Interest Rate Modelling and Derivative Pricing

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Part IV

Term Structure Modelling

Outline

HJM Modelling Framework

Hull-White Model

What are term structure models compared to Vanilla models?

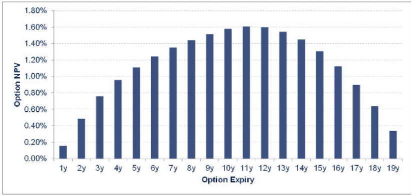
Vanilla models

- Specify dynamics for a single swap rate $S(T)$ with start/end dates T_0/T_n (and details).
- Effectively, only describes terminal distribution of $S(T)$.
- Allows pricing of European swaptions.
- Can be extended to slightly more complex options (with additional assumptions).

Term structure models

- Specify dynamics for evolution of all future zero coupon bonds $P(T, T')$ $(t \leq T \leq T')$.
- Yields (joint) distribution of *all* swap rates $S(T)$.
- Allows pricing of Bermudan swaptions and other complex derivatives.
- Typically, computationally more expensive than Vanilla model pricing.

Why do we need to model the whole term structure of interest rates?



Recall

$$V^{\text{Berm}}(t) = \text{MaxEuropean} + \text{SwitchOption}.$$

- MaxEuropean price is fully determined by Vanilla model.
- Residual SwitchOption price cannot be inferred from Vanilla model.

SwitchOption (i.e. right to postpone future exercise decisions) pricing requires modelling of full interest rate term structure.

Outline

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Heath-Jarrow-Morton specify general dynamics of zero coupon bond prices

Recall our market setting with zero coupon bonds $P(t,T)$ $(t \le T)$ and bank account $B(t) = \exp\left\{\int_0^t r(s)ds\right\}$.

Discounted bond price is martingale in risk-neutral measure.

Martingale representation theorem yields

$$d\left(\frac{P(t,T)}{B(t)}\right) = -\frac{P(t,T)}{B(t)} \cdot \sigma_P(t,T)^\top \cdot dW(t)$$

where $\sigma_P(t,T) = \sigma_P(t,T,\omega)$ is a d-dimensional process adapted to $\mathcal{F}_t$.
We also impose $\sigma_P(T,T) = 0$ (pull-to-par for bond prices with $P(T,T) = 1$).

- What are dynamics of (un-discounted) zero bonds $P(t,T)$?
- What are dynamics of forward rates $f(t,T)$?
- How to specify bond price volatility?

What are dynamics of zero bonds $P(t, T)$?

Lemma (Bond price dynamics)

Under the risk-neutral measure zero bond prices evolve according to

$$\frac{dP(t,T)}{P(t,T)} = r(t) \cdot dt - \sigma_P(t,T)^\top \cdot dW(t).$$

Proof.

Apply Ito's lemma to $d\left(P(t,T)/B(t)\right)$ and compare with dynamics of discounted bond prices.

$\square$

- Zero bond drift equals short rate $r(t)$.
- Zero bond volatility $\sigma_P(t,T)$ remains unchanged.
- How do we get $r(t)$?

What are dynamics of forward rates $f(t, T)$?

Theorem (Forward rate dynamics)

Consider a d-dimensional forward rate volatility process $\sigma_f(t, T) = \sigma_f(t, T, \omega)$ *adapted to* $\mathcal{F}_t$. *Under the risk-neutral measure the dynamics of forward rates* $f(t, T)$ *are given by*

$$df(t, T) = \sigma_f(t, T)^\top \cdot \left[\int_t^T \sigma_f(t, u)du\right] \cdot dt + \sigma_f(t, T)^\top \cdot dW(t)$$

and $f(0, T) = f^M(0, T)$. *Moreover*

$$\sigma_P(t, T) = \int_t^T \sigma_f(t, u)du.$$

- Once volatility $\sigma_f(t, T)$ is specified no-arbitrage pricing yields the drift.
- Model is auto-calibrated to initial yield curve via $f(0, T) = f^M(0, T)$.

We prove the forward rate dynamics (1/2)

Recall

$$f(t, T) = -\frac{\partial}{\partial T} \ln(P(t, T)).$$

Exchanging order of differentiation yields

$$df(t, T) = d\left[-\frac{\partial}{\partial T} \ln(P(t, T))\right] = -\frac{\partial}{\partial T} d\ln(P(t, T)).$$

Applying Ito's lemma (to $d\ln(P(t, T))$) with bond price dynamics yields

$$\begin{aligned} d\ln(P(t, T)) &= \frac{d(P(t, T))}{P(t, T)} - \frac{\sigma_P(t, T)^\top \sigma_P(t, T)}{2} \cdot dt \\ &= \left[r(t) - \frac{\sigma_P(t, T)^\top \sigma_P(t, T)}{2}\right] \cdot dt - \sigma_P(t, T)^\top \cdot dW(t). \end{aligned}$$

Differentiating $d\ln(P(t, T))$ w.r.t. T gives

$$df(t, T) = \left[\frac{\partial}{\partial T}\sigma_P(t, T)\right]^\top \sigma_P(t, T) \cdot dt + \left[\frac{\partial}{\partial T}\sigma_P(t, T)\right]^\top \cdot dW(t).$$

We prove the forward rate dynamics (2/2)

$$df(t,T) = \left[\frac{\partial}{\partial T}\sigma_P(t,T)\right]^\top \sigma_P(t,T)\cdot dt + \left[\frac{\partial}{\partial T}\sigma_P(t,T)\right]^\top \cdot dW(t).$$

Denote

$$\sigma_f(t,T) = \frac{\partial}{\partial T}\sigma_P(t,T).$$

With terminal condition $\sigma_P(T,T) = 0$ follows integral representation

$$\sigma_P(t,T) = \int_t^T \sigma_f(t,u)du.$$

Substituting back gived the result

$$df(t,T) = \sigma_f(t,T)^\top \cdot \left[\int_t^T \sigma_f(t,u)du\right] \cdot dt + \sigma_f(t,T)^\top \cdot dW(t).$$

It will be useful to have the dynamics under the forward measure as well

Lemma (Brownian motion in T-forward measure)

Consider our HJM framework with Brownian motion $W(t)$ under the risk-neutral measure and

$$\frac{dP(t,T)}{P(t,T)} = r(t) \cdot dt - \sigma_P(t,T)^\top \cdot dW(t).$$

Under the T-forward measure the bond price dynamics are

$$\frac{dP(t,T)}{P(t,T)} = \left[r(t) + \sigma_P(t,T)^\top \sigma_P(t,T) \right] \cdot dt - \sigma_P(t,T)^\top \cdot dW^T(t)$$

with $W^T(t)$ a Brownian motion (under the T-forward measure). Moreover,

$$dW^T(t) = \sigma_P(t,T) \cdot dt + dW(t).$$

T-forward measure dynamics can be shown by Ito's lemma

Abbrev. deflated bond prices $Y(t) = \frac{P(t,T)}{B(t)}$, then $\frac{dY(t)}{Y(t)} = -\sigma_P(t,T)^\top dW(t)$. Now consider $1/Y(t)$ and apply Ito's lemma

$$
\begin{aligned}
d\left(\frac{1}{Y(t)}\right) &= -\frac{dY(t)}{Y(t)^2} + \frac{1}{2}\frac{2}{Y(t)^3}\left[dY(t)\right]^2 = \frac{1}{Y(t)}\left[\left(\frac{dY(t)}{Y(t)}\right)^2 - \frac{dY(t)}{Y(t)}\right] \\
&= \frac{1}{Y(t)}\left[\sigma_P(t,T)^\top \sigma_P(t,T)dt + \sigma_P(t,T)^\top dW(t)\right] \\
&= \frac{\sigma_P(t,T)^\top}{Y(t)}\left[\sigma_P(t,T)dt + dW(t)\right].
\end{aligned}
$$

However, $1/Y(t) = B(t)/P(t,T)$ is a martingale in T-forward measure and $d\left(\frac{1}{Y(t)}\right)$ must be drift-less in T-forward measure.
Define $W^T(t)$ with

$$
dW^T(t) = \sigma_P(t,T)dt + dW(t).
$$

Then $W^T(t)$ must be a Brownian motion in the T-forward measure. Substituting $dW(t)$ in the risk-neutral bond price dynamics finally gives the dynamics under T-forward measure.

Outline

Short rate can be derived from forward rate dynamics

Corollary (Short rate specification)

In our HJM framework the short rate becomes

$$
\begin{aligned}
r(t) &= f(t,t) \\
&= f(0,t) + \\
&\quad \int_0^t \sigma_f(u,t)^\top \cdot \left[\int_u^t \sigma_f(u,s)ds\right] du + \int_0^t \sigma_f(u,t)^\top \cdot dW(u).
\end{aligned}
$$

Proof.

Follows directly from forward rate dynamics and integration from 0 to t.

$\square$

- Note that integrand in diffusion term $D(t) = \int_0^t \sigma_f(u,t)^\top \cdot dW(u)$ depends on t.
- In general, $D(t)$ is *not* a martingale.
- In general, $r(t)$ is *not* Markovian unless volatility $\sigma_f(t,T)$ is suitably restricted.

We analyse diffusion term in detail

$$D(t) = \int_0^t \sigma_f(u,t)^\top \cdot dW(u).$$

It follows

$$\begin{aligned}
D(T) &= \int_0^{t} \sigma_f(u,T)^\top \cdot dW(u) + \int_{t}^T \sigma_f(u,T)^\top \cdot dW(u) \\
&= D(t) + \int_t^T \sigma_f(u,T)^\top \cdot dW(u) \\
&\quad + \int_0^t \sigma_f(u,T)^\top \cdot dW(u) - \int_0^t \sigma_f(u,t)^\top \cdot dW(u) \\
&= D(t) + \int_t^T \sigma_f(u,T)^\top \cdot dW(u) + \int_0^t \left[\sigma_f(u,T) - \sigma_f(u,t)\right]^\top \cdot dW(u).
\end{aligned}$$

- How is $\mathbb{E}^{\mathbb{Q}}\left[D(T) \,|\, D(t)\right]$ (knowing only last state) related to $\mathbb{E}^{\mathbb{Q}}\left[D(T) \,|\, \mathcal{F}_t\right]$ (knowing full history)?
- If D is Markovian then $\mathbb{E}^{\mathbb{Q}}\left[D(T) \,|\, D(t)\right] = \mathbb{E}^{\mathbb{Q}}\left[D(T) \,|\, \mathcal{F}_t\right]$ (neccessary condition).

Compare $\mathbb{E}^{\mathbb{Q}}[D(T) \mid D(t)]$ and $\mathbb{E}^{\mathbb{Q}}[D(T) \mid \mathcal{F}_t]$

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[D(T) \mid \mathcal{F}_t] &= \mathbb{E}^{\mathbb{Q}}\left[D(t) + \int_t^T \sigma_f(u,T)^\top dW(u) \mid \mathcal{F}_t\right] \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[\int_0^t [\sigma_f(u,T) - \sigma_f(u,t)]^\top \, dW(u) \mid \mathcal{F}_t\right] \\
&= D(t) + 0 + \underbrace{\int_0^t [\sigma_f(u,T) - \sigma_f(u,t)]^\top \, dW(u)}_{I(t,T)}.
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[D(T) \mid D(t)] &= \mathbb{E}^{\mathbb{Q}}\left[D(t) + \int_t^T \sigma_f(u,T)^\top dW(u) \mid D(t)\right] \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[\int_0^t [\sigma_f(u,T) - \sigma_f(u,t)]^\top \, dW(u) \mid D(t)\right] \\
&= D(t) + 0 + \mathbb{E}^{\mathbb{Q}}\left[\int_0^t [\sigma_f(u,T) - \sigma_f(u,t)]^\top \, dW(u) \mid D(t)\right].
\end{aligned}
$$

- $\mathbb{E}^{\mathbb{Q}}[D(T) \mid D(t)] = \mathbb{E}^{\mathbb{Q}}[D(T) \mid \mathcal{F}_t]$ only if $I(t,T)$ is non-random or deterministic function of $D(t)$.

An important separability condition makes $D(t)$ Markovian

Assume

$$\sigma_f(t, T) = g(t) \cdot h(T)$$

with $g(t)$ (scalar) process adapted to $\mathcal{F}_t$ and $h(T)$ (scalar) deterministic and differentiable function.
Then

$$\begin{aligned} D(T) &= \int_0^t g(u) \cdot h(T) \cdot dW(u) + \int_t^T g(u) \cdot h(T) \cdot dW(u) \\ &= \frac{h(T)}{h(t)} \cdot D(t) + h(T) \cdot \int_t^T g(u) \cdot dW(u). \end{aligned}$$

Thus

$$\mathbb{E}^{\mathbb{Q}} \left[D(T) \,|\, D(t) \right] = \mathbb{E}^{\mathbb{Q}} \left[D(T) \,|\, \mathcal{F}_t \right] = \frac{h(T)}{h(t)} \cdot D(t).$$

Moreover

$$d\left(D(t)\right) = \frac{h'(t)}{h(t)} \cdot D(t) \cdot dt + g(t) \cdot h(t) \cdot dW(t).$$

Outline

We describe a very general but still tractable class of models

- We give a general description of a class of term structure models.
- Typically, these models are called Cheyette-type or quasi-Gaussian models; also associated with work by Ritchken and Sankarasubramanian (1995).
- Particular parameter choices will specialise general model to classical model (e.g. Hull-White model).
- More complex parameter choices yield powerfull model instances for complex interest rate derivatives.

Quasi-Gaussian models are important models in the industry.

Separable forward rate volatility

Definition (Separable forward rate volatility)

The forward rate volatility $\sigma_f(t, T)$ of an HJM model is considered of separable form if

$$\sigma_f(t, T) = g(t)h(T)$$

for a matrix-valued process $g(t) = g(t, \omega) \in \mathbb{R}^{d \times d}$ adapted to $\mathcal{F}_t$ and a vector-valued deterministic function $h(T) \in \mathbb{R}^d$.

Corollary

For a separable forward rate volatility $\sigma_f(t, T) = g(t)h(T)$ the bond price volatility $\sigma_P(t, T)$ becomes

$$\sigma_P(t, T) = g(t) \int_t^T h(u)du.$$

Forward rate representation follows directly

Lemma

For a separable forward rate volatility $\sigma_f(t,T) = g(t)h(T)$ the forward rate becomes

$$
\begin{aligned}
f(t,T) = f(0,T) + &\\
& h(T)^\top \int_0^t g(s)^\top g(s) \left(\int_s^T h(u)du \right) ds + \\
& h(T)^\top \int_0^t g(s)^\top dW(s)
\end{aligned}
$$

and

$$
r(t) = f(0,t) + h(t)^\top \left[\int_0^t g(s)^\top g(s) \left(\int_s^t h(u)du \right) ds + \int_0^t g(s)^\top dW(s) \right].
$$

Proof.

Follows directly from definition.

$\square$

We need to introduce new state variables to derive Markovian representation of short rate

Re-write

$$r(t) = f(0,t) + \mathbf{1}^\top H(t)\left[\int_0^t g(s)^\top g(s)\left(\int_s^t h(u)du\right)ds + \int_0^t g(s)^\top dW(s)\right]$$

with

$$\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \text{ and } H(t) = diag\left(h(t)\right) = \begin{pmatrix} h_1(t) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & h_d(t) \end{pmatrix}.$$

Introduce vector of state variables $x(t)$ with

$$x(t) = H(t)\left[\int_0^t g(s)^\top g(s)\left(\int_s^t h(u)du\right)ds + \int_0^t g(s)^\top dW(s)\right].$$

We derive the dynamics of the short rate

Theorem (Separable HJM short rate dynamics)

In an HJM model with separable volatility the short rate is given by $r(t) = f(0,t) + \mathbf{1}^\top x(t)$. *The vector of state variables* $x(t)$ *evolves according to* $x(0) = 0$ *and*

$$dx(t) = [y(t)\mathbf{1} - \chi(t)x(t)]\,dt + H(t)g(t)^\top dW(t)$$

with symmetric matrix of auxilliary variables $y(t)$ *as*

$$y(t) = H(t)\left(\int_0^t g(s)^\top g(s)ds\right)H(t)$$

and diagonal matrix of mean reversion parameters $\chi(t)$ *as*

$$\chi(t) = -\frac{dH(t)}{dt}H(t)^{-1}.$$

Proof follows straight forward via differentiation (1/3)

We have

$$x(t) = H(t)\underbrace{\left[\int_0^t g(s)^\top g(s)\left(\int_s^t h(u)du\right)ds + \int_0^t g(s)^\top dW(s)\right]}_{G(t)}.$$

$$\begin{aligned} dx(t) &= H'(t)\cdot G(t)\cdot dt + H(t)\cdot dG(t) \\ &= H'(t)\cdot H(t)^{-1}\cdot H(t)\cdot G(t)\cdot dt + H(t)\cdot dG(t) \\ &= -\chi(t)\cdot x(t)\cdot dt + H(t)\cdot dG(t). \end{aligned}$$

Proof follows straight forward via differentiation (2/3)

$$\begin{aligned}
dx(t) &= -\chi(t) \cdot x(t) \cdot dt + H(t) \cdot dG(t), \\
G(t) &= \int_0^t g(s)^\top g(s) \left(\int_s^t h(u)du \right) ds + \int_0^t g(s)^\top dW(s).
\end{aligned}$$

Leibnitz rule yields

$$\begin{aligned}
dG(t) = &\left[g(t)^\top g(t) \left(\int_t^t h(u)du \right) + \int_0^t g(s)^\top g(s) \frac{d}{dt} \left(\int_s^t h(u)du \right) ds \right] dt \\
&+ g(t)^\top dW(t) \\
= &\left[0 + \int_0^t g(s)^\top g(s) \cdot H(t)\mathbf{1} \cdot ds \right] dt + g(t)^\top dW(t) \\
= &\left[\left(\int_0^t g(s)^\top g(s) ds \right) H(t)\mathbf{1} \right] dt + g(t)^\top dW(t).
\end{aligned}$$

Proof follows straight forward via differentiation (3/3)

Combining results gives

$$
\begin{aligned}
dx(t) &= -\chi(t) \cdot x(t) \cdot dt + H(t) \cdot dG(t) \\
&= \left[H(t) \left(\int_0^t g(s)^\top g(s) ds \right) H(t) \mathbf{1} - \chi(t) \cdot x(t) \right] dt \\
&\quad + H(t) \cdot g(t)^\top dW(t) \\
&= \left[y(t) \cdot \mathbf{1} - \chi(t) \cdot x(t) \right] dt + H(t) \cdot g(t)^\top dW(t).
\end{aligned}
$$

- Note that $dx(t)$ depends on accumulated previous volatility via $\int_0^t g(s)^\top g(s) ds$.
- $x(t)$ is Markovian only if volatility function $g(t)$ is deterministic.
- In general, short rate dynamics can be ammended by dynamics of $y(t)$.

Short rate dynamics can be written in terms of state and auxilliary variables

Corollary (Augmented short rate dynamics)

In an HJM model with separable volatility the short rate is given via $r(t) = f(0,t) + \mathbf{1}^\top x(t)$ *with*

$$dx(t) = \left[y(t)\cdot\mathbf{1} - \chi(t)\cdot x(t)\right] dt + \sigma_r(t)^\top dW(t),$$
$$dy(t) = \left[\sigma_r(t)^\top\sigma_r(t) - \chi(t)y(t) - y(t)\chi(t)\right] dt,$$

and $x(0) = 0$, $y(0) = 0$.

Proof.

Set $\sigma_r(t) = g(t)H(t)$ and differentiate $y(t) = H(t)\left(\int_0^t g(s)^\top g(s)ds\right)H(t)$.

□

- Model class also called Cheyette or quasi-Gaussian models.
- Typically $\sigma_r(t)$ and $\chi(t)$ are specified and $\sigma_f(t,T)$ is reconstructed via

$$H'(t) = -\chi(t)H(t),\ H(0) = 1 \text{ and } g(t) = \sigma_r(t)H(t)^{-1}.$$

Forward rates and zero bonds can be written in terms of state/auxilliary variables

Theorem (Forward rate and zero bond reconstruction)

In our HJM model setting we get

$$f(t,T) = f(0,T) + \mathbf{1}^\top H(T)H(t)^{-1}\left[x(t) + y(t)G(t,T)\right]$$

and

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left\{-G(t,T)^\top x(t) - \frac{1}{2}G(t,T)^\top y(t)G(t,T)\right\}$$

with

$$G(t,T) = \int_t^T H(u)H(t)^{-1}\mathbf{1}du.$$

- We prove the first part for $f(t,T)$.
- And we sketch the proof for the second part for $P(t,T)$.

We prove the first part for $f(t,T)$ ...

$$\underbrace{\mathbf{1}^\top H(T)H(t)^{-1}x(t)}_{I_1} = h(T)^T \left[\int_0^t g(s)^\top g(s)\left(\int_s^{t} h(u)du\right)ds + \int_0^t g(s)^\top dW(s)\right]$$

$$\underbrace{\mathbf{1}^\top H(T)H(t)^{-1}y(t)G(t,T)}_{I_2} = h(T)^\top \left(\int_0^t g(s)^\top g(s)ds\right)\int_{t}^T h(u)du$$

$$\begin{aligned}
I_1 + I_2 &= h(T)^T \left[\int_0^t g(s)^\top g(s)\left(\int_s^{t} h(u)du\right)ds + \left(\int_0^t g(s)^\top g(s)ds\right)\int_{t}^T h(u)du\right] \\
&\quad + h(T)^T \int_0^t g(s)^\top dW(s) \\
&= h(T)^T \left[\int_0^t g(s)^\top g(s)\left(\int_s^{t} h(u)du + \int_{t}^T h(u)du\right)ds + \int_0^t g(s)^\top dW(s)\right] \\
&= h(T)^T \left[\int_0^t g(s)^\top g(s)\left(\int_s^T h(u)du\right)ds + \int_0^t g(s)^\top dW(s)\right] \\
&= f(t,T) - f(0,T)
\end{aligned}$$

... and sketch the proof for the second part for $P(t, T)$

$$
\begin{aligned}
P(t, T) &= \exp\left\{-\int_t^T f(t, s) ds\right\} \\
&= \exp\left\{-\int_t^T \left(f(0, s) + \mathbf{1}^\top H(s) H(t)^{-1} \left[x(t) + y(t) G(t, s)\right]\right) ds\right\} \\
&= \frac{P(0, T)}{P(0, t)} \cdot \exp\left\{-\underbrace{\left(\int_t^T \mathbf{1}^\top H(s) H(t)^{-1} ds\right)}_{G(t,T)^\top} x(t)\right\} \cdot \\
&\qquad \exp\left\{-\int_t^T \mathbf{1}^\top H(s) H(t)^{-1} y(t) G(t, s) ds\right\}
\end{aligned}
$$

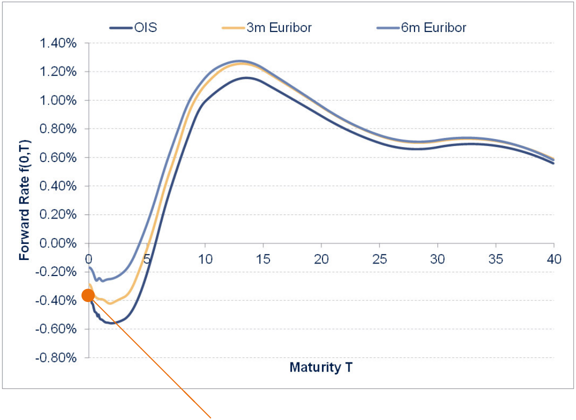
The equality

$$
\int_t^T \mathbf{1}^\top H(s) H(t)^{-1} y(t) G(t, s) ds = \frac{1}{2} G(t, T)^\top y(t) G(t, T)
$$

follows by differentiating both sides w.r.t. T.

Outline

We take a complementary view to HJM framework and consider direct modelling of the short rate $r(t)$



short rate $r(t) = f(t, t)$

We model short rate of the discount curve as offset point for future rates.

Short rate suffices to specify evolution of the full yield curve

Recall zero bond formula

$$P(t, T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left\{-\int_t^T r(s)ds\right\} \mid \mathcal{F}_t\right].$$

- Once dynamics of $r(t)$ are specified all zero bonds can be derived.

Libor rates (in multi-curve setting) are

$$L(t; T_0, T_1) = \mathbb{E}^{T_1}\left[L(T; T_0, T_1) \mid \mathcal{F}_t\right] = \left[\frac{P(t, T_0)}{P(t, T_1)} \cdot D(T_0, T_1) - 1\right]\frac{1}{\tau}.$$

- With zero bonds $P(t, T)$ (and spread factors $D(T_0, T_1)$) we can also derive future Libor rates.

Short rate is a natural choice of state variable for modelling evolution of interest rates.

Outline

Vasicek model and Ho-Lee model were the first models for the short rate

Vasicek (1977) assumed Ornstein-Uhlenbeck process

$$dr(t) = \kappa\,(\theta - r(t))\,dt + \sigma dW(t), \quad r(0) = r_0$$

for positive constants r_0, κ, θ, and σ.

- Model is not too different from HJM model representation.
- Constant parameters (in particular θ) limit ability to reproduce yield curve observed today.

Ho and Lee (1986) introduce exogenous time-dependent drift parameter,

$$dr(t) = \theta(t)dt + \sigma dW(t).$$

- Drift parameter $\theta(t)$ is used to match today's zero bonds $P(0, T)$.
- Lack of mean reversion is considered main disadvantage.
- Model was historically used with binomial tree implementation.

Hull and White (1990) extended Vasicek model by $\theta(t)$

Definition (Hull-White model)

In the Hull-White model the short rate evolves according to

$$dr(t) = [\theta(t) - a(t)r(t)]\,dt + \sigma(t)dW(t)$$

with deterministic scalar functions $\theta(t)$, $a(t)$, and $\sigma(t) > 0$.

- $\theta(t)$ is mean reversion level,
- $a(t)$ is mean reversion speed, and
- $\sigma(t)$ is short rate volatility.

- Original reference is J. Hull and A. White. Pricing interest-rate-derivative securities.
  The Review of Financial Studies, 3:573–592, 1990
- To simplify analytical tractability we will assume
  - constant mean reversion speed $a(t) = a > 0$, and
  - piece-wise constant short rate volatility function on a siutable time grid $\{t_0, \ldots, t_k\}$,

$$\sigma(t) = \sum_{i=1}^{k} \mathbb{1}_{\{t_{i-1} \le t < t_i\}} \cdot \sigma_i.$$

How do we calibrate the drift $\theta(t)$?

Lemma (Hull-White drift calibration)

In the risk-neutral specification of the Hull-White model the drift term $\theta(t)$ is given by

$$\theta(t) = \frac{\partial}{\partial T} f(0,t) + a \cdot f(0,t) + \int_0^t \left[e^{-a(t-u)} \sigma(u) \right]^2 du.$$

Here $f(0,t) = f^M(0,t)$ is exogenously specified and assumed continuously differentiable w.r.t. the maturity T.

Proof follows along the following steps

- Calculate $r(s)$ via integration.
- Integrate $I(t,T) = \int_t^T r(s)ds$ and calculate distribution of $I(t,T)$.
- Derive $\theta(t)$ such that $\mathbb{E}^{\mathbb{Q}}\left[e^{-I(0,t)} \right] = P(0,T)$.

Proof (1/4) - calculate $r(s)$

We show that for $s \geq t$

$$r(s) = e^{-a(s-t)} \left[r(t) + \int_t^s e^{a(u-t)} \left[\theta(u)du + \sigma(u)dW(u) \right] \right].$$

$$\begin{aligned} dr(s) &= -ar(s)ds + e^{-a(s-t)} \left[e^{a(s-t)} \left[\theta(s)ds + \sigma(s)dW(s) \right] \right] \\ &= \left[\theta(s) - ar(s) \right] ds + \sigma(s)dW(s). \end{aligned}$$

Use notation $[\cdot]'(t, T) = \frac{\partial}{\partial T}[\cdot]$. Set $I(t, T) = \int_t^T r(s)ds$, then $I'(t, T) = \frac{\partial I(t,T)}{\partial T} = r(T)$. We show

$$I(t, T) = G(t, T)r(t) + \int_t^T G(u, T) \left[\theta(u)du + \sigma(u)dW(u) \right]$$

with

$$G(t, T) = \int_t^T e^{-a(u-t)} du = \left[\frac{1 - e^{-a(T-t)}}{a} \right].$$

Proof (2/4) - calculate distribution $I(t, T)$

$$
\begin{aligned}
I(t,T) &= G(t,T)r(t) + \int_t^T G(u,T)\left[\theta(u)du + \sigma(u)dW(u)\right], \\
I'(t,T) &= G'(t,T)r(t) + 0 + \int_t^T G'(u,T)\left[\theta(u)du + \sigma(u)dW(u)\right] \\
&= e^{-a(T-t)}r(t) + \int_t^T e^{-a(T-u)}\left[\theta(u)du + \sigma(u)dW(u)\right] \\
&= e^{-a(T-t)}\left[r(t) + \int_t^T e^{a(u-t)}\left[\theta(u)du + \sigma(u)dW(u)\right]\right] \\
&= r(T).
\end{aligned}
$$

Integral $I(t, T)$ is normally distributed, $I(t, T) \sim N(\mu, \sigma^2)$ with

$$
\begin{aligned}
\mu(t,T) &= G(t,T)r(t) + \int_t^T G(u,T)\theta(u)du, \\
\sigma(t,T)^2 &= \int_t^T \left[G(u,T)\sigma(u)\right]^2 du.
\end{aligned}
$$

Proof (3/4) - calculate forward rate

$I(t, T) \sim N(\mu, \sigma^2)$ with

$$\mu(t, T) = G(t, T)r(t) + \int_t^T G(u, T)\theta(u)du, \quad \sigma^2(t, T) = \int_t^T [G(u, T)\sigma(u)]^2 \, du.$$

$$P(t, T) = \mathbb{E}^{\mathbb{Q}} \left[e^{-I(t,T)} \,|\, \mathcal{F}_t \right] = e^{-\mu(t,T) + \frac{1}{2}\sigma^2(t,T)}.$$

$$\begin{aligned}
f(t, T) &= -\frac{\partial}{\partial T} \ln [P(t, T)] = \frac{d}{dT} \left[\mu(t, T) - \frac{1}{2}\sigma^2(t, T) \right] \\
&= G'(t, T)r(t) + 0 + \int_t^T G'(u, T)\theta(u)du \\
&\quad - \frac{1}{2} \left[0 + \int_t^T 2G(u, T)G'(u, T)\sigma(u)^2 du \right] \\
&= G'(t, T)r(t) + \int_t^T G'(u, T)\theta(u)du - \int_t^T G'(u, T)G(u, T)\sigma(u)^2 du.
\end{aligned}$$

Proof (4/4) - derive drift $\theta(t)$

$$f(t,T) = G'(t,T)r(t) + \int_t^T G'(u,T)\theta(u)du - \int_t^T G'(u,T)G(u,T)\sigma(u)^2 du.$$

Use $G'(t,T) = e^{-a(T-t)}$ and $G''(t,T) = -aG'(t,T)$, then

$$\begin{aligned} f'(t,T) &= G''(t,T)r(t) + \theta(T) + \int_t^T G'(u,T)\theta(u)du - 0 \\ &\quad - \int_t^T \left[G''(u,T)G(u,T) + G'(u,T)^2\right]\sigma(u)^2 du \\ &= \theta(T) - af(t,T) - \int_t^T \left[G'(u,T)\sigma(u)\right]^2 du. \end{aligned}$$

This finally gives the result (with $t = 0$)

$$\begin{aligned} \theta(T) &= f'(t,T) + af(t,T) + \int_t^T \left[G'(u,T)\sigma(u)\right]^2 du \\ &= f'(0,T) + af(0,T) + \int_0^T \left[e^{-a(T-u)}\sigma(u)\right]^2 du. \end{aligned}$$

Do we really need the drift $\theta(t)$?

- Risk-neutral drift representation

$$\theta(t) = \frac{\partial}{\partial T} f(0,t) + a \cdot f(0,t) + \int_0^t \left[e^{-a(t-u)} \sigma(u) \right]^2 du$$

poses some obstacles.

- Derivative $\frac{\partial}{\partial T} f(0,t)$ may cause numerical difficulties.
- In some market situations you want to have jumps in $f(0,t)$.
- This is relevant in particular for the short end of OIS curve.

- Fortunately, for most applications we don't need drift term.

- HJM representation allows avoiding it alltogether.

Now we can also derive future zero bond prices I

Theorem (Zero bonds in Hull-White model)

In the Hull-White model future zero bond prices are given by

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left\{ -G(t,T)\left[r(t) - f(0,t)\right] - \frac{G(t,T)^2}{2} \int_0^t \left[e^{-a(t-u)} \sigma(u) \right]^2 du \right\}$$

with

$$G(t,T) = \int_t^T e^{-a(u-t)} du = \left[\frac{1 - e^{-a(T-t)}}{a} \right].$$

- The proof is a bit technical.
- We already derived the zero bond representation

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-I(t,T)} \,|\, \mathcal{F}_t \right] = e^{-\mu(t,T) + \frac{1}{2}\sigma^2(t,T)}.$$

Now we can also derive future zero bond prices II

We have from the proof of risk-neutral drift that

$$f(t,T) = G'(t,T)r(t) + \int_t^T G'(u,T)\theta(u)du - \int_t^T G'(u,T)G(u,T)\sigma^2(u)du$$

and

$$P(t,T) = e^{-G(t,T)r(t) - \int_t^T G(u,T)\theta(u)du + \frac{1}{2}\int_t^T G(u,T)^2\sigma^2(u)du}.$$

We aim at calculating the term

$$I(t,T) = -\int_t^T G(u,T)\theta(u)du + \frac{1}{2}\int_t^T G(u,T)^2\sigma^2(u)du.$$

Now we can also derive future zero bond prices III

Consider

$$
\begin{aligned}
\log\left(\frac{P(0,t)}{P(0,T)}\right) &= \left[G(0,T) - G(0,t)\right] r(0) + \int_0^T G(u,T)\theta(u)du - \int_0^t G(u,t)\theta(u)du \\
&\quad - \frac{1}{2}\left[\int_0^T G(u,T)^2\sigma^2(u)du - \int_0^t G(u,t)^2\sigma^2(u)du\right] \\
&= \left[G(0,T) - G(0,t)\right] r(0) \\
&\quad + \int_t^T G(u,T)\theta(u)du + \int_0^t \left[G(u,T) - G(u,t)\right]\theta(u)du \\
&\quad - \frac{1}{2}\left[\int_t^T G(u,T)^2\sigma^2(u)du + \int_0^t \left[G(u,T)^2 - G(u,t)^2\right]\sigma^2(u)du\right].
\end{aligned}
$$

Now we can also derive future zero bond prices IV

We use $G(u,T) - G(u,t) = G(t,T)G'(u,t)$ and re-arrange terms. Then

$$
\begin{aligned}
I(t,T) = \log\left(\frac{P(0,T)}{P(0,t)}\right) &+ G(t,T)G'(0,t)r(0) \\
&+ G(t,T)\int_0^t G'(u,t)\theta(u)du \\
&- \frac{1}{2}\int_0^t \underbrace{\left[G(u,T)+G(u,t)\right]\left[G(u,T)-G(u,t)\right]}_{[G(u,T)-G(u,t)+2G(u,t)]G(t,T)G'(u,t)} \sigma^2(u)du.
\end{aligned}
$$

We use representation for forward rate $f(t,T)$ and get

$$
\begin{aligned}
I(t,T) &= \log\left(\frac{P(0,T)}{P(0,t)}\right) + G(t,T)f(0,t) \\
&\quad - \frac{1}{2}\int_0^t \left[G(u,T)-G(u,t)\right] G(t,T)G'(u,t)\sigma^2(u)du \\
&= \log\left(\frac{P(0,T)}{P(0,t)}\right) + G(t,T)f(0,t) - \frac{G(t,T)^2}{2}\int_0^t G'(u,t)^2\sigma^2(u)du.
\end{aligned}
$$

Now we can also derive future zero bond prices V

Finally, we get the result

$$
\begin{aligned}
P(t, T) &= e^{-G(t,T)r(t)+I(t,T)} \\
&= \frac{P(0, T)}{P(0, t)} e^{-G(t,T)[r(t)-f(0,t)]-\frac{G(t,T)^2}{2}\int_0^t \left[e^{-a(t-u)}\sigma(u)\right]^2 du}.
\end{aligned}
$$

- Future zero coupon bonds depend on:
  - today's yield curve $f(0, t)$,
  - mean reversion parameter a via $G(t, T)$, and
  - short rate volatility $\sigma(t)$.
- We see that drift $\theta(t)$ is not required for future zero coupon bonds.

Outline

Recall short rate dynamics in separable HJM model

We consider a one-factor model ($d = 1$)

$$\begin{aligned} r(t) &= f(0,t) + x(t) \\ dx(t) &= [y(t) - \chi(t) \cdot x(t)]\, dt + \sigma_r(t) \cdot dW(t) \\ dy(t) &= \left[\sigma_r(t)^2 - 2 \cdot \chi(t) \cdot y(t)\right] \cdot dt \end{aligned}$$

with

$$H'(t) = -\chi(t)H(t),\ H(0) = 1 \text{ and } g(t) = H(t)^{-1}\sigma_r(t).$$

- How does this relate to Hull-White model with

$$dr(t) = [\theta(t) - a \cdot r(t)] \cdot dt + \sigma(t) \cdot dW(t)?$$

Differentiate short rate in HJM model

$$
\begin{aligned}
dr(t) &= f'(0,t)dt + dx(t) \\
&= f'(0,t)dt + [y(t) - \chi(t)x(t)]\,dt + \sigma_r(t)dW(t) \\
&= \left[f'(0,t) + y(t) - \chi(t)\left(r(t) - f(0,t)\right)\right] dt + \sigma_r(t)dW(t) \\
&= \left[\underbrace{f'(0,t) + \chi(t)f(0,t) + y(t)}_{\theta(t)} - \underbrace{\chi(t)}_{a}\, r(t)\right] dt + \underbrace{\sigma_r(t)}_{\sigma(t)} dW(t)
\end{aligned}
$$

HJM volatility parameters become

$$
H'(t) = -aH(t), \quad H(0) = 1 \Rightarrow h(t) = H(t) = e^{-at},
$$

$$
g(t) = \sigma_r(t) \cdot H(t)^{-1} = \sigma(t)e^{at}.
$$

Deterministic volatility allows calculation of auxilliary variable $y(t)$

We have

$$y'(t) = \sigma(t)^2 - 2 \cdot a \cdot y(t), \quad y(0) = 0.$$

Solving initial value problem yields

$$y(t) = \int_0^t \sigma(u)^2 \cdot e^{-2a(t-u)} du.$$

Hull-White model in HJM notation

In the HJM framework the Hull-White model becomes

$$\begin{aligned} r(t) &= f(0,t) + x(t), \\ dx(t) &= \left[\int_0^t \sigma(u)^2 \cdot e^{-2a(t-u)} du - a \cdot x(t) \right] \cdot dt + \sigma(t) \cdot dW(t), \\ x(0) &= 0. \end{aligned}$$

We will use this representation of the Hull-White model for our implementations.

This also gives HJM representation of Hull-White model

Corollary (Forward rate dynamics in Hull-White model)

In a Hull-White model the dynamics of the forward rate $f(t,T)$ become

$$df(t,T) = \sigma(t)^2 e^{-a(T-t)} \frac{1-e^{-a(T-t)}}{a} dt + \sigma(t) e^{-a(T-t)} dW(t).$$

Proof.

$$\begin{aligned}
df(t,T) &= \sigma_f(t,T) \cdot \left[\int_t^T \sigma_f(t,u)du\right] \cdot dt + \sigma_f(t,T) \cdot dW(t) \\
&= g(t)h(T) \left[\int_t^T g(t)h(u)du\right] \cdot dt + g(t)h(T) \cdot dW(t) \\
&= \sigma(t)^2 e^{-a(T-t)} \underbrace{\left[\int_t^T e^{-a(u-t)}du\right]}_{\frac{1-e^{-a(T-t)}}{a}} \cdot dt + \sigma(t) e^{-a(T-t)} \cdot dW(t).
\end{aligned}$$

□

Zero bond prices may also be computed in terms of $x(t)$

Corollary (Zero bonds in Hull-White model)

In the Hull-White model future zero coupon bonds are

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left\{ -G(t,T)x(t) - \frac{G(t,T)^2}{2} \int_0^t \left[e^{-a(t-u)} \sigma(u) \right]^2 du \right\}$$

with

$$G(t,T) = \int_t^T e^{-a(u-t)} du = \left[\frac{1 - e^{-a(T-t)}}{a} \right].$$

Proof.

Result follows either from Hull-White model zero bond formula with $x(t) = r(t) - f(0,T)$ or from zero bond formula for the separable HJM model with Hull-White results for $G(t,T)$ and $y(t)$.

$\square$

Outline

First we need the distribution of the state variable $x(t)$

We have

$$dx(t) = [y(t) - a \cdot x(t)] \cdot dt + \sigma(t) \cdot dW(t).$$

This yields for $t \geq s$

$$x(t) = e^{-a(t-s)} \left[x(s) + \int_s^t e^{a(u-s)} \left(y(u)du + \sigma(u)dW(u) \right) \right].$$

Lemma (State variable distribution)

In the HJM version of the Hull-White model we have that under the risk-neutral measure the state variable $x(t)$ is normally distributed with

$$\mathbb{E}^{\mathbb{Q}} \left[x(t) \,|\, \mathcal{F}_s \right] = e^{-a(t-s)} \left[x(s) + \int_s^t e^{a(u-s)} y(u) du \right] \text{ and}$$

$$Var \left[x(t) \,|\, \mathcal{F}_s \right] = \int_s^t \left[e^{-a(t-u)} \sigma(u) \right]^2 du.$$

Result follows directly from state variable representation for $x(t)$

Proof.

Result for $\mathbb{E}\left[x(t)\,|\,\mathcal{F}_s\right]$ follows from martingale property of Ito integral.
Variance follows from Ito isometry

$$\begin{aligned}\operatorname{Var}\left[x(t)\,|\,\mathcal{F}_s\right] &= e^{-2a(t-s)}\int_s^t \left[e^{-a(u-s)}\sigma(u)\right]^2 du \\ &= \int_s^t \left[e^{-a(t-u)}\sigma(u)\right]^2 du.\end{aligned}$$

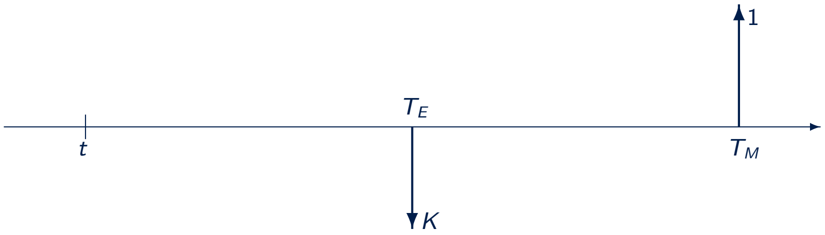
$\square$

- We will have a closer look at $\mathbb{E}^{\mathbb{Q}}\left[x(t)\,|\,\mathcal{F}_s\right] = e^{-a(t-s)}\left[x(s) + \int_s^t e^{a(u-s)}y(u)du\right]$ later on.
- Note, that we can also write

$$\operatorname{Var}\left[x(t)\,|\,\mathcal{F}_s\right] = y(t) - G'(s,t)^2 y(s).$$

Auxilliary variable $y(t)$ represents the (co-)variance process of $x(t)$.

Zero coupon bond options are important building blocks



Definition (Zero coupon bond (ZCB) option)

A zero coupon bond option is defined as an option with expiry time T_E, ZCB maturity time T_M with $T_M \geq T_E$, strike K , call/put flag $\phi \in \{1, -1\}$ and payoff

$$V^{\mathrm{ZBO}}(T_E) = \left[\phi\left(P(T_E, T_M) - K\right)\right]^+ .$$

- We are interested in present value $V^{\mathrm{ZBO}}(t)$.
- We use T_E-forward measure for valuation

$$V^{\mathrm{ZBO}}(t) = P(t, T_E) \cdot \mathbb{E}^{T_E}\left[\left[\phi\left(P(T_E, T_M) - K\right)\right]^+ \mid \mathcal{F}_t\right] .$$

$P(T_E, T_M)$ is log-normally distributed with known parameters

- We have for the forward bond price
$$\mathbb{E}^{T_E}\left[P(T_E, T_M) \,|\, \mathcal{F}_t\right] = P(t, T_M)/P(t, T_E).$$
- From
$$P(T_E, T_M) = \frac{P(t, T_M)}{P(t, T_E)} e^{-G(T_E, T_M)x(T_E) - \frac{G(T_E, T_M)^2}{2} \int_t^{T_E} \left[e^{-a(T_E - u)}\sigma(u)\right]^2 du}$$

  we get
  - $P(T_E, T_M)$ is log-normally distributed with log-normal variance
  $$\nu^2 = \mathrm{Var}\left[G(T_E, T_M)x(T_E) \,|\, \mathcal{F}_t\right] = G(T_E, T_M)^2 \int_t^{T_E} \left[e^{-a(T_E - u)}\sigma(u)\right]^2 du,$$
  - we can apply Black's formula for option pricing.

ZCO prices are given by Black's formula

Theorem (ZCO pricing formula)

The time-t price of a zero coupon bond option with expiry time T_E, ZCB maturity time T_M with $T_M \geq T_E$, strike K , call/put flag $\phi \in \{1, -1\}$ and payoff

$$V^{ZBO}(T_E) = [\phi (P(T_E, T_M) - K)]^+$$

is given by

$$V^{ZBO}(t) = P(t, T_E) \cdot Black(P(t, T_M)/P(t, T_E), K, \nu, \phi)$$

with log-normal bond price variance

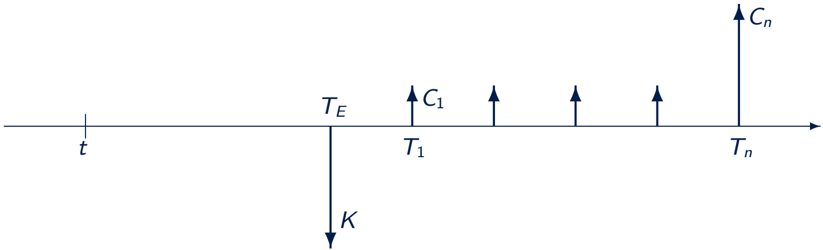
$$\nu^2 = G(T_E, T_M)^2 \int_t^{T_E} \left[e^{-a(T_E - u)} \sigma(u)\right]^2 du.$$

Proof.

Result follows from log-normal distribution property.

□

Coupon bond options are further building blocks



Payoff at option expiry T_E

$$V(T_E) = \left[\left(\sum_{i=1}^{n} C_i \cdot P(T_E, T_i) \right) - K \right]^+ .$$

Coupon bond options are options on a basket of future cash flows

Definition (Coupon bond option (CBO))

A coupon bond option is defined as an option with expiry time T_E, future cash flow payment times $T_1, \ldots, T_n$ (with $T_i > T_E$), corresponding cash flow values $C_1, \ldots, C_n$, a fixed strike price K, call/put flag $\phi \in \{1, -1\}$ and payoff

$$V^{\mathrm{CBO}}(T_E) = \left[\left(\phi\left[\left(\sum_{i=1}^{n} C_i P(T_E, T_i)\right) - K\right]\right)^+\right].$$

- We cannot price CBO directly due to the basket structure.
- However, with some (not too strong) assumptions we can represent the 'option on a basket' as a 'basket of options'.
- We use monotonicity of bond prices (for $t < T$)

$$\frac{\partial}{\partial x} P(x(t); t, T) = -G(t, T) \cdot P(x(t); t, T) < 0.$$

CBO's are transformed via Jamshidian's trick I

W.l.o.g. set $\phi = 1$ (method works for $\phi = -1$ as well).
Assume underlying bond is monotone in state variable $x(T_E)$, i.e.

$$\frac{\partial}{\partial x} \sum_{i=1}^{n} C_i P(x(T_E); T_E, T_i) = \sum_{i=1}^{n} C_i \frac{\partial}{\partial x} P(x(T_E); T_E, T_i)$$
$$= -\sum_{i=1}^{n} C_i G(T_E, T_i) P(x(T_E); T_E, T_i) < 0.$$

- Condition is satisfied, e.g. if $C_i \geq 0$.
- Small negative cash flows typically don't violate the assumption since last cash flow C_n is typically a large positive cash flow.

CBO's are transformed via Jamshidian's trick II

Then find $x^\star$ such that

$$\left(\sum_{i=1}^{n} C_i P(x^\star; T_E, T_i)\right) - K = 0$$

and set $K_i = P(x^\star; T_E, T_i)$.
We get (using monotonicity assumption)

$$\begin{aligned}
\left[\left(\sum_{i=1}^{n} C_i P(T_E, T_i)\right) - K\right]^+ &= \mathbb{1}_{\{x(T_E) \le x^\star\}} \left[\left(\sum_{i=1}^{n} C_i P(T_E, T_i)\right) - K\right] \\
&= \mathbb{1}_{\{x(T_E) \le x^\star\}} \left[\sum_{i=1}^{n} C_i P(T_E, T_i) - \sum_{i=1}^{n} C_i K_i\right] \\
&= \left[\sum_{i=1}^{n} C_i \left[P(T_E, T_i) - K_i\right] \mathbb{1}_{\{x(T_E) \le x^\star\}}\right] \\
&= \left[\sum_{i=1}^{n} C_i \left[P(T_E, T_i) - K_i\right]^+\right].
\end{aligned}$$

CBO's are transformed via Jamshidian's trick III

This gives

$$\mathbb{E}^{T_E}\left[\left[\left(\sum_{i=1}^{n} C_i P(T_E, T_i)\right) - K\right]^+\right] = \sum_{i=1}^{n} C_i \underbrace{\mathbb{E}^{T_E}\left[[P(T_E, T_i) - K_i]^+\right]}_{\text{Black's formula}}$$

or

$$\begin{aligned} V^{\mathrm{CBO}}(t) &= \sum_{i=1}^{n} C_i \cdot V_i^{\mathrm{ZBO}}(t) \\ &= \sum_{i=1}^{n} C_i \cdot P(t, T_E) \cdot \mathrm{Black}\left(P(t, T_i)/P(t, T_E), K, \nu_i, \phi\right), \\ \nu_i^2 &= G(T_E, T_i)^2 \int_t^{T_E} \left[e^{-a(T_E - u)}\sigma(u)\right]^2 du. \end{aligned}$$

CBO's are prices as sum of ZBO's

Theorem (CBO pricing formula)

Consider a CBO with expiry time T_E, future cash flow payment times $T_1, \ldots, T_n$ (with $T_i > T_E$), corresponding cash flow values $C_1, \ldots, C_n$, fixed strike price K and call/put flag $\phi \in \{1, -1\}$. Assume that the underlying bond price $\sum_{i=1}^{n} C_i P(x(T_E); T_E, T_i)$ is monotonically decreasing in the state variable $x(T_E)$. Then the time-t price of the CBO is

$$V^{CBO}(t) = \sum_{i=1}^{n} C_i \cdot V_i^{ZBO}(t)$$

where $V_i^{ZBO}(t)$ is the time-t price of a corresponding ZBO with strike $K_i = P(x^\star; T_E, T_i)$ where the break-even state $x^\star$ is given by

$$\left(\sum_{i=1}^{n} C_i P(x^\star; T_E, T_i) \right) - K = 0.$$

Proof.

Follows from derivation above.

$\square$

Outline

We have another look at the expectation(s) of $x(t)$

- For general option pricing we also need expectation $\mathbb{E}^{T}\left[x(T)\,|\,\mathcal{F}_t\right]$.
- Then we can price

$$V(t) = P(t,T)\cdot\mathbb{E}^{T}\left[V(x(T);T)\,|\,\mathcal{F}_t\right] = P(t,T)\cdot\int_{-\infty}^{+\infty} V(x;T)\cdot p_{\mu,\sigma^2}(x)\cdot dx.$$

- Here $p_{\mu,\sigma^2}(x)$ is the density of a normal distribution $N\left(\mu,\sigma^2\right)$ with

$$\mu = \mathbb{E}^{T}\left[x(T)\,|\,\mathcal{F}_t\right] \text{ and } \sigma^2 = \mathrm{Var}\left[x(T)\,|\,\mathcal{F}_t\right].$$

- Integral $\int_{-\infty}^{+\infty} V(x;T)\cdot p_{\mu,\sigma^2}(x)\cdot dx$ is typically evaluated numerically (i.e. quadrature).
- We first calculate $\mathbb{E}^{\mathbb{Q}}\left[x(T)\ \mathcal{F}_t\right]$ and then derive $\mathbb{E}^{T}\left[x(T)\ \mathcal{F}_t\right]$.

We calculate expectation in risk-neutral measure I

Recall

$$dx(t) = [y(t) - a \cdot x(t)] \cdot dt + \sigma(t) \cdot dW(t).$$

This yields for $T \geq t$

$$x(T) = e^{-a(T-t)} \left[x(t) + \int_t^T e^{a(u-t)} \left(y(u)du + \sigma(u)dW(u) \right) \right]$$

and

$$\mathbb{E}^{\mathbb{Q}} \left[x(T) \,|\, \mathcal{F}_t \right] = e^{-a(T-t)} x(t) + \int_t^T e^{-a(T-u)} y(u) du.$$

We get

$$\begin{aligned}
\int_t^T e^{-a(T-u)} y(u) du &= \int_t^T e^{-a(T-u)} \left(\int_0^u \sigma(s)^2 e^{-2a(u-s)} ds \right) du \\
&= \int_t^T e^{-a(T-u)} \left(\int_0^t \sigma(s)^2 e^{-2a(u-s)} ds \right) du \\
&\quad + \int_t^T e^{-a(T-u)} \left(\int_t^u \sigma(s)^2 e^{-2a(u-s)} ds \right) du.
\end{aligned}$$

We calculate expectation in risk-neutral measure II

We analyse the integrals individually,

$$
\begin{aligned}
I_1(t, T) &= \int_t^T e^{-a(T-u)} \left(\int_0^t \sigma(s)^2 e^{-2a(u-s)} ds \right) du \\
&= \int_t^T \left(\int_0^t e^{-a(T-u)} \sigma(s)^2 e^{-2a(u-s)} ds \right) du \\
&= \int_0^t \left(\int_t^T e^{-a(T-u)} \sigma(s)^2 e^{-2a(u-s)} du \right) ds \\
&= \int_0^t \sigma(s)^2 \left(\int_t^T e^{-a(T-u)} e^{-2a(u-s)} du \right) ds \\
&= \int_0^t \sigma(s)^2 \left[\frac{e^{-a(T-u)} e^{-2a(u-s)}}{-a} \right]_{u=t}^{T} ds \\
&= \int_0^t \frac{\sigma(s)^2}{a} \left[e^{-a(T-t)} e^{-2a(t-s)} - e^{-a(T-T)} e^{-2a(T-s)} \right] ds.
\end{aligned}
$$

Exponential terms can be further simplified as

$$
e^{-a(T-t)} e^{-2a(t-s)} - e^{-2a(T-s)} = e^{-a(T-t)} \left[1 - e^{-a(T-t)} \right] e^{-2a(t-s)}.
$$

We calculate expectation in risk-neutral measure III

This gives

$$I_1(t,T) = e^{-a(T-t)}\frac{1-e^{-a(T-t)}}{a}\int_0^t \sigma(s)^2 e^{-2a(t-s)}ds.$$

For the second integral we get

$$\begin{aligned}
I_2(t,T) &= \int_t^T e^{-a(T-u)}\left(\int_t^u \sigma(s)^2 e^{-2a(u-s)}ds\right)du \\
&= \int_t^T \left(\int_t^u e^{-a(T-u)}\sigma(s)^2 e^{-2a(u-s)}ds\right)du \\
&= \int_t^T \left(\int_s^T e^{-a(T-u)}\sigma(s)^2 e^{-2a(u-s)}du\right)ds \\
&= \int_t^T \sigma(s)^2 \left(\int_s^T e^{-a(T-u)}e^{-2a(u-s)}du\right)ds \\
&= \int_t^T \sigma(s)^2 \left[\frac{e^{-a(T-u)}e^{-2a(u-s)}}{-a}\right]_{u=s}^T ds \\
&= \int_t^T \frac{\sigma(s)^2}{a}\left[e^{-a(T-s)}e^{-2a(s-s)} - e^{-a(T-T)}e^{-2a(T-s)}\right]ds.
\end{aligned}$$

We calculate expectation in risk-neutral measure IV

Again we simplify exponential terms

$$e^{-a(T-s)} - e^{-2a(T-s)} = e^{-a(T-s)} \left[1 - e^{-a(T-s)} \right].$$

This gives

$$I_2(t,T) = \int_t^T \sigma(s)^2 e^{-a(T-s)} \frac{1 - e^{-a(T-s)}}{a} ds.$$

In summary, we get

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[x(T) \mid \mathcal{F}_t\right] &= e^{-a(T-t)} x(t) + I_1(t,T) + I_2(t,T) \\
&= e^{-a(T-t)} \left[x(t) + \frac{1 - e^{-a(T-t)}}{a} \int_0^t \sigma(s)^2 e^{-2a(t-s)} ds \right] \\
&\quad + \int_t^T \sigma(s)^2 e^{-a(T-s)} \frac{1 - e^{-a(T-s)}}{a} ds.
\end{aligned}$$

We calculate expectation in terminal measure I

Recall change of measure

$$dW^T(t) = dW(t) + \sigma_P(t,T)dt.$$

We have

$$\sigma_P(t,T) = \sigma(t)G(t,T) = \sigma(t) \cdot \frac{1 - e^{-a(T-t)}}{a}.$$

This gives

$$dx(t) = \left[y(t) - \sigma(t)^2 G(t,T) - a \cdot x(t)\right] \cdot dt + \sigma(t) \cdot dW^T(t)$$

and

$$x(T) = e^{-a(T-t)} \left[x(t) + \int_t^T e^{a(u-t)} \left(\left[y(u) - \sigma(u)^2 G(u,T)\right] du + \sigma(u) dW^T(u)\right)\right].$$

We find that

$$\mathbb{E}^T\left[x(T) \mid \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[x(T) \mid \mathcal{F}_t\right] - \int_t^T \sigma(u)^2 e^{-a(T-u)} G(u,T) du.$$

We calculate expectation in terminal measure II

It turns out that

$$\int_t^T \sigma(u)^2 e^{-a(T-u)} G(u,T) du = \int_t^T \sigma(u)^2 e^{-a(T-u)} \frac{1-e^{-a(T-u)}}{a} du = I_2(t,T).$$

As a result, we get

$$\mathbb{E}^T \left[x(T) \,|\, \mathcal{F}_t \right] = e^{-a(T-t)} \left[x(t) + \frac{1-e^{-a(T-t)}}{a} \int_0^t \sigma(s)^2 e^{-2a(t-s)} ds \right]$$

or more formally

$$\mathbb{E}^T \left[x(T) \,|\, \mathcal{F}_t \right] = G'(t,T) \left[x(t) + G(t,T) y(t) \right].$$

Outline

All the formulas serve the purpose of model calibration and derivative pricing

Model Calibration	Derivative Pricing
zero bond option (ZBO)	future zero bonds $P(x(t); t, T)$
coupon bond option (CBO)	expectation $\mathbb{E}^T\left[x(T) \,\vert\, \mathcal{F}_t\right]$ and variance $\mathrm{Var}\left[x(T) \,\vert\, \mathcal{F}_t\right]$
European swaption	payoff pricing $V(t) = P(t, T) \cdot \mathbb{E}^T\left[V(x(T); T) \,\vert\, \mathcal{F}_t\right]$

Bond option pricing is realised via ZBO's and CBO's

Zero Bond Option (ZBO)

Zero bond with expiry T_E, maturity T_M, strike K and call/put flag ϕ

$$
\begin{aligned}
V^{\mathrm{ZBO}}(0) &= P(0, T_E) \cdot \mathrm{Black}\left(P(0, T_M)/P(0, T_E), K, \nu, \phi\right), \\
\nu^2 &= G(T_E, T_M)^2 y(T_E).
\end{aligned}
$$

Coupon Bond Option (CBO)

Coupon bond option with strike K and underlying bond $\sum_{i=1}^{n} C_i \cdot P(T_E, T_i)$,

$$
V^{\mathrm{CBO}}(t) = \sum_{i=1}^{n} C_i \cdot V_i^{\mathrm{ZBO}}(t)
$$

where ZBO's $V_i^{\mathrm{ZBO}}(t)$ with expiry T_E, maturity T_i, and strike $K_i = P(x^\star, T_E, T_i)$ and $x^\star$ s.t.

$$
\sum_{i=1}^{n} C_i \cdot P(x^\star; T_E, T_i) = K.
$$

General derivative pricing requires state variable expectation and variance

Zero Bonds (as building blocks for payoffs $V(x(T);T)$)

$$P(x(T);T,S) = \frac{P(0,S)}{P(0,T)} \exp\left\{ -G(T,S)x(T) - \frac{G(T,S)^2}{2} y(T) \right\}.$$

General Derivative Pricing

$$V(t) = P(t,T) \cdot \mathbb{E}^T \left[V(x(T);T) \,|\, \mathcal{F}_t \right] = P(t,T) \cdot \int_{-\infty}^{+\infty} V(x;T) \cdot p_{\mu,\sigma^2}(x) \cdot dx$$

with $p_{\mu,\sigma^2}(x)$ density of a Normal distribution $N\left(\mu,\sigma^2\right)$ with

$$\mu = \mathbb{E}^T \left[x(T) \,|\, \mathcal{F}_t \right] = G'(t,T) \left[x(t) + G(t,T) y(t) \right]$$

and

$$\sigma^2 = \mathrm{Var}\left[x(T) \,|\, \mathcal{F}_t \right] = y(T) - G'(t,T)^2 y(t).$$

Fortunately, we only need a small set of model functions for implementation

- Discount factors $P(0, T)$ from input yield curve.
- Function $G(t, T)$ with

$$G(t, T) = \frac{1 - e^{-a(T-t)}}{a}.$$

- Function $G'(t, T)$ with

$$G'(t, T) = e^{-a(T-t)}.$$

- Auxilliary variable $y(t)$ with

$$y(t) = \int_0^t \left[e^{-a(t-u)}\sigma(u)\right]^2 du = \sum_{j=1}^{k} \frac{e^{-2a(t-t_j)} - e^{-2a(t-t_{j-1})}}{2a}\sigma_j^2$$

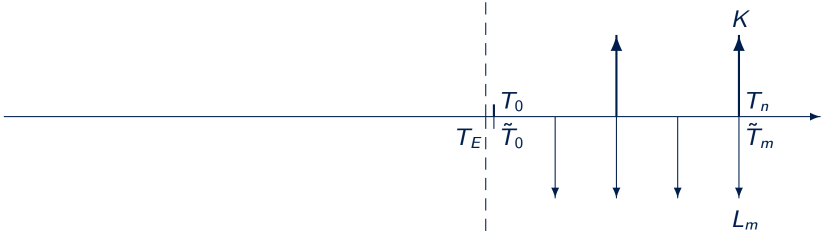
where we assume $\sigma(t)$ piece-wise constant on a grid $0 = t_0, t_1, \ldots, t_k = t$.

Outline

It remains to show how Hull-Wite model is applied to European swaptions

Model Calibration	Derivative Pricing
zero bond option (ZBO)	future zero bonds $P(x(t); t, T)$
coupon bond option (CBO)	expectation $\mathbb{E}^T\left[x(T) \,\vert\, \mathcal{F}_t\right]$ and variance $\mathrm{Var}\left[x(T) \,\vert\, \mathcal{F}_t\right]$
European swaption	payoff pricing $V(t) = P(t, T) \cdot \mathbb{E}^T\left[V(x(T); T) \,\vert\, \mathcal{F}_t\right]$

Recall that Swaption is option to enter into a swap at a future time



- At option exercise time T_E present value of swap is

$$V^{\text{Swap}}(T_E) = \underbrace{K \cdot \sum_{i=1}^{n} \tau_i \cdot P(T_E, T_i)}_{\text{future fixed leg}} - \underbrace{\sum_{j=1}^{m} L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) \cdot \tilde{\tau}_j \cdot P(T_E, \tilde{T}_j)}_{\text{future float leg}}.$$

- Option to enter represents the right but not the obligation to enter swap.
- Rational market participant will exercise if swap present value is positive, i.e.

$$V^{\text{Swpt}}(T_E) = \max\left\{ V^{\text{Swap}}(T_E), 0 \right\}.$$

How do we get the swaption payoff compatible to our Hull-White model formulas?

$$V^{\text{Swap}}(T_E) = \underbrace{K \cdot \sum_{i=1}^{n} \tau_i \cdot P(T_E, T_i)}_{\text{future fixed Leg}} - \underbrace{\sum_{j=1}^{m} L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) \cdot \tilde{\tau}_j \cdot P(T_E, \tilde{T}_j)}_{\text{future float leg}}$$

- Fixed leg can be expressed in terms of future state variable $x(T_E)$ via $P(x(T_E); T_E, T_i)$
- Float leg contains future forward Libor rates $L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta)$ from (future) projection curve
- However, Hull-White model only provides representation of discount factors, i.e. $P(T_E, \tilde{T}_j)$

We need to model the relation between future Libor rates $L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta)$ and discount factors $P(T_E, \tilde{T}_j)$.

We do have all ingredients from our deterministic multi-curve model

Recall the definition of (future) forward Libor rate

$$L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta) = \mathbb{E}^{\tilde{T}_{j-1}+\delta}\left[L^{\delta}(\tilde{T}_{j-1}, \tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta) \mid \mathcal{F}_{T_E}\right]$$
$$= \left[\frac{P(T_E, \tilde{T}_{j-1})}{P(T_E, \tilde{T}_{j-1}+\delta)} \cdot D(\tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta) - 1\right] \frac{1}{\tau(\tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta)}$$

with tenor basis spread discount factor

$$D(\tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta) = \frac{Q(T_E, \tilde{T}_{j-1})}{Q(T_E, \tilde{T}_{j-1}+\delta)}$$

and discount factors $Q(T_E, T)$ arising from credit (or funding) risk embedded in Libor rates $L^{\delta}(\cdot)$.

- Key assumption is that $D(\tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta)$ is deterministic or independent of T_E.
- Then

$$D(\tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta) = \frac{Q(0, \tilde{T}_{j-1})}{Q(0, \tilde{T}_{j-1}+\delta)} = \frac{P^{\delta}(0, \tilde{T}_{j-1})}{P^{\delta}(0, \tilde{T}_{j-1}+\delta)} \cdot \frac{P(0, \tilde{T}_{j-1}+\delta)}{P^{\delta}(0, \tilde{T}_{j-1})}.$$

We use basis spread model to simplify Libor coupons

- Basis spread discount factor

$$D(\tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) = \frac{P^{\delta}(0, \tilde{T}_{j-1})}{P^{\delta}(0, \tilde{T}_{j-1} + \delta)} \cdot \frac{P(0, \tilde{T}_{j-1} + \delta)}{P^{\delta}(0, \tilde{T}_{j-1})}$$

  is calculated from today's projection curve $P^{\delta}(0, T)$ and discount curve $P^{\delta}(0, T)$.
- Further assume *natural* Libor payment dates and consistent year fractions

$$\tilde{T}_j = \tilde{T}_{j-1} + \delta, \quad \tau(\tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) = \tilde{\tau}_j.$$

- Libor coupon becomes

$$\begin{aligned} L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_j) \cdot \tilde{\tau}_j \cdot P(T_E, \tilde{T}_j) &= \left[\frac{P(T_E, \tilde{T}_{j-1})}{P(T_E, \tilde{T}_j)} \cdot D(\tilde{T}_{j-1}, \tilde{T}_j) - 1 \right] \frac{1}{\tilde{\tau}_j} \cdot \tilde{\tau}_j \cdot P(T_E, \tilde{T}_j) \\ &= P(T_E, \tilde{T}_{j-1}) \cdot D(\tilde{T}_{j-1}, \tilde{T}_j) - P(T_E, \tilde{T}_j). \end{aligned}$$

We can write the float leg ... I

$$
\begin{aligned}
V^{\mathrm{Swap}}(T_E) &= \underbrace{K \cdot \sum_{i=1}^{n} \tau_i \cdot P(T_E, T_i)}_{\text{future fixed leg}} - \underbrace{\sum_{j=1}^{m} L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) \cdot \tilde{\tau}_j \cdot P(T_E, \tilde{T}_j)}_{\text{future float leg}} \\
&= K \cdot \sum_{i=l}^{n} \tau_i \cdot P(T_E, T_i) - \sum_{j=1}^{m} P(T_E, \tilde{T}_{j-1}) \cdot D(\tilde{T}_{j-1}, \tilde{T}_j) - P(T_E, \tilde{T}_j) \\
&= K \cdot \sum_{i=1}^{n} \tau_i \cdot P(T_E, T_i) \\
&\quad - \left[P(T_E, \tilde{T}_0) \cdot D(\tilde{T}_0, \tilde{T}_1) - P(T_E, \tilde{T}_m) + \sum_{j=2}^{m} P(T_E, \tilde{T}_{j-1}) \cdot \left[D(\tilde{T}_{j-1}, \tilde{T}_j) - 1 \right] \right] \\
&= K \cdot \sum_{i=1}^{n} \tau_i \cdot P(T_E, T_i) \\
&\quad - \left[P(T_E, \tilde{T}_0) - P(T_E, \tilde{T}_m) + \sum_{j=1}^{m} P(T_E, \tilde{T}_{j-1}) \cdot \left[D(\tilde{T}_{j-1}, \tilde{T}_j) - 1 \right] \right].
\end{aligned}
$$

We can write the float leg ... II

Reordering terms yields

$$V^{\mathrm{Swap}}(T_E) = -\underbrace{P(T_E,\tilde{T}_0)}_{\text{strike paid at } T_0} + \underbrace{\sum_{i=1}^{n} K\cdot\tau_i\cdot P(T_E,T_i)}_{\text{fixed rate coupons}}$$

$$\underbrace{-\sum_{j=1}^{m} P(T_E,\tilde{T}_{j-1})\cdot\left[D(\tilde{T}_{j-1},\tilde{T}_j)-1\right]}_{\text{negative spread coupons}} + \underbrace{P(T_E,\tilde{T}_m)}_{\text{notional payment}}$$

$$= \sum_{k=0}^{n+m+1} C_k\cdot P(T_E,\bar{T}_k)$$

with

$$C_0 = -1,\ C_i = K\cdot\tau_i\,(i=1,\ldots,n),\ C_{n+j} = -\left[D(\tilde{T}_{j-1},\tilde{T}_j)-1\right],\ (j=1,\ldots,m),$$

$$\text{and } C_{n+m+1} = 1,$$

and corresponding payment times $\bar{T}_k$.

Swaptions are equivalent to coupon bond options

Corollary (Equivalence between Swaption and bond option)

Consider a European Swaption with receiver/payer flag $\phi \in \{1, -1\}$ payoff

$$V^{Swpt}(T_E) = \left[\phi \left\{ K \cdot \sum_{i=1}^{n} \tau_i \cdot P(T_E, T_i) - \sum_{j=1}^{m} L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) \cdot \tilde{\tau}_j \cdot P(T_E, \tilde{T}_j) \right\} \right]^+ .$$

Under our deterministic basis spread assumption the swaption payoff is equal to a call/put bond option payoff

$$V^{CBO}(T_E) = \left[\phi \left\{ \sum_{k=0}^{n+m+1} C_k \cdot P(T_E, \bar{T}_k) \right\} \right]^+$$

with zero strike and cash flows C_k and times $\bar{T}_k$ as elaborated above. Moreover, if the underlying bond payoff is monotonic then

$$V^{Swpt}(t) = V^{CBO}(t) = \sum_{k=0}^{n+m+1} C_k \cdot V_k^{ZBO}(t)$$

with corresponding zero bond option parameters.

We give some comments regarding the CBO maping

- Note that $C_0 = -1$ is a *large* negative cash flow.
- However, $\frac{\partial}{\partial x}\left[-P(T_E, \tilde{T}_0)\right] \approx -G(T_E, T_0)$ is small because $T_E - T_0$ is small.
- If $T_E = \tilde{T}_0$, i.e. no spot offset between option expiry and swap start time, then
  - set CBO strike $K = D(\tilde{T}_0, \tilde{T}_1)$,
  - remove first negative spread coupon C_{n+1} from cash flow list.
- In practice monotonicity assumption

$$\frac{\partial}{\partial x}\left[\sum_{k=0}^{n+m+1} C_k \cdot P(T_E, \bar{T}_k)\right] < 0$$

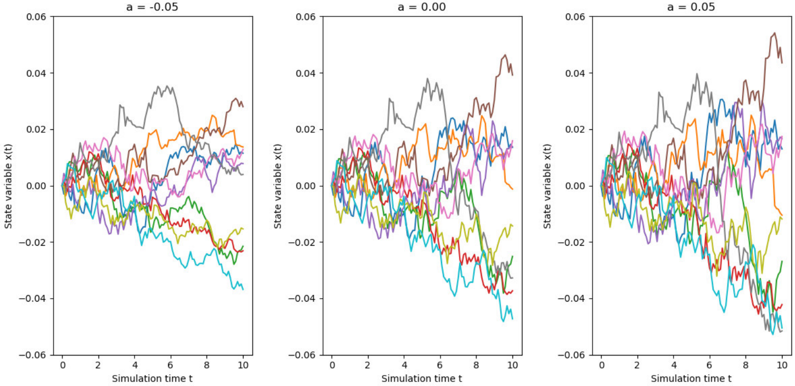
is typically no issue.

In Hull-White model calibration we will use CBO formula to match Hull-White model prices versus Vanilla model swaption prices.

Outline

How do the simulated paths *look like*?

- Model short rate volatility σ calibrated to $100bp$ flat volatility at $5y$ and $10y$, mean reversion $a \in \{-5\%, 0\%, 5\%\}$ [5]



- Higher mean reversion yields more *forward volatility.*

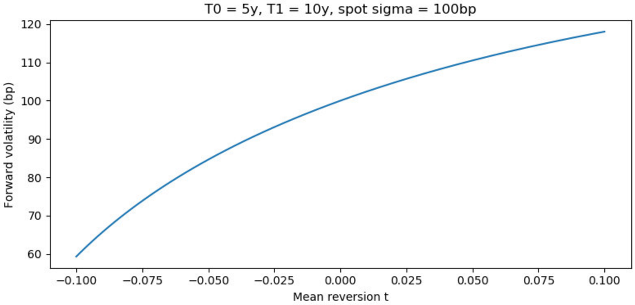
[5] Zero mean reversion is effectively approximated via $a = 1bp$. This does not change the overall behavior and avoids special treatment in formulas.

Forward volatility dependence on mean reversion can also be derived analytically

Denote forward volatility as

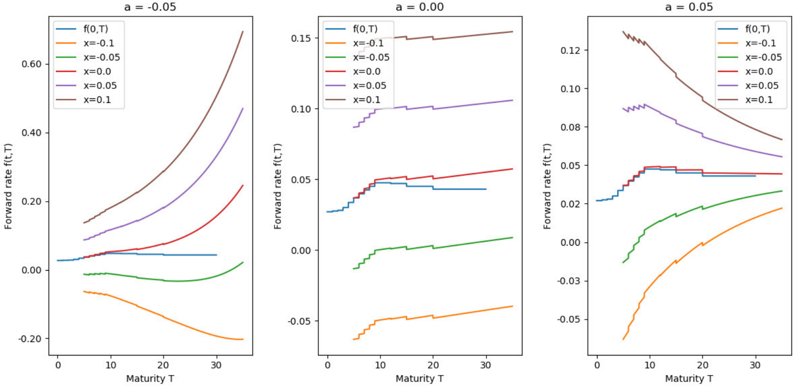
$$\sigma_{\text{Fwd}}(T_0, T_1) = \frac{\text{Var}\left[x(T_1) \,|\, \mathcal{F}_{T_0}\right]}{\sqrt{T_1 - T_0}} = \frac{y(T_1) - G'(T_0, T_1)^2 y(T_0)}{\sqrt{T_1 - T_0}}$$

- Suppose spot volatilities $\sigma_{\text{Fwd}}(0, T_1)$ and $\sigma_{\text{Fwd}}(0, T_0)$ (and thus $y(T_0)$ and $y(T_1)$ are fixed)
- If mean reversion a increases then $G'(T_0, T_1) = e^{-a(T_1 - T_0)}$ decreases
- Thus forward volatility $\sigma_{\text{Fwd}}(T_0, T_1)$ increases



Which kind of curves can we simulate with Hull-White model?

- Models use flat short rate volatility $\sigma = 100bp$ and mean reversion $a \in \{-5\%, 0\%, 5\%\}$ [6]



- Model *works* with negative mean reversion - however, yield curves are exploding

[6] Zero mean reversion is effectively approximated via $a = 1bp$. This does not change the overall behavior and avoids special treatment in formulas.

What are relevant properties of a model for option pricing?

- Vanilla models require input (ATM volatility) parameters for expiry-tenor-pairs.
  - Which shape of ATM volatilities for expiry-tenor-pairs are predicted by Hull-White model?
- SABR model allows modelling of volatility smile.
  - Which shapes of volatility smile can be modelled with Hull-White model?
  - How does the smile change if we change the model parameters?
- We aim at applying the Hull-White model to price Bermudan swaptions.
  - How do the model parameters impact prices of exotic derivatives?

For now we focus on model-implied volatilities (ATM and smile). The impact of model parameters on Bermudans is analysed later.

Model properties for option pricing are assessed by analysing model-implied volatilities

Model-implied normal volatility

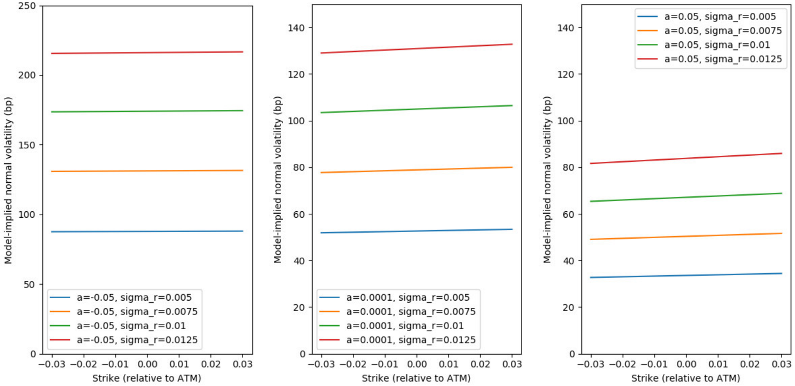
Consider a swaption with expiry/start/end-dates $T_E/T_0/T_n$ and strike rate K. For a given Hull-White model the model-implied normal volatility is calculated as

$$\sigma(T_0, T_n, K) = \text{Bachelier}^{-1}\left(S(t), K, V^{\text{CBO}}(t)/An(t), \phi\right)/\sqrt{T_E - t}.$$

Here, $S(t)$ and $An(t)$ are the forward swap rate and annuity of the underlying swap with start/end-date T_0/T_n. $V^{\text{CBO}}(t)$ is the Hull-White model price of a coupon bond option equivalent to the input swaption.

Which shapes of volatility smile can be modelled and how does the smile change if we change the model parameters?

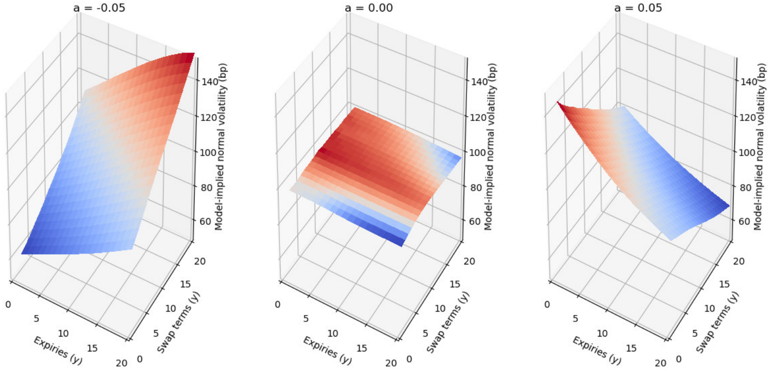
- Models use flat short rate volatility $\sigma \in \{50bp, 75bp, 100bp, 125bp\}$ and mean reversion $a \in \{-5\%, 0\%, 5\%\}$:



- We can only model flat smile - this is a major model limitation!
- Model-implied volatility decreases if mean reversion increases.

Which shape of ATM volatilities for expiry-tenor-pairs are predicted by Hull-White model?

- Models use flat short rate volatility σ - calibrated to 10y-10y swaption with $100bp$ volatility
- Mean reversion $a \in \{-5\%, 0\%, 5\%\}$:



- Mean reversion impacts slope of ATM volatilities in expiry and swap term dimension.

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