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June 10, 2022

Interest Rate Modelling and Derivative Pricing, SS 2022

Exercise 4

1. **Affine term-structure models and HJM model representation**
The continuously-compounded spot rate is defined as

$$R(t,T) = -\frac{\ln[P(t,T)]}{T-t}.$$

Affine term structure models are characterised by the fact that continuously-compounded spot rates take the form

$$R(t,T) = \alpha(t,T) + \beta(t,T) \cdot r(t)$$

for deterministic functions α and β.

(a) Show that for affine models future zero bond prices can be represented as

$$P(t,T) = A(t,T) \cdot e^{-B(t,T)\cdot r(t)}$$

with deterministic functions A and B.

(b) Assume a one-factor diffusion for the short rate in the risk-neutral measure via

$$dr(t) = [\theta(t) - \chi \cdot r(t)] \cdot dt + \sigma(t, r(t)) \cdot dW(t).$$

Show that the dynamics of the forward rates $f(t,T)$ become

$$df(t,T) = [\ldots] \cdot dt + \frac{\partial B(t,T)}{\partial T} \cdot \sigma(t, r(t)) \cdot dW(t)$$

(c) Now consider the case $\sigma(t, r(t)) = \sqrt{a + b \cdot r(t)}$. What are the dynamics of the model in terms of the state variables $x(t)$ and $y(t)$?

2. **Swap rate dynamics in separable HJM model**
Consider the one-factor separable HJM model with $r(t) = f(0,t) + x(t)$ and risk-neutral dynamics

$$\begin{aligned} dx(t) &= [y(t) - \chi \cdot x(t)]\, dt + \sigma_r(t) dW(t), \\ dy(t) &= \left[\sigma_r(t)^2 - 2\chi \cdot y(t)\right] dt. \end{aligned}$$

The forward swap rate (in the single curve setting) is (for $t \le T_0$)

$$S(t) = \frac{P(t,T_0) - P(t,T_N)}{\sum_{i=1}^{N} \tau_i P(t,T_i)}.$$

Recall that zero bonds can be written in terms of state variables via $P(t,T) = P(x(t), y(t); t, T)$.

(a) What are the dynamics of the swap rate in the annuity measure given that HJM model?

(b) Assume the short rate volatility $\sigma_r(t)$ is a deterministic function. Can the swap rate best be approximated by a normal, lognormal or shifted log-normal model? What is the corresponding volatility parameters in such an approximated model?