

# Interest Rate Modelling and Derivative Pricing

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Summer term, 2022

## Part V

# Bermudan Swaption Pricing

# Outline

Bermudan Swaptions

Pricing Methods for Bermudans

Density Integration Methods

PDE and Finite Differences

American Monte Carlo

# Outline

## Bermudan Swaptions

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# Let's have another look at the cancellation option

## Interbank swap deal example

Pays 3% on 100mm EUR

Start date: Oct 30, 2020

End date: Oct 30, 2040

(annually, 30/360 day count, modified following, Target calendar)



Pays 6-months Euribor floating rate on 100mm EUR

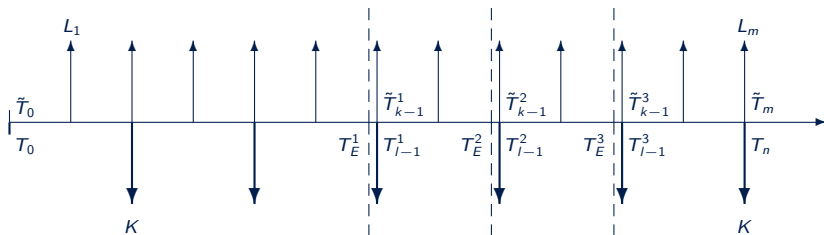
Start date: Oct 30, 2020

End date: Oct 30, 2040

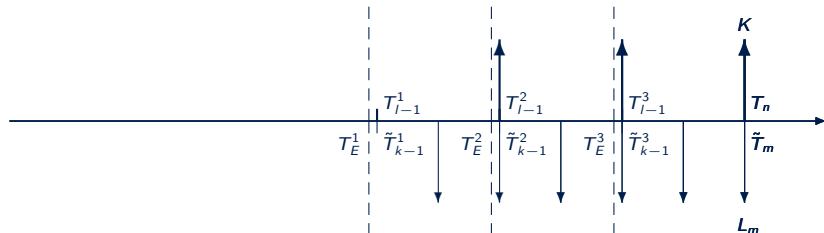
(semi-annually, act/360 day count, modified following, Target calendar)

Bank A may decide to **early terminate deal in 10, 11, 12,..years.**

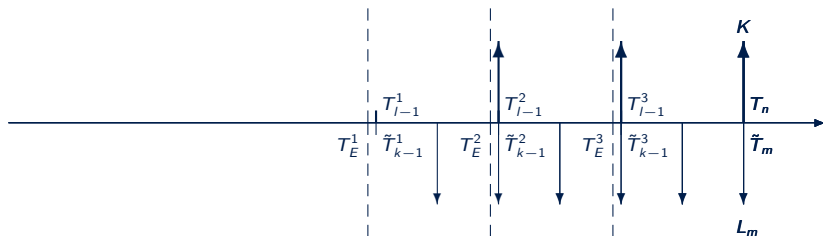
## What does such a *Bermudan call right* mean?



[Bermudan cancellable swap] = [full swap] + [Bermudan option on opposite swap]



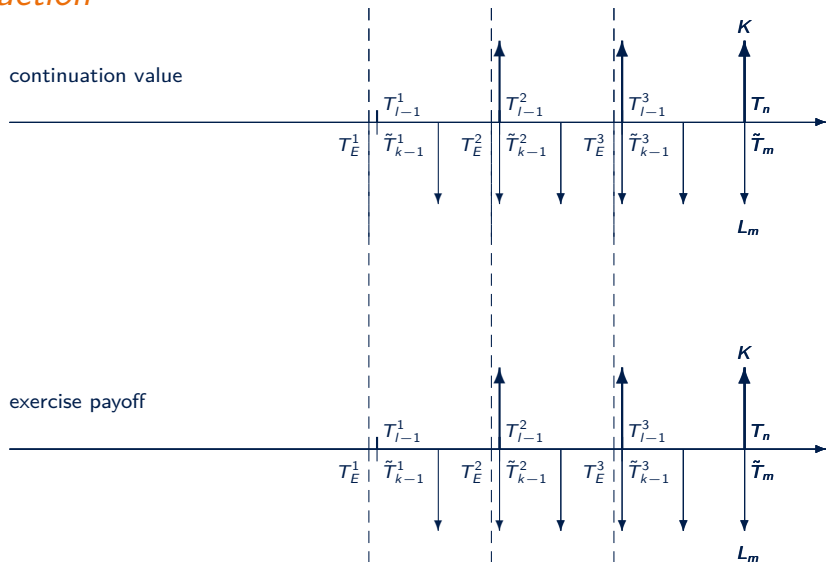
# What is a Bermudan swaption?



## Bermudan swaption

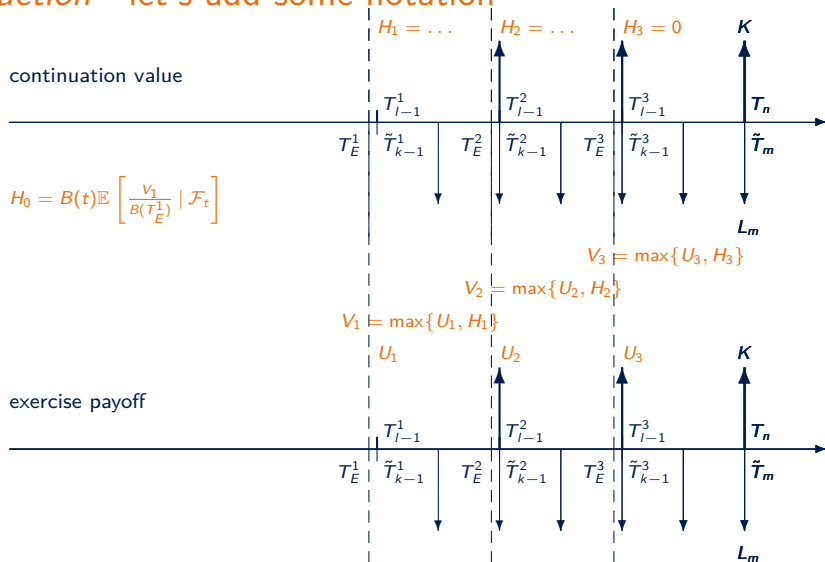
A Bermudan swaption is an option to enter into a Vanilla swap with fixed rate  $K$  and final maturity  $T_n$  at various exercise dates  $T_E^1, T_E^2, \dots, T_E^{\bar{k}}$ . If there is only one exercise date (i.e.  $\bar{k} = 1$ ) then the Bermudan swaption equals a European swaption.

## A Bermudan swaption can be priced via *backward induction*





A Bermudan swaption can be priced via *backward induction* - let's add some notation



## First we specify the future payoff cash flows

- ▶ Choose a numeraire  $B(t)$  and corresponding cond. expectations  $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t]$ .
- ▶ Underlying payoff  $U_k$  if option is exercised

$U_k$

$$\begin{aligned}
 &= B(T_E^k) \sum_{T_i \geq T_E^k} \mathbb{E}_{T_E^k} \left[ \frac{X_i(T_i)}{B(T_i)} \right] \\
 &= B(T_E^k) \underbrace{\left[ \sum_{T_i \geq T_E^k} K_{T_i} P(T_E^k, T_i) - \sum_{\tilde{T}_j \geq T_E^k} L^\delta(T_E^k, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) \tilde{\tau}_j P(T_E^k, \tilde{T}_j) \right]}_{\text{future fixed leg minus future float leg}} \\
 &= B(T_E^k) \left[ \sum_{T_i \geq T_E^k} K_{T_i} P(T_E^k, T_i) - [P(T_E^k, \tilde{T}_{j_k}) - P(T_E^k, \tilde{T}_m)] \right. \\
 &\quad \left. - \sum_{\tilde{T}_j \geq T_E^k} P(T_E^k, \tilde{T}_{j-1}) [D(\tilde{T}_{j-1}, \tilde{T}_j) - 1] \right].
 \end{aligned}$$

## Then we specify the continuation value and optimal exercise (1/2)

- ▶ Continuation value  $H_k(t)$  ( $T_E^k \leq t \leq T_E^{k+1}$ ) represents the **time- $t$  value of the remaining option** if not exercised.
- ▶ Option becomes worthless if not exercised at last exercise date  $T_E^{\bar{k}}$ . Thus last continuation value  $H_{\bar{k}}(T_E^{\bar{k}}) = 0$ .
- ▶ Recall that Bermudan option gives the right but not the obligation to enter into underlying at exercise.
- ▶ Rational agent will choose the maximum of payoff and continuation at exercise, i.e.

$$V_k = \max \{ U_k, H_k(T_E^k) \}.$$

## Then we specify the continuation value and optimal exercise (2/2)

$$V_k = \max \{U_k, H_k(T_E^k)\}.$$

- ▶  $V_k$  represents the Bermudan option value at exercise  $T_E^k$ . Thus we also must have for the continuation value

$$H_{k-1}(T_E^k) = V_k.$$

- ▶ Derivative pricing formula yields

$$\begin{aligned} H_{k-1}(T_E^{k-1}) &= B(T_E^{k-1}) \cdot \mathbb{E}_{T_E^{k-1}} \left[ \frac{H_{k-1}(T_E^k)}{B(T_E^k)} \right] \\ &= B(T_E^{k-1}) \cdot \mathbb{E}_{T_E^{k-1}} \left[ \frac{V_k}{B(T_E^k)} \right]. \end{aligned}$$

# We summarize the Bermudan pricing algorithm

## Backward induction for Bermudan options

Consider a Bermudan option with  $\bar{k}$  exercise dates  $T_E^k$  ( $k = 1, \dots, \bar{k}$ ) and underlying future payoffs with (time- $T_E^k$ ) prices  $U_k$ .

Denote  $H_k(t)$  the option's continuation value for  $T_E^k \leq t \leq T_E^{k+1}$  and set  $H_{\bar{k}}(T_E^{\bar{k}}) = 0$ . Also set  $T_E^0 = t$  (i.e. pricing time today).

The option price can be derived via the recursion

$$\begin{aligned} H_k(T_E^k) &= B(T_E^k) \cdot \mathbb{E}_{T_E^k} \left[ \frac{H_k(T_E^{k+1})}{B(T_E^{k+1})} \right] \\ &= B(T_E^k) \cdot \mathbb{E}_{T_E^k} \left[ \frac{\max \{U_{k+1}, H_{k+1}(T_E^{k+1})\}}{B(T_E^{k+1})} \right]. \end{aligned}$$

for  $k = \bar{k} - 1, \dots, 0$ . The Bermudan option price is given by

$$V^{\text{Berm}}(t) = H_0(t) = H_0(T_E^0).$$

## Some more comments regarding Bermudan pricing ...

- ▶ Recursion for Bermudan pricing can be formally derived via theory of optimal stopping and Hamilton-Jacobi-Bellman (HJB) equation.
- ▶ For more details, see Sec. 18.2.2 in Andersen/Piterbarg (2010).
- ▶ For a single exercise date  $\bar{k} = 1$  we get

$$H_0(t) = B(t) \cdot \mathbb{E}_t \left[ \frac{\max \{U_1, 0\}}{B(T_E^1)} \right].$$

This is the general pricing formula for a European swaption (if  $U_1$  represents a Vanilla swap).

- ▶ In principle, recursion  $H_k(T_E^k) = B(T_E^k) \cdot \mathbb{E}_{T_E^k} \left[ \frac{\max \{U_{k+1}, H_{k+1}(T_E^{k+1})\}}{B(T_E^{k+1})} \right]$  holds for any payoffs  $U_k$ . However, computation

$$U_k = B(T_E^k) \sum_{T_i \geq T_E^k} \mathbb{E}_{T_E^k} \left[ \frac{X_i(T_i)}{B(T_i)} \right]$$

might pose additional challenges if cash flows  $X_i(T_i)$  are more complex.

## How do we price a Bermudan in practice?

- ▶ In principle, recursion algorithm for  $H_k()$  is straight forward.
- ▶ Computational challenge is calculating conditional expectations

$$H_k(T_E^k) = B(T_E^k) \cdot \mathbb{E}_{T_E^k} \left[ \frac{\max \{U_{k+1}, H_{k+1}(T_E^{k+1})\}}{B(T_E^{k+1})} \right].$$

- ▶ Note, that this problem is an instance of the general option pricing problem

$$V(T_0) = B(T_0) \cdot \mathbb{E} \left[ \frac{V(T_1)}{B(T_1)} \mid \mathcal{F}_{T_0} \right].$$

We can apply general option pricing methods to *roll-back* the Bermudan payoff.

# Outline

Bermudan Swaptions

**Pricing Methods for Bermudans**

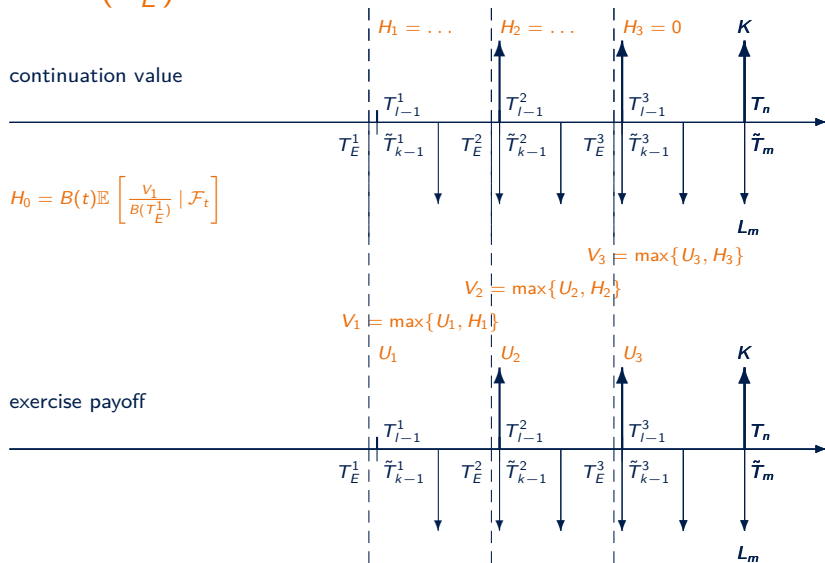
Density Integration Methods

PDE and Finite Differences

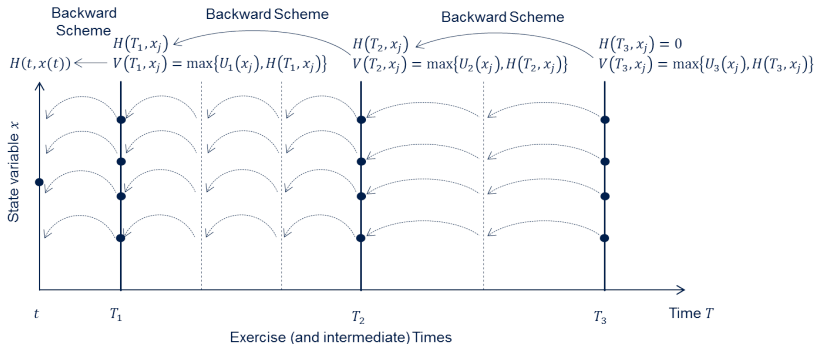
American Monte Carlo



Note that  $U_k$ ,  $V_k$  and  $H_k$  depend on underlying state variable  $x(T_E^k)$



Typically we need to discretise variables  $U_k$ ,  $V_k$  and  $H_k$  on a grid of underlying state variables



Forthcoming, we discuss several methods to roll-back the payoffs.

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Pricing Methods for Bermudans

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American Monte Carlo

# Outline

## Density Integration Methods

- General Density Integration Method

- Gauss–Hermite Quadrature

- Cubic Spline Interpolation and Exact Integration

## Key idea is using the conditional density function in the Hull-White model

Recall that

$$V(T_0) = B(T_0) \cdot \mathbb{E} \left[ \frac{V(T_1)}{B(T_1)} \mid \mathcal{F}_{T_0} \right].$$

We choose the  $T_1$ -maturing zero coupon bond  $P(t, T_1)$  as numeraire.  
Then

$$\begin{aligned} V(T_0) &= P(T_0, T_1) \cdot \mathbb{E}^{T_1} [V(T_1) \mid \mathcal{F}_{T_0}] \\ &= P(x(T_0); T_0, T_1) \cdot \int_{-\infty}^{+\infty} V(x; T_1) \cdot p_{\mu, \sigma^2}(x) \cdot dx. \end{aligned}$$

State variable  $x = x(T_1)$  is normally distributed with known mean and variance.

## Hull-White model results yield density parameters of the state variable $x(T_1)$

$$V(T_0) = P(x(T_0); T_0, T_1) \cdot \int_{-\infty}^{+\infty} V(x; T_1) \cdot p_{\mu, \sigma^2}(x) \cdot dx.$$

State variable  $x = x(T_1)$  is normally distributed with mean

$$\mu = \mathbb{E}^{T_1} [x(T_1) | \mathcal{F}_{T_0}] = G'(T_0, T_1) [x(T_0) + G(T_0, T_1)y(T_0)]$$

and variance

$$\sigma^2 = \text{Var} [x(T_1) | \mathcal{F}_{T_0}] = y(T_1) - G'(T_0, T_1)^2 y(T_0).$$

Thus

$$p_{\mu, \sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\}$$

and

$$V(T_0) = P(x(T_0); T_0, T_1) \cdot \int_{-\infty}^{+\infty} \frac{V(x; T_1)}{\sqrt{2\pi\sigma^2}} \cdot \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\} dx.$$

## Integral against normal density needs to be computed numerically by quadrature methods

$$V(T_0) = P(x(T_0); T_0, T_1) \cdot \int_{-\infty}^{+\infty} \frac{V(x; T_1)}{\sqrt{2\pi\sigma^2}} \cdot \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx.$$

- ▶ We can apply general purpose quadrature rules to function

$$f(x) = \frac{V(x; T_1)}{\sqrt{2\pi\sigma^2}} \cdot \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$

- ▶ Select a grid  $[x_0, \dots, x_N]$  and approximate e.g. via
- ▶ Trapezoidal rule

$$\int_{-\infty}^{+\infty} f(x) \cdot dx \approx \sum_{i=1}^N \frac{1}{2} [f(x_{i-1}) + f(x_i)] (x_i - x_{i-1})$$

- ▶ or Simpson's rule with equidistant grid ( $h = x_i - x_{i-1}$ ) and even sub-intervalls, then

$$\int_{-\infty}^{+\infty} f(x) \cdot dx \approx \frac{h}{3} \cdot \left[ f(x_0) + 2 \sum_{j=1}^{N/2-1} f(x_{2j}) + 4 \sum_{j=1}^{N/2} f(x_{2j-1}) + f(x_N) \right].$$

## There are some details that need to be considered - Select your integration domain carefully

- ▶ Infinite integral is approximated by definite integral

$$\int_{-\infty}^{+\infty} f(x) \cdot dx \approx \int_{x_0}^{x_N} f(x) \cdot dx \approx \dots$$

- ▶ Finite integration boundaries need to be chosen carefully by taking into account variance of  $x(t)$ .
- ▶ One approach is calculating variance to option expiry  $T_1$ ,  $\hat{\sigma}^2 = \text{Var}[x(T)] = y(T_1)$  and set

$$x_0 = -\lambda \cdot \hat{\sigma} \quad \text{and} \quad x_N = \lambda \cdot \hat{\sigma}.$$

- ▶ Note, that  $\mathbb{E}^{T_1}[x(T_1)] = 0$ , thus we do not apply a shift to the  $x$ -grid.



## There are some details that need to be considered - Take care of the break-even point

- ▶ Note that convergence of quadrature rules depends on smoothness of integrand  $f(x)$ .
- ▶ Recall that

$$f(x) = V(x) \cdot p_{\mu, \sigma^2}(x) = \max \{ U_{k+1}(x), H_{k+1}(x; T_E^{k+1}) \} \cdot p_{\mu, \sigma^2}(x).$$

- ▶ Max-function is not smooth at  $U_{k+1}(x) = H_{k+1}(x; T_E^{k+1})$ .

Determine  $x^*$  (via interpolation of  $H_{k+1}(\cdot)$  and numerical root search) such that

$$U_{k+1}(x^*) = H_{k+1}(x^*; T_E^{k+1})$$

and split integration

$$\int_{-\infty}^{+\infty} f(x) \cdot dx = \int_{-\infty}^{x^*} f(x) \cdot dx + \int_{x^*}^{+\infty} f(x) \cdot dx.$$

## Can we exploit the structure of the integrand?

$$V(T_0) = P(x(T_0); T_0, T_1) \cdot \int_{-\infty}^{+\infty} \frac{V(x; T_1)}{\sqrt{2\pi\sigma^2}} \cdot \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx.$$

- ▶ Integral against normal distribution density can be solved more efficiently:
  1. Use Gauss–Hermite quadrature.
  2. Interpolate only  $V(x; T_1)$  by cubic spline and integrate exact.

# Outline

## Density Integration Methods

General Density Integration Method

Gauss–Hermite Quadrature

Cubic Spline Interpolation and Exact Integration

## Gauss–Hermite quadrature is an efficient integration method for smooth integrands

- ▶ Gauss–Hermite quadrature is a particular form of Gaussian quadrature.
- ▶ Choose a degree parameter  $d$ , and approximate

$$\int_{-\infty}^{+\infty} f(x) \cdot e^{-x^2} dx \approx \sum_{k=1}^d w_k \cdot f(x_k)$$

with  $x_k$  ( $i = 1, 2, \dots, d$ ) being the roots of the physicists' version of the Hermite polynomial  $H_d(x)$  and  $w_k$  are weights with

$$w_k = \frac{2^{d-1} d! \sqrt{\pi}}{d^2 [H_{d-1}(x_k)]^2}.$$

- ▶ Roots and weights can be obtained, e.g. via stored values and look-up tables.

## Variable transformation allows application of Gauss–Hermite quadrature to Hull-White model integration

We get

$$\begin{aligned} & \int_{-\infty}^{+\infty} \frac{V(x; T_1)}{\sqrt{2\pi\sigma^2}} \cdot \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} V(\sqrt{2}\sigma x + \mu; T_1) \cdot e^{-x^2} dx \\ &\approx \frac{1}{\sqrt{\pi}} \sum_{k=1}^d w_k \cdot V(\sqrt{2}\sigma x_k + \mu; T_1). \end{aligned}$$

Some constraints need to be considered:

- ▶ Payoff  $V(\cdot, T_1)$  is only available on the  $x$ -grid at  $T_1$ , thus  $V(\cdot, T_1)$  needs to be interpolated.
- ▶ Gauss-Hermite quadrature does not take care of non-smooth payoff at break-even state  $x^*$ , thus  $d$  needs to be sufficiently large to mitigate impact.

# Outline

## Density Integration Methods

General Density Integration Method

Gauss–Hermite Quadrature

Cubic Spline Interpolation and Exact Integration

## If we apply cubic spline interpolation anyway then we can also integrate exactly

Approximate  $V(\cdot, T_1)$  via cubic spline on the grid  $[x_0, \dots, x_N]$  as

$$V(x, T_1) \approx C(x) = \sum_{i=0}^{N-1} \mathbb{1}_{\{x_i \leq x < x_{i+1}\}} \sum_{k=0}^d c_{i,k} \cdot (x - x_i)^k.$$

Then

$$\begin{aligned} \int_{-\infty}^{+\infty} V(x; T_1) \cdot p_{\mu, \sigma^2}(x) \cdot dx &\approx \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \sum_{k=0}^d c_{i,k} \cdot (x - x_i)^k \cdot p_{\mu, \sigma^2}(x) \cdot dx \\ &= \sum_{i=0}^{N-1} \sum_{k=0}^d c_{i,k} \cdot \int_{x_i}^{x_{i+1}} (x - x_i)^k \cdot p_{\mu, \sigma^2}(x) \cdot dx. \end{aligned}$$

Thus, all we need is

$$l_{i,k} = \int_{x_i}^{x_{i+1}} (x - x_i)^k \cdot p_{\mu, \sigma^2}(x) \cdot dx.$$

## We transform variables to make integration easier

First we apply the variable transformation  $\bar{x} = (x - \mu)/\sigma$ . This yields  $p_{\mu, \sigma^2}(x) = p_{0,1}(\bar{x})/\sigma$  and

$$\begin{aligned} I_{i,k} &= \int_{\bar{x}_i}^{\bar{x}_{i+1}} (\sigma\bar{x} + \mu - x_i)^k \cdot p_{0,1}(\bar{x}) \cdot \frac{d\bar{x}}{\sigma} \\ &= \int_{\bar{x}_i}^{\bar{x}_{i+1}} \sigma^k (\bar{x} - \bar{x}_i)^k \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\bar{x}^2}{2}\right\}}_{\text{standard normal density}} d\bar{x} \end{aligned}$$

with the shifted grid points  $\bar{x}_i = (x_i - \mu)/\sigma$ .

Denote  $\Phi(\cdot)$  the cumulated standard normal distribution function. Then

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\bar{x}^2}{2}\right\} \quad \text{and} \quad \Phi''(x) = -x\Phi'(x).$$

As a sub-step we aim at solving the integral

$$\int_{\bar{x}_i}^{\bar{x}_{i+1}} \bar{x}^k \cdot \Phi'(\bar{x}) \cdot d\bar{x}.$$



We use cubic splines ( $d = 3$ ) to keep formulas reasonably simple I

It turns out that

$$F_0(\bar{x}) = \int \Phi'(\bar{x}) d\bar{x} = \Phi(\bar{x}),$$

$$F_1(\bar{x}) = \int \bar{x}\Phi'(\bar{x}) d\bar{x} = -\Phi'(\bar{x}),$$

$$F_2(\bar{x}) = \int \bar{x}^2\Phi'(\bar{x}) d\bar{x} = \Phi(\bar{x}) - \bar{x} \cdot \Phi'(\bar{x}),$$

$$F_3(\bar{x}) = \int \bar{x}^3\Phi'(\bar{x}) d\bar{x} = -(\bar{x}^2 + 2) \cdot \Phi'(\bar{x}).$$

This yields for  $I_{i,0}$

$$I_{i,0} = \int_{\bar{x}_i}^{\bar{x}_{i+1}} \Phi'(\bar{x}) \cdot dx = F_0(\bar{x}_{i+1}) - F_0(\bar{x}_i)$$

We use cubic splines ( $d = 3$ ) to keep formulas reasonably simple II

and for  $l_{i,1}$

$$\begin{aligned}l_{i,1} &= \int_{\bar{x}_i}^{\bar{x}_{i+1}} \sigma (\bar{x} - \bar{x}_i) \cdot \Phi'(\bar{x}) \cdot dx \\ &= \sigma \cdot \int_{\bar{x}_i}^{\bar{x}_{i+1}} \bar{x} \cdot \Phi'(\bar{x}) \cdot dx - \sigma \cdot \bar{x}_i \cdot l_{i,0} \\ &= \sigma \cdot [F_1(\bar{x}_{i+1}) - F_1(\bar{x}_i)] - \sigma \cdot \bar{x}_i \cdot l_{i,0}.\end{aligned}$$

## We use cubic splines ( $d = 3$ ) to keep formulas reasonably simple III

We may proceed similarly for  $l_{i,2}$

$$\begin{aligned}l_{i,2} &= \int_{\bar{x}_i}^{\bar{x}_{i+1}} \sigma^2 (\bar{x} - \bar{x}_i)^2 \cdot \Phi'(\bar{x}) \cdot dx \\&= \int_{\bar{x}_i}^{\bar{x}_{i+1}} \sigma^2 [\bar{x}^2 - 2\bar{x}_i\bar{x} + \bar{x}_i^2] \cdot \Phi'(\bar{x}) \cdot dx \\&= \sigma^2 [F_2(\bar{x}_{i+1}) - F_2(\bar{x}_i)] - 2\sigma^2\bar{x}_i [F_1(\bar{x}_{i+1}) - F_1(\bar{x}_i)] + \sigma^2\bar{x}_i^2 l_{i,0} \\&= \sigma^2 [F_2(\bar{x}_{i+1}) - F_2(\bar{x}_i)] - 2\sigma\bar{x}_i [l_{i,1} + \sigma \cdot \bar{x}_i \cdot l_{i,0}] + \sigma^2\bar{x}_i^2 l_{i,0} \\&= \sigma^2 [F_2(\bar{x}_{i+1}) - F_2(\bar{x}_i)] - 2\sigma\bar{x}_i l_{i,1} - \sigma^2\bar{x}_i^2 l_{i,0}\end{aligned}$$

We use cubic splines ( $d = 3$ ) to keep formulas reasonably simple IV

and  $l_{i,3}$

$$\begin{aligned}l_{i,3} &= \int_{\bar{x}_i}^{\bar{x}_{i+1}} \sigma^3 (\bar{x} - \bar{x}_i)^3 \cdot \Phi'(\bar{x}) \cdot dx \\&= \int_{\bar{x}_i}^{\bar{x}_{i+1}} \sigma^3 [\bar{x}^3 - 3\bar{x}_i\bar{x}^2 + 3\bar{x}_i^2\bar{x} - \bar{x}_i^3] \cdot \Phi'(\bar{x}) \cdot dx \\&= \sigma^3 [F_3(\bar{x}_{i+1}) - F_3(\bar{x}_i)] - 3\sigma^3\bar{x}_i [F_2(\bar{x}_{i+1}) - F_2(\bar{x}_i)] \\&\quad + 3\sigma^3\bar{x}_i^2 [F_1(\bar{x}_{i+1}) - F_1(\bar{x}_i)] - \sigma^3\bar{x}_i^3 l_{i,0}.\end{aligned}$$

Substituting terms as before yields

$$\begin{aligned}l_{i,3} &= \sigma^3 [F_3(\bar{x}_{i+1}) - F_3(\bar{x}_i)] - 3\sigma\bar{x}_i [l_{i,2} + 2\sigma\bar{x}_i l_{i,1} + \sigma^2\bar{x}_i^2 l_{i,0}] \\&\quad + 3\sigma^2\bar{x}_i^2 [l_{i,1} + \sigma \cdot \bar{x}_i \cdot l_{i,0}] - \sigma^3\bar{x}_i^3 l_{i,0} \\&= \sigma^3 [F_3(\bar{x}_{i+1}) - F_3(\bar{x}_i)] - 3\sigma\bar{x}_i l_{i,2} - 3\sigma^2\bar{x}_i^2 l_{i,1} - \sigma^3\bar{x}_i^3 l_{i,0}.\end{aligned}$$

## Let's summarise the formulas...

We get

$$\begin{aligned} V(T_0) &= P(x(T_0); T_0, T_1) \cdot \int_{-\infty}^{+\infty} V(x; T_1) \cdot p_{\mu, \sigma^2}(x) \cdot dx \\ &\approx P(x(T_0); T_0, T_1) \cdot \sum_{i=0}^{N-1} \sum_{k=0}^3 c_{i,k} \cdot l_{i,k} \end{aligned}$$

with

$$l_{i,0} = F_0(\bar{x}_{i+1}) - F_0(\bar{x}_i)$$

$$l_{i,1} = \sigma \cdot [F_1(\bar{x}_{i+1}) - F_1(\bar{x}_i)] - \sigma \cdot \bar{x}_i \cdot l_{i,0}$$

$$l_{i,2} = \sigma^2 [F_2(\bar{x}_{i+1}) - F_2(\bar{x}_i)] - 2\sigma \bar{x}_i l_{i,1} - \sigma^2 \bar{x}_i^2 l_{i,0}$$

$$l_{i,3} = \sigma^3 [F_3(\bar{x}_{i+1}) - F_3(\bar{x}_i)] - 3\sigma \bar{x}_i l_{i,2} - 3\sigma^2 \bar{x}_i^2 l_{i,1} - \sigma^3 \bar{x}_i^3 l_{i,0}$$

and anti-derivative functions  $F_k(x)$  as before.

## Integrating a cubic spline versus a normal density function occurs in various contexts of pricing methods

- ▶ Method already yields good accuracy for smaller number of grid points.
- ▶ For larger number of grid points accuracy benefit compared to e.g. Simpson integration seems not too much.
- ▶ Either way, use special treatment of break-even point  $x^*$ .
- ▶ Computational effort can become significant for larger number of grid points.
  - ▶ Bermudan pricing requires  $N^2$  evaluations of  $\Phi(\cdot)$  and  $\Phi'(\cdot)$  per exercise.

# Outline

Bermudan Swaptions

Pricing Methods for Bermudans

Density Integration Methods

**PDE and Finite Differences**

American Monte Carlo

# PDE methods for finance and pricing are extensively studied in the literature

- ▶ We present the basic principles and some aspects relevant for Bermudan bond option pricing.
- ▶ Further reading:
  - ▶ L. Andersen and V. Piterbarg. *Interest rate modelling, volume I to III*. Atlantic Financial Press, 2010, Sec. 2.
  - ▶ D. Duffy. *Finite Difference Methods in Financial Engineering*. Wiley Finance, 2006



# Outline

## PDE and Finite Differences

Derivative Pricing PDE in Hull-White Model

State Space Discretisation via Finite Differences

Time-integration via  $\theta$ -Method

Alternative Boundary Conditions for Bond Option Payoffs

Summary of PDE Pricing Method

## We can adapt the Black-Scholes equation to our Hull-White model setting

- ▶ Recall that state variable  $x(t)$  follows the risk-neutral dynamics

$$dx(t) = \underbrace{[y(t) - a \cdot x(t)]}_{\mu(t, x(t))} dt + \sigma(t) \cdot dW(t).$$

- ▶ Consider an option with price  $V = V(t, x(t))$ , option expiry time  $T$  and payoff  $V(T, x(T)) = g(x(T))$ .
- ▶ Derivative pricing formula yields that discounted option price is a martingale, i.e.

$$d\left(\frac{V(t, x(t))}{B(t)}\right) = \sigma_V(t, x(t)) \cdot dW(t).$$

How can we use this to derive a PDE?

## Apply Ito's Lemma to the discounted option price

We get

$$d\left(\frac{V(t, x(t))}{B(t)}\right) = \frac{dV(t, x(t))}{B(t)} + V(t)d\left(\frac{1}{B(t)}\right).$$

With  $d(B(t)^{-1}) = -r(t) \cdot B(t)^{-1} \cdot dt$  follows

$$d\left(\frac{V(t, x(t))}{B(t)}\right) = \frac{1}{B(t)} [dV(t, x(t)) - r(t) \cdot V(t) \cdot dt].$$

Applying Ito's Lemma to option price  $V(t, x(t))$  gives

$$\begin{aligned} dV(t, x(t)) &= V_t \cdot dt + V_x \cdot dx(t) + \frac{1}{2} V_{xx} \cdot [dx(t)]^2 \\ &= \left[ V_t + V_x \cdot \mu(t, x(t)) + \frac{1}{2} V_{xx} \cdot \sigma(t)^2 \right] dt + V_x \cdot \sigma(t) \cdot dW(t) \end{aligned}$$

with partial derivatives  $V_t = \partial V(t, x(t)) / \partial t$ ,  $V_x = \partial V(t, x(t)) / \partial x$  and  $V_{xx} = \partial^2 V(t, x(t)) / \partial x^2$ .

## Combining results yields dynamics of discounted option price

$$d\left(\frac{V(t, x(t))}{B(t)}\right) = \frac{1}{B(t)} \underbrace{\left[ V_t + V_x \cdot \mu(t, x(t)) + \frac{1}{2} V_{xx} \cdot \sigma(t)^2 - r(t) \cdot V \right]}_{\mu_V(t, x(t))} dt + \underbrace{\frac{V_x \cdot \sigma(t)}{B(t)}}_{\sigma_V(t, x(t))} \cdot dW(t).$$

Martingale property of  $\frac{V(t, x(t))}{B(t)}$  requires that drift vanishes. That is

$$\mu_V(t, x(t)) = V_t + V_x \cdot \mu(t, x(t)) + \frac{1}{2} V_{xx} \cdot \sigma(t)^2 - r(t) \cdot V = 0.$$

Substituting  $\mu(t, x(t)) = y(t) - a \cdot x(t)$  and  $r(t) = f(0, t) + x(t)$  yields pricing PDE.

## We get the parabolic pricing PDE with terminal condition

### Theorem (Derivative pricing PDE in Hull-White model)

Consider our Hull-White model setup and a derivative security with price process  $V(t, x(t))$  that pays at time  $T$  the payoff  $V(T, x(T)) = g(x(T))$ . Further assume  $V(T, x(T))$  has finite variance and is attainable.

Then for  $t < T$  the option price

$$V(t, x(t)) = B(t) \cdot \mathbb{E}^{\mathbb{Q}} \left[ \frac{V(T, x(T))}{B(T)} \mid \mathcal{F}_t \right]$$

follows the PDE

$$V_t(t, x) + [y(t) - a \cdot x] \cdot V_x(t, x) + \frac{\sigma(t)^2}{2} \cdot V_{xx}(t, x) = [x + f(0, t)] \cdot V(t, x)$$

with terminal condition

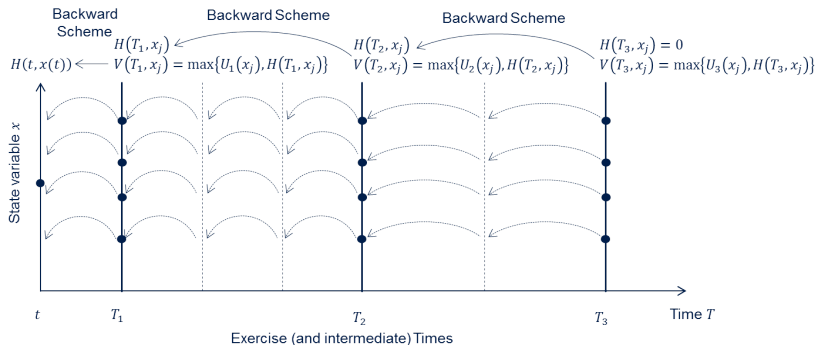
$$V(T, x) = g(x).$$

### Proof.

Follows from derivation above.



# How does this help for our Bermudan option pricing problem?



- We need option prices on a grid of state variables  $[x_0, \dots, x_N]$

Solve Hull-White option pricing PDE backwards from exercise to exercise.

# Pricing PDE is typically solved via finite difference scheme and time integration

- ▶ Use *method of lines (MOL)* to solve parabolic PDE:
  - ▶ First discretise state space.
  - ▶ Then integrate resulting system of ODEs with terminal condition in time-direction.
- ▶ We discuss basic (or general purpose) approach to solve PDE; for a detailed treatment see Andersen/Piterbarg (2010) or Duffy (2006).
- ▶ Some aspects may require special attention in the context of Hull-White model:
  - ▶ more problem-specific boundary discretisation,
  - ▶ non-equidistant grids with finer grid around break-even state  $x^*$ ,
  - ▶ upwinding schemes to deal with potentially dominant impact of convection term  $[y(t) - a \cdot x] \cdot V_x(t, x)$  at the grid boundaries of  $x$ .

# Outline

## PDE and Finite Differences

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**State Space Discretisation via Finite Differences**

Time-integration via  $\theta$ -Method

Alternative Boundary Conditions for Bond Option Payoffs

Summary of PDE Pricing Method



## How do we discretise state space?

- ▶ PDE for  $V(t, x(t))$  is defined on infinite domain  $(-\infty, +\infty)$ .
  - ▶ We don't get explicit boundary conditions from PDE derivation.
  - ▶ Thus, we require payoff-specific approximation.
  - ▶ Finally, we are interested in  $V(0, 0)$ .

- ▶ We use equidistant  $x$ -grid  $x_0, \dots, x_N$  with grid size  $h_x$  centered around zero and scaled via standard deviation of  $x(T)$  at final maturity  $T$ ,

$$x_0 = -\lambda \cdot \hat{\sigma} \quad \text{and} \quad x_N = \lambda \cdot \hat{\sigma}$$

with  $\hat{\sigma}^2 = \text{Var}[x(T)] = y(T)$  and  $\lambda \approx 5$ .

- ▶ Why not shift the grid by expectation  $\mathbb{E}[x(T)]$  (as suggested in the literature)?
  - ▶ Pricing PDE is independent of pricing measure (used for derivation).
  - ▶ There is no *natural* measure under which  $\mathbb{E}[x(T)]$  should be calculated.
  - ▶ In  $T$ -forward measure  $\mathbb{E}^T[x(T)] = 0$  anyway.

## Differential operators in state-dimension are discretised via central finite differences

For now leave time  $t$  continuous. We use notation  $V(\cdot, x)$ .

For inner grid points  $x_i$  with  $i = 1, \dots, N - 1$

$$V_x(\cdot, x_i) = \frac{V(\cdot, x_{i+1}) - V(\cdot, x_{i-1})}{2h_x} + \mathcal{O}(h_x^2) \quad \text{and}$$

$$V_{xx}(\cdot, x_i) = \frac{V(\cdot, x_{i+1}) - 2V(\cdot, x_i) + V(\cdot, x_{i-1}))}{h_x^2} + \mathcal{O}(h_x^2).$$

At the boundaries we impose condition

$$V_{xx}(\cdot, x_0) = \lambda_0 \cdot V_x(\cdot, x_0) \quad \text{and} \quad V_{xx}(\cdot, x_N) = \lambda_N \cdot V_x(\cdot, x_N).$$

This yields one-sided first order partial derivative approximations

$$V_x(\cdot, x_0) \approx \frac{2[V(\cdot, x_1) - V(\cdot, x_0)]}{(2 + \lambda_0 h_x) h_x} \quad \text{and} \quad V_x(\cdot, x_N) \approx \frac{2[V(\cdot, x_N) - V(\cdot, x_{N-1})]}{(2 - \lambda_N h_x) h_x}.$$

## Some initial comments regarding choice of $\lambda_{0,N}$

- ▶ Often,  $\lambda_{0,N} = 0$  (also suggested in the literature).
- ▶ With  $\lambda_{0,N} = 0$  we have  $V_{xx}(\cdot, x_0) = V_{xx}(\cdot, x_N) = 0$  and

$$V_x(\cdot, x_0) = \frac{V(\cdot, x_1) - V(\cdot, x_0)}{h_x} + \mathcal{O}(h_x^2) \quad \text{and}$$

$$V_x(\cdot, x_N) = \frac{V(\cdot, x_N) - V(\cdot, x_{N-1})}{h_x} + \mathcal{O}(h_x^2).$$

- ▶ However, for bond options the choice  $V_{xx}(\cdot, x_0) = V_{xx}(\cdot, x_N) = 0$  might be a poor approximation.
- ▶ We will discuss an alternative choice for  $\lambda_{0,N}$  later.

## Now consider PDE for each grid point individually

Define the vector-valued function  $v(t)$  via

$$v(t) = [v_0(t), \dots, v_N(t)]^\top = [V(t, x_0), \dots, V(t, x_N)]^\top \in \mathbb{R}^{N+1}.$$

Then state discretisation yields for inner points  $x_i$  with  $i = 1, \dots, N-1$ ,

$$v'_i(t) + [y(t) - ax_i] \frac{v_{i+1}(t) - v_{i-1}(t)}{2h_x} + \frac{\sigma(t)^2}{2} \frac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{h_x^2} = [x_i + f(0, t)] v_i(t)$$

and for the boundaries

$$v'_0(t) + \left[ y(t) - ax_0 + \lambda_0 \frac{\sigma(t)^2}{2} \right] \frac{2[v_1(t) - v_0(t)]}{(2 + \lambda_0 h_x) h_x} = [x_0 + f(0, t)] v_0(t),$$

$$v'_N(t) + \left[ y(t) - ax_N + \lambda_N \frac{\sigma(t)^2}{2} \right] \frac{2[v_N(t) - v_{N-1}(t)]}{(2 - \lambda_N h_x) h_x} = [x_N + f(0, t)] v_N(t).$$

As before, we have the terminal condition

$$v_i(T) = g(x_i).$$

Parabolic PDE is transformed into linear system of ODEs with terminal condition.

## It is more convenient to write system of ODEs in matrix-vector notation (1/2)

We get

$$v'(t) = M(t) \cdot v(t) = \begin{bmatrix} c_0 & u_0 & & & \\ l_1 & \ddots & \ddots & & \\ & \ddots & \ddots & u_{N-1} & \\ & & l_N & c_N & \end{bmatrix} \cdot v(t)$$

with time-dependent inner components  $c_i, l_i, u_i$  ( $i = 1, \dots, N - 1$ ),

$$c_i = \frac{\sigma(t)^2}{h_x^2} + x_i + f(0, t),$$
$$l_i = -\frac{\sigma(t)^2}{2h_x^2} + \frac{y(t) - ax_i}{2h_x},$$
$$u_i = -\frac{\sigma(t)^2}{2h_x^2} - \frac{y(t) - ax_i}{2h_x}.$$

## It is more convenient to write system of ODEs in matrix-vector notation (2/2)

Boundary elements of  $M(t)$  become

$$c_0 = \frac{2 \left[ y(t) - ax_0 + \lambda_0 \frac{\sigma(t)^2}{2} \right]}{(2 + \lambda_0 h_x) h_x} + x_0 + f(0, t),$$

$$c_N = - \frac{2 \left[ y(t) - ax_N + \lambda_N \frac{\sigma(t)^2}{2} \right]}{(2 - \lambda_N h_x) h_x} + x_0 + f(0, t),$$

$$u_0 = - \frac{2 \left[ y(t) - ax_0 + \lambda_0 \frac{\sigma(t)^2}{2} \right]}{(2 + \lambda_0 h_x) h_x},$$

$$l_N = \frac{2 \left[ y(t) - ax_N + \lambda_N \frac{\sigma(t)^2}{2} \right]}{(2 - \lambda_N h_x) h_x}.$$

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# Linear system of ODEs can be solved by standard methods

We have

$$v'(t) = f(t, v(t)) = M(t) \cdot v(t).$$

We demonstrate time discretisation based on  $\theta$ -method. Consider equidistant time grid  $t = t_0, \dots, t_M = T$  with step size  $h_t$  and approximation

$$\frac{v(t_{j+1}) - v(t_j)}{h_t} \approx f(t_{j+1} - \theta h_t, (1 - \theta)v(t_{j+1}) + \theta v(t_j))$$

for  $\theta \in [0, 1]$ .

- ▶ In general, approximation yields method of order  $\mathcal{O}(h_t)$ .
- ▶ For  $\theta = \frac{1}{2}$ , approximation yields method of order  $\mathcal{O}(h_t^2)$ .

For our linear ODE we set  $v^j = v(t_j)$ ,  $M_\theta = M(t_{j+1} - \theta h_t)$  and get the scheme

$$\frac{v^{j+1} - v^j}{h_t} = M_\theta [(1 - \theta)v^{j+1} + \theta v^j].$$



## We get a recursion for the $\theta$ -method

Rearranging terms yields

$$[I + h_t \theta M_\theta] v^j = [I - h_t (1 - \theta) M_\theta] v^{j+1}.$$

If  $[I + h_t \theta M_\theta]$  is regular then we can solve for  $v^j$  via

$$v^j = [I + h_t \theta M_\theta]^{-1} [I - h_t (1 - \theta) M_\theta] v^{j+1}.$$

Terminal condition is

$$v^M = [g(x_0), \dots, g(x_N)]^\top.$$

- ▶ Unless  $\theta = 0$  (Explicit Euler scheme) we need to solve a linear equation system.
- ▶ Fortunately, matrix  $[I + h_t \theta M_\theta]$  is tri-diagonal; solution requires  $\mathcal{O}(M)$  operations.
- ▶  $\theta$ -method is  $A$ -stable for  $\theta \geq \frac{1}{2}$ .
- ▶ However, oscillations in solution may occur unless  $\theta = 1$  (Implicit Euler scheme,  $L$ -stable).

# Outline

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Summary of PDE Pricing Method

## Let's have another look at the boundary condition ...

We look at an example of a zero coupon bond option with payoff

$$V(x, T) = [P(x, T, T') - K]^+.$$

For  $x \ll 0$  option is far in-the-money and  $V(x, t)$  can be approximated by intrinsic value  $V(x, t) \approx \tilde{V}(x, t)$  with

$$\tilde{V}(x, t) = [P(x, t, T') - K]^+ = \left[ \frac{P(0, T')}{P(0, t)} e^{-G(t, T)x - \frac{1}{2}G(t, T)^2 y(t)} - K \right]^+.$$

This yields

$$\frac{\partial}{\partial x} \tilde{V}(x, t) = -G(t, T) [\tilde{V}(x, t) + K]$$

and

$$\frac{\partial^2}{\partial x^2} \tilde{V}(x, t) = \underbrace{-G(t, T)}_{\lambda} \frac{\partial}{\partial x} \tilde{V}(x, t).$$

Alternatively, for  $x \gg 0$  option is far out-of-the-money and

$$\frac{\partial^2}{\partial x^2} \tilde{V}(x, t) = \frac{\partial}{\partial x} \tilde{V}(x, t) = 0.$$

## We adapt approximation to our option pricing problem

- ▶ In principle, for a coupon bond underlying we could estimate  $\lambda = \lambda(t)$  via option intrinsic value  $\tilde{V}(x, t)$  and

$$\lambda(t) = \left[ \frac{\partial^2}{\partial x^2} \tilde{V}(x, t) \right] / \frac{\partial}{\partial x} \tilde{V}(x, t) \quad \text{for} \quad \frac{\partial}{\partial x} \tilde{V}(x, t) \neq 0,$$

otherwise  $\lambda(t) = 0$ .

- ▶ We take a more rough approach by approximating  $\lambda$  based only on previous solution

$$\begin{aligned} \lambda_{0,N} &= \left[ \frac{\partial^2}{\partial x^2} V(x, t) \right] / \frac{\partial}{\partial x} V(x, t) \\ &\approx \left[ \frac{\partial^2}{\partial x^2} V(x_{1,N-1}, t + h_t) \right] / \frac{\partial}{\partial x} V(x_{1,N-1}, t + h_t) \\ &\approx \frac{v_{0,N-2}^{j+1} - 2v_{1,N-1}^{j+1} + v_{2,N}^{j+1}}{h_x^2} / \frac{v_{2,N}^{j+1} - v_{0,N-2}^{j+1}}{2h_x} \end{aligned}$$

for  $v_{2,N}^{j+1} - v_{0,N-2}^{j+1} / (2h_x) \neq 0$ , otherwise  $\lambda_{0,N} = 0$ .

It turns out that accuracy of one-sided first order derivative approximation is of order  $\mathcal{O}(h_x^2)$  !

### Lemma

Assume  $V = V(x)$  is twice continuously differentiable. Moreover, consider grid points  $x_{-1}, x_0, x_1$  with equal spacing  $h_x = x_1 - x_0 = x_0 - x_{-1}$ . If there is a  $\lambda_0 \in \mathbb{R}$  such that

$$V''(x_0) = \lambda_0 \cdot V'(x_0)$$

then

$$V'(x_0) = \frac{2[V(x_1) - V(x_0)]}{(2 + \lambda_0 h_x) h_x} + \mathcal{O}(h_x^2).$$

### Proof:

Denote  $v_i = V(x_i)$ . We have from standard Taylor approximation

$$V''(x_0) = \frac{v_{-1} - 2v_0 + v_1}{h_x^2} + \mathcal{O}(h_x^2) \quad \text{and} \quad V'(x_0) = \frac{v_1 - v_{-1}}{2h_x} + \mathcal{O}(h_x^2).$$

It turns out that accuracy of one-sided first order derivative approximation is of order  $\mathcal{O}(h_x^2)$  !!

From  $V''(x_0) = \lambda \cdot V'(x_0)$  follows

$$\frac{v_{-1} - 2v_0 + v_1}{h_x^2} + \mathcal{O}(h_x^2) = \lambda_0 \left[ \frac{v_1 - v_{-1}}{2h_x} + \mathcal{O}(h_x^2) \right].$$

Multiplying with  $2h_x^2$  gives the relation

$$2(v_{-1} - 2v_0 + v_1) + \mathcal{O}(h_x^4) = \lambda_0 h_x (v_1 - v_{-1}) + \mathcal{O}(h_x^4).$$

Reordering terms yields

$$(2 + \lambda_0 h_x) v_{-1} = 4v_0 + (\lambda_0 h_x - 2) v_1 + \mathcal{O}(h_x^4).$$

And solving for  $v_{-1}$  gives

$$v_{-1} = [4v_0 + (\lambda_0 h_x - 2) v_1] / (2 + \lambda_0 h_x) + \mathcal{O}(h_x^4).$$

## It turns out that accuracy of one-sided first order derivative approximation is of order $\mathcal{O}(h_x^2)$ III

Now, we substitute  $v_{-1}$  in the approximation for  $V'(x)$ . This gives

$$\begin{aligned}V'(x_0) &= \frac{v_1 - \left[ \frac{4v_0 + (\lambda_0 h_x - 2)v_1}{2 + \lambda_0 h_x} + \mathcal{O}(h_x^4) \right]}{2h_x} + \mathcal{O}(h_x^2) \\&= \frac{(2 + \lambda_0 h_x)v_1 - [4v_0 + (\lambda_0 h_x - 2)v_1]}{2(2 + \lambda_0 h_x)h_x} + \mathcal{O}(h_x^2) + \mathcal{O}(h_x^3) \\&= \frac{2v_1 - 4v_0 + 2v_1}{2(2 + \lambda_0 h_x)h_x} + \mathcal{O}(h_x^2) \\&= \frac{2(v_1 - v_0)}{(2 + \lambda_0 h_x)h_x} + \mathcal{O}(h_x^2).\end{aligned}$$

- ▶ With constraint  $V''(x_0) = \lambda \cdot V'(x_0)$  we can eliminate explicit dependence on second derivative  $V''(x_0)$  and outer grid point  $v_{-1} = V(x_{-1})$ .

It turns out that accuracy of one-sided first order derivative approximation is of order  $\mathcal{O}(h_x^2)$  IV

- ▶ Analogous result can be derived for upper boundary and down-ward approximation of first derivative.
- ▶ Resulting scheme is still second order accurate in state space direction.



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Summary of PDE Pricing Method

## We summarise the PDE pricing method

1. Discretise state space  $x$  on a grid  $[x_0, \dots, x_N]$  and specify time step size  $h_t$  and  $\theta \in [0, 1]$ .
2. Determine the terminal condition  $v^{j+1} = \max \{U_{j+1}, H_{j+1}\}$  for the current valuation step.
3. Set up discretised linear operator  $M_\theta$  of the resulting ODE system  $\frac{d}{dt}v = M_\theta \cdot v$ .
4. Incorporate appropriate product-specific boundary conditions.
5. Set up linear system  $[I + h_t\theta M_\theta] v^j = [I - h_t(1 - \theta) M_\theta] v^{j+1}$ .
6. Solve linear system for  $v^j$  by tri-diagonal matrix solver.
7. Repeat with step 3. until next exercise date or  $t_j = 0$ .

# Outline

Bermudan Swaptions

Pricing Methods for Bermudans

Density Integration Methods

PDE and Finite Differences

American Monte Carlo

# Monte Carlo methods are widely applied in various finance applications

- ▶ We demonstrate the basic principles for
  - ▶ path integration of Ito processes
  - ▶ exact simulation of Hull-White model paths
- ▶ There are many aspects that should also be considered, see e.g.
  - ▶ L. Andersen and V. Piterbarg. *Interest rate modelling, volume I to III*. Atlantic Financial Press, 2010, Sec. 3.
  - ▶ P. Glasserman. *Monte Carlo Methods in Financial Engineering*. Springer, 2003

# Outline

## American Monte Carlo

Introduction to Monte Carlo Pricing

Monte Carlo Simulation in Hull-White Model

Regression-based Backward Induction

# Monte Carlo (MC) pricing is based on the Strong Law of Large Numbers

## Theorem (Strong Law of Large Numbers)

Let  $Y_1, Y_2, \dots$  be a sequence of independent identically distributed (i.i.d.) random variables with finite expectation  $\mu < \infty$ . Then the sample mean  $\bar{Y}_n = \frac{1}{n} \sum_{i=1}^n Y_i$  converges to  $\mu$  a.s. That is

$$\lim_{n \rightarrow \infty} \bar{Y}_n = \mu \quad \text{a.s.}$$

- ▶ We aim at calculating  $V(t) = N(t) \cdot \mathbb{E}^N [V(T)/N(T) | \mathcal{F}_t]$ .
- ▶ For MC pricing simulate future discounted payoffs  $\left\{ \frac{V(T; \omega_i)}{N(T; \omega_i)} \right\}_{i=1, 2, \dots, n}$ .
- ▶ And estimate

$$V(t) = N(t) \cdot \frac{1}{n} \sum_{i=1}^n \frac{V(T; \omega_i)}{N(T; \omega_i)}.$$

Keep in mind that sample mean is still a random variable governed by central limit theorem (1/2)

### Theorem (Central Limit Theorem)

Let  $Y_1, Y_2, \dots$  be a sequence of i.i.d. random variables with finite expectation  $\mu < \infty$  and standard deviation  $\sigma < \infty$ . Denote the sample mean  $\bar{Y}_n = \frac{1}{n} \sum_{i=1}^n Y_i$ . Then

$$\frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0, 1).$$

Moreover, for the variance estimator  $s_n^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y}_n)^2$  we also have

$$\frac{\bar{Y}_n - \mu}{s_n/\sqrt{n}} \xrightarrow{d} N(0, 1).$$

Keep in mind that sample mean is still a random variable governed by central limit theorem (2/2)

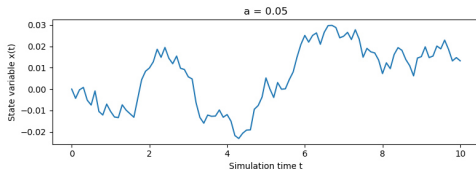
$$\frac{\bar{Y}_n - \mu}{s_n/\sqrt{n}} \xrightarrow{d} N(0, 1).$$

- ▶ Here,  $N(0, 1)$  is the standard normal distribution.
- ▶  $\xrightarrow{d}$  denotes convergence in distribution, i.e.  $\lim_{n \rightarrow \infty} F_n(x) = F(x)$  for the corresponding cumulative distribution functions and all  $x \in \mathbb{R}$  at which  $F(x)$  is continuous.
- ▶  $s_n/\sqrt{n}$  is the standard error of the sample mean  $\bar{Y}_n$ .

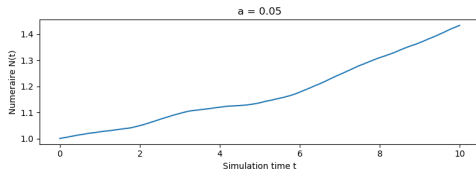


# How do we get our samples $V(T; \omega_i)/N(T; \omega_i)$ ?

1. Simulate state variables  $x(t)$  on relevant dates  $t$ :



2. Simulate numeraire  $N(t)$  on relevant dates  $t$ :



3. Calculate payoff  $V(T, x(T))$  at observation/pay date  $T$ .

## We need to simulate our state variables on the relevant observation dates

Consider the general dynamics for a process given as SDE

$$dX(t) = \mu(t, X(t)) \cdot dt + \sigma(t, X(t)) \cdot dW(t).$$

- ▶ Typically, we know initial value  $X(t)$  ( $t = 0$ ).
- ▶ We need  $X(T)$  for some future time  $T > t$ .
- ▶ In Hull-White model and risk-neutral measure formulation we have

$$\mu(t, X(t)) = y(t) - a \cdot X(t), \quad \text{and,} \quad \sigma(t, X(t)) = \sigma(t).$$

There are several standard methods to solve above SDE. We will briefly discuss Euler method and Milstein method.

## Euler method for SDEs is similar to Explicit Euler method for ODEs

- ▶ Specify a grid of simulation times  $t = t_0, t_1, \dots, t_M = T$ .

- ▶ Calculate sequence of state variables

$$X_{k+1} = X_k + \mu(t_k, X_k)(t_{k+1} - t_k) + \sigma(t_k, X_k)[W(t_{k+1}) - W(t_k)].$$

- ▶ Drift  $\mu(t_k, X_k)$  and volatility  $\sigma(t_k, X_k)$  are evaluated at current time  $t_k$  and state  $X_k$ .
- ▶ Increment of Brownian motion  $W(t_{k+1}) - W(t_k)$  is normally distributed, i.e.

$$W(t_{k+1}) - W(t_k) = Z_k \cdot \sqrt{t_{k+1} - t_k} \quad \text{with} \quad Z_k \sim N(0, 1).$$

## Milstein method refines the simulation of the diffusion term (1/2)

- ▶ Again, specify a grid of simulation times  $t = t_0, t_1, \dots, t_M = T$ .
- ▶ Calculate sequence of state variables

$$X_{k+1} = X_k + \mu(t_k, X_k)(t_{k+1} - t_k) + \sigma(t_k, X_k)[W(t_{k+1}) - W(t_k)] \\ + \frac{1}{2}\sigma(t_k, X_k)\frac{\partial\sigma(t_k, X_k)}{\partial X} \left[ (W(t_{k+1}) - W(t_k))^2 - (t_{k+1} - t_k) \right].$$

- ▶ Drift  $\mu(t_k, X_k)$  and volatility  $\sigma(t_k, X_k)$  are evaluated at current time  $t_k$  and state  $X_k$ .

## Milstein method refines the simulation of the diffusion term (2/2)

- ▶ Requires calculation of derivative of volatility  $\frac{\partial}{\partial x}\sigma(t_k, X_k)$  w.r.t. state variable.
- ▶ Increment of Brownian motion  $W(t_{k+1}) - W(t_k)$  is normally distributed, i.e.

$$W(t_{k+1}) - W(t_k) = Z_k \cdot \sqrt{t_{k+1} - t_k} \quad \text{with} \quad Z_k \sim N(0, 1).$$

- ▶ With  $\Delta_k = t_{k+1} - t_k$  iteration becomes

$$\begin{aligned} X_{k+1} = & X_k + \mu(t_k, X_k)\Delta_k + \sigma(t_k, X_k)Z_k\sqrt{\Delta_k} \\ & + \frac{1}{2}\sigma(t_k, X_k)\frac{\partial\sigma(t_k, X_k)}{\partial X}(Z_k^2 - 1)\Delta_k. \end{aligned}$$

## How can we measure convergence of the methods?

- ▶ We distinguish **strong order** of convergence and **weak order** of convergence.
- ▶ Consider a discrete SDE solution  $\{X_k^h\}_{k=0}^M$  with  $X_k^h \approx X(t + kh)$ ,  $h = \frac{T-t}{M}$ .

### Definition (Strong order of convergence)

The discrete solution  $X_M^h$  at final maturity  $T = t + hM$  converges to the exact solution  $X(T)$  with strong order  $\beta$  if there exists a constant  $C$  such that

$$\mathbb{E} [ |X_M^h - X(T)| ] \leq C \cdot h^\beta.$$

- ▶ Strong order of convergence focuses on convergence on the individual paths.
- ▶ Euler method has strong order of convergence of  $\frac{1}{2}$  (given sufficient conditions on  $\mu(\cdot)$  and  $\sigma(\cdot)$ ).
- ▶ Milstein method has strong order of convergence of 1 (given sufficient conditions on  $\mu(\cdot)$  and  $\sigma(\cdot)$ ).

# For derivative pricing we are typically interested in weak order of convergence

We need some context for weak order of convergence

- ▶ A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is polynomially bounded if  $|f(x)| \leq k(1 + |x|)^q$  for constants  $k$  and  $q$  and all  $x$ .
- ▶ The set  $\mathcal{C}_{\mathcal{P}}^n$  represents all functions that are  $n$ -times continuously differentiable and with 1st to  $n$ th derivative polynomially bounded.

## Definition (Weak order of convergence)

The discrete solution  $X_M^h$  at final maturity  $T = t + hM$  converges to the exact solution  $X(T)$  with weak order  $\beta$  if there exists a constant  $C$  such that

$$|\mathbb{E}[f(X_M^h)] - \mathbb{E}[f(X(T))]| \leq C \cdot h^\beta \quad \forall f \in \mathcal{C}_{\mathcal{P}}^{2\beta+2}$$

for sufficiently small  $h$ .

- ▶ Think of  $f$  as a payoff function, then weak order of convergence is related to convergence in price.
- ▶ Euler method and Milstein method can be shown to have weak order 1 convergence (given sufficient conditions on  $\mu$  and  $\sigma$ ).

# Some comments regarding weak order of convergence

## Error estimate

$$|\mathbb{E}[f(X_M^h)] - \mathbb{E}[f(X(T))]| \leq C \cdot h^\beta$$

requires considerable assumptions regarding smoothness of  $\mu(\cdot)$ ,  $\sigma(\cdot)$  and test functions  $f(\cdot)$ .

- ▶ In practice payoffs are typically non-smooth at the strike.
- ▶ This limits applicability of more advanced schemes with theoretical higher order of convergence.
- ▶ A fairly simple approach of a higher order scheme is based on Richardson extrapolation:
  - ▶ this method is also applied to ODEs,
  - ▶ see Glassermann (2000), Sec. 6.2.4 for details.
- ▶ Typically, numerical testing is required to assess convergence in practice.



# The choice of pricing measure is crucial for numeraire simulation

Consider risk-neutral measure, then

$$\begin{aligned} N(T) = B(T) &= \exp \left\{ \int_0^T r(s) ds \right\} = \exp \left\{ \int_0^T [f(0, s) + x(s)] ds \right\} \\ &= P(0, T)^{-1} \exp \left\{ \int_0^T x(s) ds \right\}. \end{aligned}$$

Requires simulation or approximation of  $\int_0^T x(s) ds$ .

Suppose  $x(t_k)$  is simulated on a time grid  $\{t_k\}_{k=0}^M$  then we approximate integral via Trapezoidal rule

$$\int_0^T x(s) ds \approx \sum_{i=1}^M \frac{x(t_{k-1}) + x(t_k)}{2} (t_k - t_{k-1}).$$

Numeraire simulation is done in parallel to state simulation

$$N(t_k) = \frac{P(0, t_{k-1})}{P(0, t_k)} \cdot N(t_{k-1}) \cdot \exp \left\{ \frac{x(t_{k-1}) + x(t_k)}{2} (t_k - t_{k-1}) \right\}.$$

## Alternatively, we can simulate in $T$ -forward measure for a fixed future time $T$

Select a future time  $\bar{T}$  sufficiently large. Then  $N(0) = P(0, \bar{T})$ .  
At any pay time  $T \leq \bar{T}$  numeraire is directly available via zero coupon bond formula

$$N(T) = P(x(T), T, \bar{T}) = \frac{P(0, \bar{T})}{P(0, T)} e^{-G(T, T')x(T) - \frac{1}{2}G(T, T')^2 y(T)}.$$

However,  $\bar{T}$ -forward measure simulation needs consistent model formulation or change of measure.

In particular

$$\underbrace{dW^{\bar{T}}(t)}_{\text{B.M. in } \bar{T}\text{-forward measure}} = \underbrace{\sigma_P(t, \bar{T})}_{\text{ZCB volatility}} \cdot dt + \underbrace{dW(t)}_{\text{B.M. in risk-neutral measure}}.$$

## Another commonly used numeraire for simulation is the discretely compounded bank account

- ▶ Consider a grid of simulation times  $t = t_0, t_1, \dots, t_M = T$ .
- ▶ Assume we start with 1 EUR at  $t = 0$ , i.e.  $N(0) = 1$ .
- ▶ At each  $t_k$  we take numeraire  $N(t_k)$  and buy zero coupon bond maturing at  $t_{k+1}$ . That is

$$N(t) = P(t, t_{k+1}) \cdot \frac{N(t_k)}{P(t_k, t_{k+1})} \quad \text{for } t \in [t_k, t_{k+1}].$$

Explicitly, define **discretely compounded bank account** as  $\bar{B}(0) = 1$  and

$$\bar{B}(t) = P(t, t_{k+1}) \prod_{t_k < t} \frac{1}{P(t_k, t_{k+1})}.$$

We get

$$d \left( \frac{\bar{B}(t)}{P(t, t_{k+1})} \right) = \prod_{t_k < t} \frac{1}{P(t_k, t_{k+1})} \cdot d \left( \frac{P(t, t_{k+1})}{P(t, t_{k+1})} \right) = 0 \quad \text{for } t \in [t_k, t_{k+1}].$$

Simulating in  $\bar{B}$ -measure is equivalent to simulating in rolling  $t_{k+1}$ -forward measure.

# Outline

## American Monte Carlo

Introduction to Monte Carlo Pricing

Monte Carlo Simulation in Hull-White Model

Regression-based Backward Induction

# Do we really need to solve the Hull-White SDE numerically?

Recall dynamics in  $T$ -forward measure

$$dx(t) = [y(t) - \sigma(t)^2 G(t, T) - a \cdot x(t)] \cdot dt + \sigma(t) \cdot dW^T(t).$$

That gives

$$x(T) = e^{-a(T-t)}.$$

$$\left[ x(t) + \int_t^T e^{a(u-t)} ([y(u) - \sigma(u)^2 G(u, T)] du + \sigma(u) dW^T(u) \right].$$

As a result  $x(T) \sim N(\mu, \sigma^2)$  (conditional on  $t$ ) with

$$\mu = \mathbb{E}^T [x(T) | \mathcal{F}_t] = G'(t, T) [x(t) + G(t, T)y(t)] \quad \text{and}$$

$$\sigma^2 = \text{Var} [x(T) | \mathcal{F}_t] = y(T) - G'(t, T)^2 y(t).$$

We can simulate exactly

$$x(T) = \mu + \sigma \cdot Z \quad \text{with} \quad Z \sim N(0, 1).$$

## Expectation calculation via $\mu = \mathbb{E}^T [x(T) | \mathcal{F}_t]$ requires careful choice of numeraire

Consider grid of simulation times  $t = t_0, t_1, \dots, t_M = T$ .

We simulate

$$x(t_{k+1}) = \mu_k + \sigma_k \cdot Z_k$$

with

$$\mu_k = G'(t_k, t_{k+1}) [x(t_k) + G(t_k, t_{k+1})y(t_k)],$$

$$\sigma_k^2 = y(t_{k+1}) - G'(t_k, t_{k+1})^2 y(t_k), \quad \text{and}$$

$$Z_k \sim N(0, 1).$$

Grid point  $t_{k+1}$  must coincide with forward measure for  $\mathbb{E}^{t_{k+1}}[\cdot]$  for each individual step  $k \rightarrow k + 1$ .

Numeraire must be discretely compounded bank account  $\bar{B}(t)$  and

$$\bar{B}(t_{k+1}) = \frac{\bar{B}(t_k)}{P(x(t_k), t_k, t_{k+1})}.$$

Recursion for  $x(t_{k+1})$  and  $\bar{B}(t_{k+1})$  fully specifies path simulation for pricing.

## Some comments regarding Hull-White MC simulation ...

- ▶ We could also simulate in risk-neutral measure or  $\bar{T}$ -forward measure.
  - ▶ This might be advantageous if also FX or equities are modelled/simulated.
  - ▶ Requires adjustment of conditional expectation  $\mu_k$  and numeraire  $N(t_k)$  calculation.
  - ▶ Variance  $\sigma_k^2$  is invariant to change of measure in Hull-White model.
- ▶ Repeat path generation for as many paths  $1, \dots, n$  as desired (or computationally feasible).
- ▶ For Bermudan pricing we need to simulate  $x$  and  $N$  (at least) at exercise dates  $T_E^1, \dots, T_E^{\bar{k}}$ .
- ▶ For calculation of  $Z_k$  use
  - ▶ pseudo-random numbers or
  - ▶ Quasi-Monte Carlo sequences.

as proxies for independent  $N(0, 1)$  random variables across time steps and paths.

## We illustrate MC pricing by means of a coupon bond option example

Consider coupon bond option expiring at  $T_E$  with coupons  $C_i$  paid at  $T_i$  ( $i = 1, \dots, u$ , incl. strike and notional).

- ▶ Set  $t_0 = 0$ ,  $t_1 = T_E/2$  and  $t_2 = T_E$  (two steps for illustrative purpose).
- ▶ Compute  $2n$  independent  $N(0, 1)$  pseudo random numbers  $Z^1, \dots, Z^{2n}$ .
- ▶ For all paths  $j = 1, \dots, n$  calculate:
  - ▶  $\mu_0^j$ ,  $\sigma_0$  and  $\bar{B}^j(t_1)$ ; note  $\mu_0^j$  and  $\bar{B}^j(t_1)$  are equal for all paths  $j$  since  $x(t_0) = 0$ ,
  - ▶  $x_1^j = \mu_0^j + \sigma_0 \cdot Z^j$ ,
  - ▶  $\mu_1^j$ ,  $\sigma_1$  and  $\bar{B}^j(t_2)$ ; note now  $\mu_1^j$  and  $\bar{B}^j(t_2)$  depend on  $x_1^j$ ,
  - ▶  $x_2^j = \mu_1^j + \sigma_1 \cdot Z^{n+j}$ ,
  - ▶ payoff  $V^j(t_2) = [\sum_{i=1}^u C_i \cdot P(x_2^j, t_2, T_i)]^+$  at  $t_2 = T_E$ .
- ▶ Calculate option price (note  $\bar{B}(0) = 1$ )

$$V(0) = \bar{B}(0) \cdot \frac{1}{n} \sum_{j=1}^n \frac{V^j(t_2)}{\bar{B}^j(t_2)}.$$



# Outline

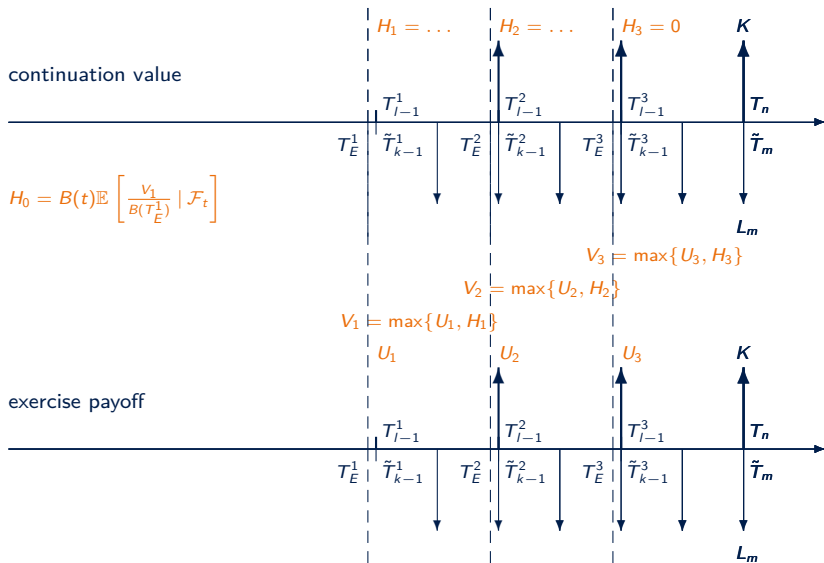
## American Monte Carlo

Introduction to Monte Carlo Pricing

Monte Carlo Simulation in Hull-White Model

Regression-based Backward Induction

# Let's return to our Bermudan option pricing problem

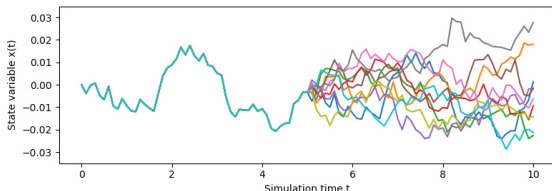


## In this setting we need to calculate future conditional expectations

- ▶ Assume we already simulated paths for state variables  $x_k$ , underlyings  $U_k$  and numeraire  $B_k$  for all relevant dates  $t_k$ .
- ▶ We need continuation values  $H_k$  defined recursively via  $H_{\bar{k}} = 0$  and

$$H_k = B_k \mathbb{E}_k \left[ \frac{\max \{U_{k+1}, H_{k+1}\}}{B_{k+1}} \right].$$

- ▶ In principle, we could use nested Monte Carlo:



- ▶ In practice, nested Monte Carlo is typically computationally not feasible.

# A key idea of American Monte Carlo is approximating conditional expectation via regression

Conditional expectation

$$H_k = \mathbb{E}_k \left[ \frac{B_k}{B_{k+1}} \max \{ U_{k+1}, H_{k+1} \} \right]$$

is a function of the path  $x(t)$  for  $t \leq t_k$ .

For non-path-dependent underlyings  $U_k$ ,  $H_k$  can be written as function of  $x_k = x(t_k)$ , i.e.

$$H_k = H_k(x_k).$$

We aim at finding a regression operator

$$\mathcal{R}_k = \mathcal{R}_k [Y]$$

which we can use as proxy for  $H_k$ .

## What do we mean by regression operator?

Denote  $\zeta(\omega) = [\zeta_1(\omega), \dots, \zeta_q(\omega)]^\top$  a set of **basis functions** (vector of random variables).

Let  $Y = Y(\omega)$  be a target random variable.

Assume we have outcomes  $\omega_1, \dots, \omega_{\bar{n}}$  with **control variables**  $\zeta(\omega_1), \dots, \zeta(\omega_{\bar{n}})$  and **observations**  $Y(\omega_1), \dots, Y(\omega_{\bar{n}})$ .

A **regression operator**  $\mathcal{R}[Y]$  is defined via

$$\mathcal{R}[Y](\omega) = \zeta(\omega)^\top \beta$$

where the regression coefficients  $\beta$  solve linear least squares problem

$$\left\| \begin{bmatrix} \zeta(\omega_1)^\top \beta - Y(\omega_1) \\ \vdots \\ \zeta(\omega_{\bar{n}})^\top \beta - Y(\omega_{\bar{n}}) \end{bmatrix} \right\|^2 \rightarrow \min.$$

Linear least squares system can be solved e.g. via QR factorisation or SVD.

## A basic pricing scheme is obtained by replacing conditional expectation of future payoff by regression operator

Approximate  $\tilde{H}_k \approx H_k$  via  $\tilde{H}_{\bar{k}} = H_{\bar{k}} = 0$  and

$$\tilde{H}_k = \mathcal{R}_k \left[ \frac{B_k}{B_{k+1}} \max \{ U_{k+1}, \tilde{H}_{k+1} \} \right] \quad \text{for } k = \bar{k} - 1, \dots, 1.$$

- ▶ Critical piece of this methodology is (for each step  $k$ )
  - ▶ choice of regression variables  $\zeta_1, \dots, \zeta_q$  and
  - ▶ calibration of regression operator  $\mathcal{R}_k$  with coefficients  $\beta$ .
- ▶ Regression variables  $\zeta_1, \dots, \zeta_q$  must be calculated based on information up to  $t_k$ .
  - ▶ They must not look into the future to avoid upward bias.
- ▶ Control variables  $\zeta(\omega_1), \dots, \zeta(\omega_{\bar{n}})$  and observations  $Y(\omega_1), \dots, Y(\omega_{\bar{n}})$  for calibration should be simulated on paths independent from pricing.
  - ▶ Using same paths for calibration and payoff simulation also incorporates information on the future.

# What are typical basis functions?

## State variable approach

Set  $\zeta_i = x(t_k)^{i-1}$  for  $i = 1, \dots, q$ . Typical choice is  $q \approx 4$  (i.e. polynomials of order 3). For multi-dimensional models we would set

$$\zeta_i = \prod_{j=1}^d x_j(t_k)^{p_{i,j}} \quad \text{with} \quad \sum_{j=1}^d p_{i,j} \leq r.$$

- ▶ Very generic and easy to incorporate.

## Explanatory variable approach

Identify variables  $y_1, \dots, y_{\bar{d}}$  relevant for the underlying option. Set basis functions as monomials

$$\zeta_i = \prod_{j=1}^{\bar{d}} y_j(t_k)^{p_{i,j}} \quad \text{with} \quad \sum_{j=1}^{\bar{d}} p_{i,j} \leq r.$$

- ▶ Can be chosen option-specific and incorporate information prior to  $t_k$ .
- ▶ Typical choices are co-terminal swap rates or Libor rates (observed at  $t_k$ ).

## Regression of the full underlying can be a bit rough - we may restrict regression to exercise decision only

For a given path consider

$$\begin{aligned} H_k &= \frac{B_k}{B_{k+1}} \max \{ U_{k+1}, H_{k+1} \} \\ &= \frac{B_k}{B_{k+1}} \left[ \mathbb{1}_{\{U_{k+1} > H_{k+1}\}} U_{k+1} + (1 - \mathbb{1}_{\{U_{k+1} > H_{k+1}\}}) H_{k+1} \right]. \end{aligned}$$

Use regression to calculate  $\mathbb{1}_{\{U_{k+1} > H_{k+1}\}}$ .

Calculate  $\mathcal{R}_{k+1} = \mathcal{R}_{k+1} [U_{k+1} - H_{k+1}]$ , set  $H_{\bar{k}} = 0$  and

$$H_k = \frac{B_k}{B_{k+1}} \left[ \mathbb{1}_{\{\mathcal{R}_{k+1} > 0\}} U_{k+1} + (1 - \mathbb{1}_{\{\mathcal{R}_{k+1} > 0\}}) H_{k+1} \right] \quad \text{for } k = \bar{k}-1, \dots, 1.$$

- ▶ Think of  $\mathbb{1}_{\{\mathcal{R}_{k+1} > 0\}}$  as an exercise strategy (which might be sub-optimal).
- ▶ This approach is sometimes considered more accurate than regression on regression.
- ▶ For further reference, see also Longstaff/Schwartz (2001).



## We summarise the American Monte Carlo method

1. Simulate  $n$  paths of state variables  $x_k^j$ , underlyings  $U_k^j$  and numeraires  $B_k^j$  ( $j = 1, \dots, n$ ) for all relevant times  $t_k$  ( $k = 1, \dots, \bar{k}$ ).
2. Set  $H_{\bar{k}}^j = 0$ .
3. For  $k = \bar{k} - 1, \dots, 1$  iterate:
  - 3.1 Calculate control variables  $\{\zeta_i^j = \zeta_i(\omega_j)\}_{i=1, \dots, q}^{j=1, \dots, \hat{n}}$  and regression variables  $Y^j = U_k^j - H_k^j$  for the first  $\hat{n}$  paths ( $\hat{n} \approx \frac{1}{4}n$ ).
  - 3.2 Calibrate regression operator  $\mathcal{R}_{k+1} = \mathcal{R}_{k+1}[Y]$  which gives coefficients  $\beta$ .
  - 3.3 Calculate control variables  $\{\zeta_i^j = \zeta_i(\omega_j)\}_{i=1, \dots, q}^{j=\hat{n}+1, \dots, n}$  for remaining paths and (for all paths)

$$H_k^j = \frac{B_k^j}{B_{k+1}^j} \left[ \mathbb{1}_{\{\mathcal{R}_{k+1}(\omega_j) > 0\}} U_{k+1}^j + \left( 1 - \mathbb{1}_{\{\mathcal{R}_{k+1}(\omega_j) > 0\}} \right) H_{k+1}^j \right].$$

4. Calculate discounted payoffs for the paths  $j = \hat{n} + 1, \dots, n$  not used for regression

$$H_0^j = \frac{B_k^j}{B_{k+1}^j} \max \left\{ U_1^j, H_1^j \right\}.$$

5. Derive average  $V(0) = \frac{1}{n - \hat{n}} \sum_{j=\hat{n}+1}^n H_0^j$ .

## Some comments regarding AMC for Bermudans in Hull-White model

- ▶ AMC implementations can be very bespoke and problem specific.
  - ▶ See literature for more details.
- ▶ More explanatory variables or too high polynomial degree for regression may deteriorate numerical solution.
  - ▶ This is particularly relevant for 1-factor models like Hull-White.
  - ▶ Single state variable or co-terminal swap rate should suffice.
- ▶ AMC with Hull-White for Bermudans is *not* the method of choice.
  - ▶ PDE and integration methods are directly applicable.
  - ▶ AMC is much slower and less accurate compared to PDE and integration.

AMC is the method of choice for high-dimensional models and/or path-dependent products.

# Outline

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