbb{A}_n|} \sum_{a \in \mathbb{A}_n} g_a(\varrho_i, J, \theta) - \mu[\varrho_i] \right| + \epsilon = \epsilon.$$

A reverse inequality holds for the lim inf. This proves the assertion, since $\epsilon > 0$ is arbitrary. □

5.2. *Impact of the boundary conditions*

We now establish a quantitative bound on the impact of boundary conditions on individual equilibrium actions in finite systems.

Proposition 5.3. *Let $x, y \in S$. Under the assumption of Theorem 4.1 there exists, for every $\epsilon > 0$, some constants $N, k \in \mathbb{N}$, that do not depend on x, y, such that*

$$|g_a^{n,x}(J, \theta) - g_a^{n,y}(J, \theta)| < \epsilon$$

for all $(J, \theta) \in \mathcal{N}^c$, each $n \geq N$ and every $a \in \mathbb{A}_{n-k}$.

Proof. We fix a pair $(J, \theta) \in \mathcal{N}^c$ of interaction patterns and taste shocks and recall that T^a denotes the canonical a-fold shift operator on $\mathbb{A}$. To simplify the notation, we suppress the dependence of equilibrium actions, best reply functions, etc. on (J, θ). We assume with no loss of generality that the diameter of the action set is at most one, and consider only the case $M = 1$ of a nearest neighbor interaction. Similar arguments apply for arbitrary $M \in \mathbb{N}$.

Our idea is then to view the equilibrium action $g_a^{n,y}$ as the result of a sequential myopic best reply dynamics when in every period the agents chose their optimal actions in reaction to the choices of all the other players in the previous period, given the boundary condition y on $\mathbb{A}_n^c$. More precisely, we define a deterministic dynamic process $\{G_t^n\}_{t \in \mathbb{N}}$, $G_t^n = (G_t^n(a))_{a \in \mathbb{A}}$ by

$$G_0^n(a) = \begin{cases} g_a^{n,x} & \text{if } a \in \mathbb{A}_n \\ y^a & \text{if } a \notin \mathbb{A}_n \end{cases} \quad \text{and} \quad G_{t+1}^n(a) = \begin{cases} h_0(\{G_t^n(b)\}_{b \neq a}, \varrho_t^n, \cdot) \circ T^a & \text{if } a \in \mathbb{A}_n \\ y^a & \text{if } a \notin \mathbb{A}_n \end{cases}$$

for $t \geq 1$ where ϱ_t^n denotes the average associated with the actions $G_t^n(a)$ taken by the agents $a \in \mathbb{A}_n$ at time t. In the first period $t = 1$ only the $2^d(n-1)^{d-1}$ agents on the boundary of $\mathbb{A}_n$ change their actions. The resulting impact on the average choice is of the order n^{-1}. In the second period all agents react to the change of the average. In addition, all the agents that are at most 2 away from the boundary modify their choices in reaction to changes of their neighbors' actions, etc. By Proposition 4 in Glaeser and Scheinkman (2002) the sequence $\{G_t^n\}_{t \in \mathbb{N}}$ converges to the unique equilibrium $g^{n,y}$ for $\mathcal{E}^{n,y}$ because the finite system satisfies our MSI condition. In terms of the quantities

$$L_t^n(a) := G_t^n(a) - G_{t-1}^n(a) \quad \text{and} \quad L_t^n := \frac{1}{|\mathbb{A}_n|} \sum_{a \in \mathbb{A}_n} |L_t^n(a)|$$

we have

$$|g_a^{n,y} - g_a^{n,x}| \leq \sum_{t \geq 1} |L_t^n(a)|.$$

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The goal is thus to establish an upper bound for the random variables $L_t^n(a)$ and to prove that $\sum_{t\geq 1}|L_t^n(a)|$ is small whenever n is large enough. To this end, we proceed in four steps.

(Step i) We first introduce the random variables $L^{a,b} := L^{b-a} \circ T^a$ that measure the impact the previous action of the agent b has on agent a's current best reply. Since

$$G_{t+1}^n(a) = h_0(\{G_t^n(b)\}_{b\neq a}, \varrho_t^n) \circ T^a \quad \text{and} \quad G_t^n(a) = h_0(\{G_{t-1}^n(b)\}_{b\neq a}, \varrho_t^n) \circ T^a.$$

Lipschitz continuity of the best reply function yields

$$|L_{t+1}^n(a)| \leq \sum_{b\neq a} L^{a,b}|G_{t+1}^n(b) - G_t^n(b)| + L^\varrho|\varrho_t^n - \varrho_{t-1}^n| \leq \sum_{b\neq a} L^{a,b}|L_t^n(b)| + L^\varrho L_t^n.$$

Write

$$A_t^n := \left\{ a \in \mathbb{A}_n : \min_{b\notin\mathbb{A}_n} |a-b| \leq t \right\}$$

for the set of all agents that are at most t away from the boundary of $\mathbb{A}_n$. Furthermore, we define a sequence $\{\lambda_t^n\}_{t\in\mathbb{N}}$ via:

$$\lambda_1^n := L_1^n \quad \text{and} \quad \lambda_{t+1}^n := \alpha\lambda_t^n + \frac{|A_{t+1}^n|}{|\mathbb{A}_n|}\alpha^{t+1} \quad (t = 1, 2, \ldots),$$

where $\alpha = \alpha_1 + \alpha_2 < 1$ denotes the bound on the impact of other agents' choices on an individual player's optimal action. In the next step we show that the following estimate for the random variables $L_t^n(a)$ and L_t^n in terms of the quantities λ_t^n holds almost surely:

$$|L_t^n(a)| \leq \begin{cases} \lambda_t^n + \alpha^t & \text{if } t \in A_t^n \\ \lambda_t^n & \text{otherwise} \end{cases} \quad \text{and} \quad L_t^n \leq \lambda_t^n. \tag{13}$$

(Step ii) We prove (13) by induction. For $t = 1$ the assertion follows from the MSI condition along with the fact that the diameter of the action space is at most one. Hence we assume that (13) holds for all $t \leq T$. Consider then the case $a \notin A_{T+1}^n$ so that $b \notin A_T^n$ for all the neighbors $b \in N(a)$. In this case the impact of the boundary condition on the agent's action at time T is felt only indirectly through its impact on the average choice. The induction hypothesis along with out weak interaction condition yields

$$|L_{T+1}^n(a)| \leq \sum_{b\neq a} L^{a,b}|L_T^n(b)| + L^\varrho L_T^n \leq \sum_{b\neq a} L^{a,b}\lambda_T^n + L^\varrho\lambda_T^n \leq \alpha\lambda_T^n \leq \lambda_{T+1}^n. \tag{14}$$

If $a \in A_{T+1}^n$ then $b \in A_T^n$ for at least one of the agent's neighbors $b \in N(a)$. In such a situation the impact of the boundary condition on a is felt both directly through its impact on some neighbors' actions and indirectly through its impact on the average. Thus, the induction hypothesis yields

$$|L_{T+1}^n(a)| \leq \sum_{b\neq a} L^{a,b}|L_T^n(b)| + L^\varrho L_T^n \leq \sum_{b\neq a} L^{a,b}\left[\lambda_T^n + \alpha^T\right] + L^\varrho L_T^n$$
$$= \alpha\lambda_T^n + \alpha^{T+1} \leq \lambda_{T+1}^n + \alpha^{T+1}. \tag{15}$$

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From Eqs. (14) and (15) we now obtain that (13) holds for all $t \in \mathbb{N}$ because

$$L_{T+1}^n = \frac{1}{|\mathbb{A}_n|} \sum_{a \in \mathbb{A}_n} |L_{T+1}^n(a)| \le \alpha \lambda_T^n + \frac{|A_{T+1}^n|}{|\mathbb{A}_n|} \alpha^{T+1} = \lambda_{T+1}^n.$$

(Step iii) Next, we establish the existence of a constant $K < \infty$ that satisfies

$$\sum_{t \ge 1} \lambda_t^n \le \frac{K}{n}.$$

To this end, observe first that $|\mathbb{A}_n| = (2n+1)^d$. For $n \ge t$ the number of agents that are at most t away from the boundary can thus be estimated by

$$|A_t^n| = (2n+1)^d - (2(n-t)+1)^d \le \hat{K} t^d n^{d-1},$$

where the constant $\hat{K} < \infty$ depends only on the dimension d of the integer lattice. Hence we obtain for all $t \in \mathbb{N}$ that

$$\frac{|A_t^n|}{|\mathbb{A}_n|} \le \hat{K} \frac{t^d}{n}.$$

This shows that

$$\lambda_t^n \le \alpha \lambda_{t-1}^n + \hat{K} \alpha^t \frac{t^d}{n} \le \alpha^2 \lambda_{t-2}^n + \hat{K} \alpha^t \frac{(t-1)^d}{n} + \hat{K} \alpha^t \frac{t^d}{n} \le \cdots \le \alpha^{t-1} \lambda_1^n + \hat{K} \frac{\alpha^t}{n} \sum_{i=0}^{t} i^d.$$

Since $\lambda_1^n = L_1^n$ is of the order n^{-1} there exists a constant $\tilde{K}$ such that

$$\lambda_t^n \le \tilde{K} \frac{\alpha^t}{n} + \tilde{K} \frac{\alpha^t}{n} \sum_{i=0}^{t} i^d \le \tilde{K} \alpha^t \frac{1 + t^{d+1}}{n}.$$

By the quotient criteria $K := \tilde{K} \sum_{t \ge 0} \alpha^t (1 + t^{d+1}) < \infty$. This yields

$$\sum_{t \ge 0} \lambda_t^n \le \frac{K}{n}.$$

(Step iv) We are now ready to establish an upper bound on the impact of the boundary condition on the equilibrium actions of an agent $a \in \mathbb{A}$. From (i) to (iii) we obtain

$$|g_a^{n,x} - g_a^{n,y}| \le \sum_{t \ge 1} L_t^n(a) \le \sum_{t \ge 1} \lambda_t^n + \alpha^{n-|a|} \sum_{t \ge 1} \alpha^t \le \frac{K}{n} + \frac{\alpha^{n-|a|}}{1-\alpha}. \tag{16}$$

This shows that uniformly in $a \in \mathbb{A}_n$ the impact of the boundary condition on the agents' equilibrium actions converges to zero as the number of agents tends to infinity. □

Remark 5.4. Our estimate (16) shows that the impact of the boundary condition on equilibrium actions can de decomposed into two parts. The global impact through the dependence of choice on average actions decreases linearly as $n \to \infty$. The local impact through the dependence of neighbors' actions decreases exponentially with an agent's distance to the boundary.

The same arguments as in the previous proposition can also be applied to microscopic equilibria.

Corollary 5.5. *Let $g^{n,x}(\varrho, \cdot)$ be the unique equilibrium configuration of $\mathcal{E}^{n,x}$ with boundary condition x if all the agents optimize their behavior under the assumption that the average action is given by ϱ. Under the assumptions of Theorem 4.1 there exists, for every $\epsilon > 0$, some constants $N, k \in \mathbb{N}$ such that*

$$|g_a^{n,x}(\varrho, J, \theta) - g_a^{n,y}(\varrho, J, \theta)| < \epsilon \quad \mathbb{P}-\text{a.s.}$$

for all $n \geq N$ and each $a \in \mathbb{A}_{n-k}$. The constants k and N are independent of the boundary conditions and the perceived average action.

The fact that the impact of boundary conditions on individual equilibrium actions in finite systems decreases to zero uniformly in the players' distance to the boundary of the set $\mathbb{A}_n$ allows us to prove that accumulation points of empirical averages are almost surely independent of the boundary conditions.

Corollary 5.6. *Let $g^{n,x}(J, \theta)$ be the unique equilibrium configuration of $\mathcal{E}^{n,x}$ with boundary condition x given (J, θ), and let $\varrho^{n,x}(J, \theta)$ be the associated average action. Any accumulation point of the sequence $\{\varrho^{n,x}(J, \theta)\}_{n \in \mathbb{N}}$ is almost surely independent of x.*

Proof. The sequence $\{\varrho^{n,x}(J, \theta)\}_{n \in \mathbb{N}}$ takes values in the compact set X, and so there exists a subsequence $\{n_k\}_{k \in \mathbb{N}}$ and a constant ϱ^x that may both depend on the realizations of the random variables J and θ such that

$$\lim_{k\to\infty} \varrho^{n_k,x}(J, \theta) = \varrho^x.$$

In order to show that ϱ^x does not depend on x, we fix a second boundary condition, $y \in S$, and consider the sequence of empirical averages $\{\varrho^{n,y}(J, \theta)\}_{n \in \mathbb{N}}$ associated with the unique equilibrium profiles $\{g^{n,y}(J, \theta)\}_{n \in \mathbb{N}}$. This sequence converges along a suitable subsequence $\{n_l\}_{l \in \mathbb{N}}$ of $\{n_k\}_{k \in \mathbb{N}}$. By Proposition 5.3 there exists $k \in \mathbb{N}$ and $\epsilon > 0$ such that

$$\lim_{l\to\infty} \left| \frac{1}{|\mathbb{A}_{n_l}|} \sum_{a \in \mathbb{A}_{n_l}} \{g^{n_l,x}(J, \theta) - g^{n_l,y}(J, \theta)\} \right| = \lim_{l\to\infty} \left| \frac{1}{|\mathbb{A}_{n_l-k}|} \sum_{a \in \mathbb{A}_{n_l-k}} \{g^{n_l,x}(J, \theta) - g^{n_l,y}(J, \theta)\} \right| < \epsilon \quad \mathbb{P}-\text{a.s.}$$

This shows that the sequence $\{\varrho^{n,y}(J, \theta)\}_{n \in \mathbb{N}}$ converges along the entire sequence $\{n_k\}_{k \in \mathbb{N}}$ to $\varrho^x(J, \theta)$. Hence any accumulation point of the sequence of average actions is independent of boundary conditions. □

5.3. From average to individual convergence

We now show that convergence of empirical averages implies convergence of individual equilibrium actions. The proof is based on an interplay between microscopic equilibria and equilibria for finite systems and the interplay between microscopic equilibria in finite and infinite systems. More precisely, the vector of equilibrium actions $g^{n,x}(J, \theta)$ of the finite system $\mathcal{E}^{n,x}$ with associated average action $\varrho^{n,x}(J, \theta)$ may be viewed as the unique microscopic equilibrium configuration $g^{n,x}(\varrho^{n,x}(J, \theta), J, \theta)$ where all the agents in $\mathbb{A}_n$ optimize their behavior under the assumption that the average action is given by $\varrho^{n,x}(J, \theta)$, i.e.,

$$g_a^{n,x}(J, \theta) = g_a^{n,x}(\varrho^{n,x}(J, \theta), J, \theta) \quad \text{for all} \quad a \in \mathbb{A}_n. \tag{17}$$

By analogy, for any ϱ, the microscopic equilibrium $g(\varrho, J, \theta)$ of the infinite system $\mathcal{E}$ can be regarded as the microscopic equilibrium $g^{n,g(\varrho,J,\theta)}(\varrho, J, \theta)$ of finite systems with boundary conditions $g(\varrho, J, \theta)$:

$$g_a(\varrho, J, \theta) = g_a^{n,g(\varrho,J,\theta)}(\varrho, J, \theta) \quad \text{for all} \quad a \in \mathbb{A} \quad \text{and each} \quad n \in \mathbb{N}. \tag{18}$$

Proposition 5.7. *Assume that the sequence of average actions* $\{\varrho^{n,x}(J, \theta)\}_{n \in \mathbb{N}}$ *associated with the unique equilibria* $g^{n,x}(J, \theta)$ *of the finite systems* $\mathcal{E}^{n,x}$ *converges to* $\varrho(J, \theta)$ *along some subsequence* $\{n_k\}_{k \in \mathbb{N}}$ *and let* $g(\varrho(J, \theta), J, \theta)$ *be the microscopic equilibrium action profile of the infinite system associated with* $\varrho(J, \theta)$. *The equilibrium action profiles of the finite systems converge almost surely to* $g(\varrho(J, \theta), J, \theta)$ *on the level of individual actions*:

$$\lim_{k\to\infty} g_a^{n_k,x}(J, \theta) = g_a(\varrho(J, \theta), J, \theta) \quad \textit{for any} \quad a \in \mathbb{A}.$$

In particular, accumulation points of the sequence $\{(g^{n,x}(J, \theta), \varrho^{n,x}(J, \theta))\}_{n \in \mathbb{N}}$ *of equilibria and associated empirical distributions are almost surely independent of the boundary condition.*

Proof. We regard $g^{n,x}(J, \theta)$ as the unique equilibrium action profile $g^{n,x}(\varrho^{n,x}(J, \theta), J, \theta,)$ of $\mathcal{E}^{n,x}$ associated with $\varrho^{n,x}(J, \theta)$. The same arguments as in the proof of Lemma 5.1 show that

$$\lim_{k\to\infty} |g_a^{n_k,x}(\varrho^{n,x}(J, \theta), J, \theta) - g_a^{n_k,x}(\varrho(J, \theta), J, \theta)| = 0 \quad \text{for all} \quad a \in \mathbb{A}$$

almost surely. As a result it is enough to prove that

$$\lim_{k\to\infty} |g_a^{n_k,x}(\varrho(J, \theta), J, \theta) - g_a(\varrho(J, \theta), J, \theta)| = 0. \tag{19}$$

In view of (18) the specific choice of the boundary condition $g = g(\varrho(J, \theta), J, \theta)$ yields

$$g_a(\varrho(J, \theta), J, \theta) = g_a^{n,g}(\varrho(J, \theta), J, \theta) \quad \text{for all} \quad n \in \mathbb{N}. \tag{20}$$

Applying Corollary 5.5 to $\varrho = \varrho(J, \theta)$ we obtain for any $\epsilon > 0$ constants $N, k \in \mathbb{N}$ such that almost surely

$$|g_a^{n_k,x}(\varrho(J, \theta), J, \theta) - g_a^{n,g}(\varrho(J, \theta), J, \theta)| < \epsilon \tag{21}$$

for all $n \geq N$ and each $a \in \mathbb{A}_{n-k}$. This proves the convergence in (19), due to (20). □

5.4. Proof of Theorem 4.1

To finish the proof of Theorem 4.1 we first recall from Lemma 5.2 that there exists a set $\mathcal{N}$ of measure zero such that for any ϱ and every pair $(J, \theta) \notin \mathcal{N}$ of interaction patterns and taste shocks, the empirical average associated with the microscopic equilibrium with respect to ϱ exists and is independent of (J, θ). For every such pair it follows from Propositions 5.3 and 5.7 that any accumulation point $(g^x(J, \theta), \varrho^x(J, \theta))$ of the sequence $\{(g^{n,x}(J, \theta), \varrho^{n,x}(J, \theta))\}_{n \in \mathbb{N}}$ of equilibrium action profiles and associated empirical averages is independent of the boundary condition:

$$g_a^x(J, \theta) \equiv g_a(J, \theta) \quad \text{and} \quad \varrho^x(J, \theta) \equiv \varrho(J, \theta). \tag{22}$$

It remains to prove that the average $\varrho(J, \theta)$ does not depend on (J, θ) and that $\varrho(J, \theta)$ is the empirical average associated with the unique equilibrium of the infinite system.

To this end, we consider the average action $\varrho^{n,g(J,\theta)}(J, \theta)$ associated with the equilibrium action profile $g^{n,g(J,\theta)}(J, \theta)$ of the finite system $\mathcal{E}^{n,g(J,\theta)}$ with boundary condition $g(J, \theta)$ defined by (22). By Proposition 5.7

$$\lim_{k\to\infty} g_a^{n_k,g(J,\theta)}(J, \theta) = g_a(\varrho(J, \theta), J, \theta)$$

for every $a \in \mathbb{A}$. Since $g^{n,g(J,\theta)}(J, \theta)$ may be viewed as the unique equilibrium action profile in $\mathcal{E}^{n,g(J,\theta)}$ if the agents optimize their behavior under the assumption that the average action is given by $\varrho^{n,g(J,\theta)}(J, \theta)$ we have

$$g^{n,g(J,\theta)}(J, \theta) = g^{n,g(J,\theta)}(\varrho^{n,g(J,\theta)}(J, \theta), J, \theta).$$

In view of (18) similar arguments as in the proofs of Lemma 5.1 show that

$$|g_a^{n_k,g(J,\theta)}(J, \theta) - g_a(\varrho(J, \theta), J, \theta)| < \epsilon$$

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MATECO-1398; No. of Pages 15
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for all sufficiently large k and all agents $a \in \mathbb{A}_{n_k}$. In particular, the average $\varrho(J, \theta)$ satisfies

$$\varrho(J,\theta) = \lim_{k\to\infty} \varrho^{n_k, g(J,\theta)}(J,\theta) = \lim_{k\to\infty} \frac{1}{|\mathbb{A}_{n_k}|} \sum_{a \in \mathbb{A}_{n_k}} g_a^{n,g(J,\theta)}(J,\theta) = \lim_{k\to\infty} \frac{1}{|\mathbb{A}_{n_k}|} \sum_{a \in \mathbb{A}_{n_k}} g_a(\varrho(J,\theta), J, \theta). \tag{23}$$

By Lemma 5.2 we have for all $(\hat{J}, \hat{\theta}) \in \mathcal{N}^c$ that the microscopic equilibrium with respect to $\varrho(J, \theta)$ satisfies

$$\lim_{n\to\infty} \frac{1}{|\mathbb{A}_n|} \sum_{a \in \mathbb{A}_n} g_a(\varrho(J,\theta), \hat{J}, \hat{\theta}) = \mu[\varrho(J,\theta)]. \tag{24}$$

Choosing $(\hat{J}, \hat{\theta}) = (J, \theta)$ in (24) we see that the limit in (23) exists along the whole sequence and that $\varrho(J, \theta)$ satisfies the fixed point condition

$$\varrho = \mu[\varrho].$$

The assumption of moderate social influence guarantees that this map a *unique* fixed point, due to Lemma 35 in Horst and Scheinkman (2006). This shows that $\varrho(J, \theta)$ is almost surely independent of (J, θ).

6. Conclusion

In this paper, we established a convergence result for equilibria in systems of social interactions when the number of agents growth to infinity. We assumed that the infinite system satisfies an ergodicity property. Under a moderate social influence condition, which restricts the influence of an agent's choices on the optimal decisions of other agents, the finite and infinite systems have a unique equilibrium. We showed that the equilibria of finite systems converge almost surely to the equilibrium of the infinite systems. The convergence takes place both locally, i.e. at the level of individual choices and globally, i.e. at the level of average actions. Our convergence result can thus be seen as a justification for the analysis of infinite systems.

References

Brock, W.A., Durlauf, S.N., 2001. Discrete choice with social interactions. Review of Economic Studies 68 (2), 235–260.

Durlauf, S.N., (1997). Statistical mechanics approaches to socioeconomic behavior. Reprint No. 455. SSRI.

Föllmer, H., Horst, U., 2001. Convergence of locally and globally interacting Markov chains. Stochastic Processes and their Application 96, 99–121.

Glaeser, E., Scheinkman, J.A., 2002. Non-market interactions. In: Dewatripont, M., Hansen, L.P., Turnovsky, S. (Eds.), Advances in Economics and Econometrics: Theory and Applications, Eight World Congress. Cambridge University Press.

Horst, U., 2002. Asymptotics of locally interacting Markov chains with global signals. Advances in Applied Probablity 34 (2), 1–25.

Horst, U., Scheinkman, J., 2006. Equilibria in systems of social interactions. Journal of Economic Theory 130, 44–77.

Morris, S., 1997. Interaction Games: A Unified Analysis of Incomplete Information, Local Interaction and Random Matching. Unpublished. University of Pennsylvania.

Morris, S., Shin, H.S., 2005. Heterogeneity and uniqueness in interaction games. In: Blume, L., Durlauf, S. (Eds.), The Economy as an Evolving Complex System. Oxford University Press.

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