Exercise 1

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1. Let U be a Hilbert space and $A \in L(U)$ be a symmetric operator. Then

$$\|A\|_{L(U)} = \sup_{x \in U, \|x\|_U = 1} |\langle Ax, x \rangle|.$$

2. Let U be a separable Banach space and U^* be its dual space. Prove there exists a sequence $(f_n)_n \subset U^*$ separating points in U, i.e., for any $x, y \in U$ with $x \neq y$, there exists f_n such that $f_n(x) \neq f_n(y)$.

3. Let the assumption in 2. hold. Prove for any $x \in U$ it holds that

$$\|x\|_U = \sup_n f_n(x),$$

where $(f_n)_n$ is the sequence in 2.

4. Let U and H be two Hilbert spaces. For any $B \in L_1(U, H)$ and $A \in L(H, U)$, prove

$$Trace(BA) \leq \|B\|_{L_1(U,H)} \|A\|_{L(H,U)}.$$