Interest Rate Modelling and Derivative Pricing

Sebastian Schlenkrich

HU Berlin, Department of Mathematics

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Part VII

Sensitivity Calculation

Outline

Outline

Why do we need sensitivities?

Consider a (differentiable) pricing model $V = V(p)$ based on some input parameter p. Sensitivity of V w.r.t. changes in p is

$$V'(p) = \frac{dV(p)}{dp}.$$

- Hedging and risk management.
- Market risk measurement.
- Many more applications for accounting, regulatory reporting, ...

Sensitivity calculation is a crucial function for banks and financial institutions.

Derivative pricing is based on hedging and risk replication

Recall fundamental derivative replication result

$$V(t) = V\left(t, X(t)\right) = \phi(t)^\top X(t) \text{ for all } t \in [0, T],$$

- $V(t)$ price of a contingent claim,
- $\phi(t)$ permissible trading strategy,
- $X(t)$ assets in our market.

How do we find the trading strategy?

Consider portfolio $\pi(t) = V(t, X(t)) - \phi(t)^\top X(t)$ and apply Ito's lemma

$$d\pi(t) = \mu_\pi \cdot dt + \left[\nabla_X \pi(t)\right]^\top \cdot \sigma_X^\top dW(t).$$

From replication property follows $d\pi(t) = 0$ for all $t \in [0, T]$. Thus, in particular

$$0 = \nabla_X \pi(t) = \nabla_X V(t, X(t)) - \phi(t).$$

This gives Delta-hedge

$$\phi(t) = \nabla_X V(t, X(t)).$$

Market risk calculation relies on accurate sensitivities (1/2)

Consider portfolio value $\pi(t)$, time horizon Δt and returns

$$\Delta\pi(t) = \pi(t) - \pi(t - \Delta t).$$

Market risk measure Value at Risk (VaR) is the lower quantile q of distribution of portfolio returns $\Delta\pi(t)$ given a confidence level $1 - \alpha$, formally

$$\text{VaR}_\alpha = \inf\left\{q \quad s.t.\, \mathbb{P}\left\{\Delta\pi(t) \leq q \,|\, \pi(t)\right\} > \alpha\right\}.$$

Delta-Gamma VaR calculation method consideres $\pi(t) = \pi\left(X(t)\right)$ in terms of risk factors $X(t)$ and approximates

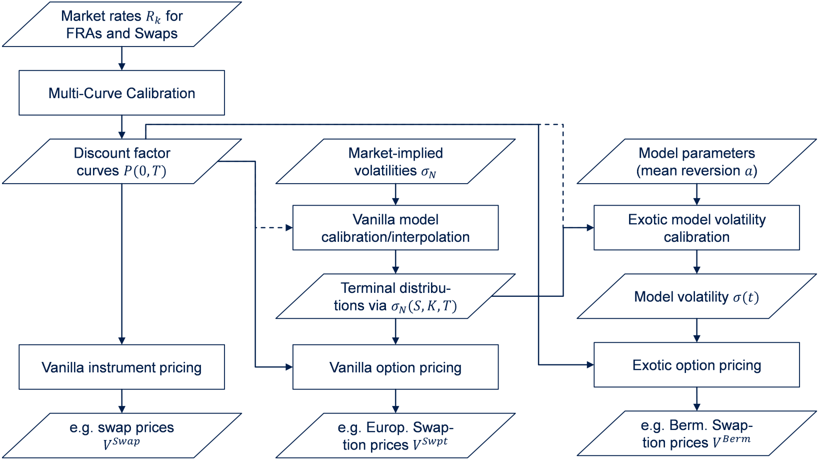
$$\Delta\pi \approx \left[\nabla_X \pi\left(X\right)\right]^\top \Delta X + \frac{1}{2}\Delta X^\top \left[H_X \pi\left(X\right)\right] \Delta X.$$

Market risk calculation relies on accurate sensitivities (2/2)

$$\Delta\pi \approx \left[\nabla_X \pi(X)\right]^\top \Delta X + \frac{1}{2}\Delta X^\top \left[H_X \pi(X)\right] \Delta X.$$

- VaR is calculated based on joint distribution of risk factor returns $\Delta X = X(t + \Delta t) - X(t)$ and sensitivities $\nabla_X \pi$ (gradient) and $H_X \pi$ (Hessian).
- Bank portfolio π may consist of linear instruments (e.g. swaps), Vanilla options (e.g. European swaptions) and exotic instruments (e.g. Bermudans).
- Common interest rate risk factors are FRA rates, par swap rates, ATM volatilities.

Sensitivity specification needs to take into account data flow and dependencies



Depending on context, risk factors can be market parameters or model parameters.

In practice, sensitivities are scaled relative to pre-defined risk factor shifts

Scaled sensitivity ΔV becomes

$$\Delta V = \frac{dV(p)}{dp} \cdot \Delta p \approx V(p + \Delta p) - V(p).$$

Typical scaling (or risk factor shift sizes) Δp are

- $1bp$ for interest rate shifts,
- $1bp$ for implied normal volatilities,
- 1% for implied lognormal or shifted lognormal volatilities.

Par rate Delta and Gamma are sensitivity w.r.t. changes in market rates (1/2)

Bucketed Delta and Gamma

Let $\bar{R} = [R_k]_{k=1,\ldots q}$ be the list of market quotes defining the inputs of a yield curve. The bucketed par rate delta of an instrument with model price $V = V(\bar{R})$ is the vector

$$\Delta_R = 1bp \cdot \left[\frac{\partial V}{\partial R_1}, \ldots, \frac{\partial V}{\partial R_q} \right].$$

Bucketed Gamma is calculated as

$$\Gamma_R = [1bp]^2 \cdot \left[\frac{\partial^2 V}{\partial R_1^2}, \ldots, \frac{\partial^2 V}{\partial R_q^2} \right].$$

- For multiple projection and discounting yield curves, sensitivities are calculated for each curve individually.

Par rate Delta and Gamma are sensitivity w.r.t. changes in market rates (2/2)

Parallel Delta and Gamma

Parallel Delta and Gamma represent sensitivities w.r.t. simultanous shifts of all market rates of a yield curve. With $\mathbf{1} = [1, \ldots 1]^\top$ we get

$$\bar{\Delta}_R = \mathbf{1}^\top \Delta_R = 1bp \cdot \sum_k \frac{\partial V}{\partial R_k} \approx \frac{V(\bar{R} + 1bp \cdot \mathbf{1}) - V(\bar{R} - 1bp \cdot \mathbf{1})}{2} \quad \text{and}$$

$$\bar{\Gamma}_R = \mathbf{1}^\top \Gamma_R = [1bp]^2 \cdot \sum_k \frac{\partial^2 V}{\partial R_k^2} \approx V(\bar{R} + 1bp \cdot \mathbf{1}) - 2V(\bar{R}) + V(\bar{R} - 1bp \cdot \mathbf{1}).$$

Vega is the sensitivity w.r.t. changes in market volatilities (1/2)

Bucketed ATM Normal Volatility Vega

Denote $\bar{\sigma} = \left[\sigma_N^{k,l}\right]$ the matrix of market-implied At-the-money normal volatilites for expiries $k = 1, \ldots, q$ and swap terms $l = 1, \ldots, r$.

Bucketed ATM Normal Volatility Vega of an instrument with model price $V = V(\bar{\sigma})$ is specified as

$$\text{Vega} = 1bp \cdot \left[\frac{\partial V}{\partial \sigma_N^{k,l}}\right]_{k=1,\ldots,q,\ l=1,\ldots,r}.$$

Vega is the sensitivity w.r.t. changes in market volatilities (2/2)

Parallel ATM Normal Volatility Vega

Parallel ATM Normal Volatility Vega represents sensitivity w.r.t. a parallel shift in the implied ATM swaption volatility surface. That is

$$\begin{aligned}
\overline{\text{Vega}} &= 1bp \cdot \mathbf{1}^\top [\text{Vega}]\, \mathbf{1} \\
&= 1bp \cdot \sum_{k,l} \frac{\partial V}{\partial \sigma_N^{k,l}} \\
&\approx \frac{V(\bar{\sigma} + 1bp \cdot \mathbf{1}\,\mathbf{1}^\top) - V(\bar{\sigma} - 1bp \cdot \mathbf{1}\,\mathbf{1}^\top)}{2}.
\end{aligned}$$

- Volatility smile sensitivities are often specified in terms of Vanilla model parameter sensitivities.
- For example, in SABR model, we can calculate sensitivities with respect to α, β, ρ and ν.

Outline

Crutial part of sensitivity calculation is evaluation or approximation of partial derivatives

Consider again general pricing function $V = V(p)$ in terms of a scalar parameter p. Assume differentiability of V w.r.t. p and sensitivity

$$\Delta V = \frac{dV(p)}{dp} \cdot \Delta p.$$

Finite Difference Approximation

Finite difference approximation with step size h is

$$\frac{dV(p)}{dp} \approx \frac{V(p+h) - V(p)}{h} \approx \frac{V(p) - V(p-h)}{h} \quad \text{(one-sided), or}$$

$$\frac{dV(p)}{dp} \approx \frac{V(p+h) - V(p-h)}{2h} \quad \text{(two-sided).}$$

- Simple to implement and calculate; only pricing function evaluation.
- Typically used for black-box pricing functions.

We do a case study for European swaption Vega I

Recall pricing function

$$V^{\text{Swpt}} = Ann(t) \cdot \text{Bachelier}\left(S(t), K, \sigma\sqrt{T-t}, \phi\right)$$

with

$$\text{Bachelier}(F, K, \nu, \phi) = \nu \cdot \left[\Phi(h) \cdot h + \Phi'(h)\right], \quad h = \frac{\phi\,[F-K]}{\nu}.$$

First, analyse Bachelier formula. We get

$$\begin{aligned}
\frac{d}{d\nu}\text{Bachelier}(\nu) &= \frac{\text{Bachelier}(\nu)}{\nu} + \nu\left[\left(\Phi'(h)\,h + \Phi(h)\right)\frac{dh}{d\nu} - \Phi'(h)\,h\frac{dh}{d\nu}\right] \\
&= \frac{\text{Bachelier}(\nu)}{\nu} + \nu\Phi(h)\,\frac{dh}{d\nu}.
\end{aligned}$$

With $\frac{dh}{d\nu} = -\frac{h}{\nu}$ follows

$$\frac{d}{d\nu}\text{Bachelier}(\nu) = \Phi(h) \cdot h + \Phi'(h) - \Phi(h) \cdot h = \Phi'(h).$$

We do a case study for European swaption Vega II

Moreover, second derivative (Volga) becomes

$$\frac{d^2}{d\nu^2}\text{Bachelier}\left(\nu\right) = -h\Phi'\left(h\right)\frac{dh}{d\nu} = \frac{h^2}{\nu}\Phi'\left(h\right).$$

This gives for ATM options with $h = 0$ that

- Volga $\frac{d^2}{d\nu^2}\text{Bachelier}\left(\nu\right) = 0$.
- ATM option price is approximately linear in volatility ν.

Differentiating once again yields (we skip details)

$$\frac{d^3}{d\nu^3}\text{Bachelier}\left(\nu\right) = \left(h^2 - 3\right)\frac{h^2}{\nu^2}\Phi'\left(h\right).$$

It turns out that Volga has a maximum at moneyness

$$h = \pm\sqrt{3}.$$

We do a case study for European swaption Vega III

Swaption Vega becomes

$$\frac{d}{d\sigma} V^{\mathrm{Swpt}} = An(t) \cdot \frac{d}{d\nu} \mathrm{Bachelier}\,(\nu) \cdot \sqrt{T - t}.$$

Test case

- Rates flat at 5%, implied normal volatilities flat at $100bp$.
- 10y into 10y European payer swaption (call on swap rate).
- Strike at $5\% + 100bp \cdot \sqrt{10y} \cdot \sqrt{3} = 10.48\%$ (maximizing Volga).

What is the problem with finite difference approximation? I

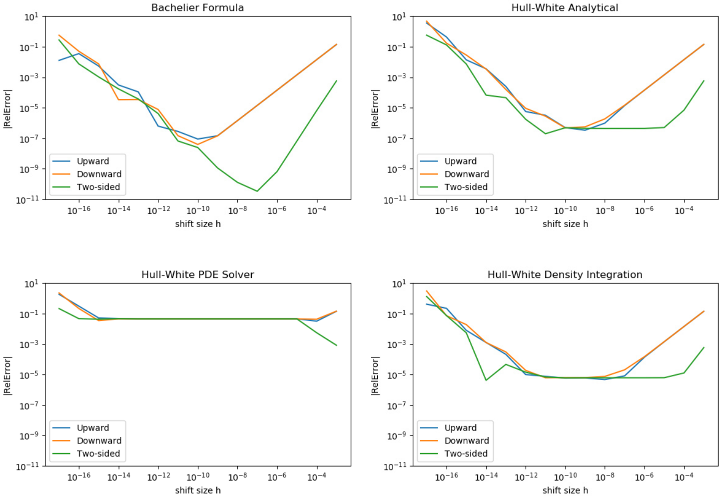
- There is a non-trivial trade-off between convergence and numerical accuracy.
- We have analytical Vega formula from Bachelier formula and implied normal volatility

$$\text{Vega} = An(t) \cdot \Phi'(h) \cdot \sqrt{T - t}.$$

- Compare one-sided (upward and downward) and two-sided finite difference approximation Vega_{FD} using
  - Bachelier formula,
  - Analytical Hull-White coupon bond option formula,
  - Hull-White model via PDE solver (Crank-Nicolson, 101 grid points, 3 stdDevs wide, 1m time stepping),
  - Hull-White model via density integration (C^2-spline exact with break-even point, 101 grid points, 5 stdDevs wide).
- Compare absolute relative error (for all finite difference approximations)

$$|\text{RelErr}| = \left| \frac{\text{Vega}_{FD}}{\text{Vega}} - 1 \right|$$

What is the problem with finite difference approximation? II



Optimal choice of FD step size h is very problem-specific and depends on discretisation of numerical method.

Outline

Derivative pricing usually involves model calibration (1/2)

Consider swap pricing function V^{Swap} as a function of yield curve model parameters z, i.e.

$$V^{\text{Swap}} = V^{\text{Swap}}(z).$$

Model parameters z are itself derived from market quotes R for par swaps and FRAs. That is

$$z = z(R).$$

This gives mapping

$$R \mapsto z \mapsto V^{\text{Swap}} = V^{\text{Swap}}\left(z(R)\right).$$

Interest rate Delta becomes

$$\Delta_R = 1bp \cdot \underbrace{\frac{dV^{\text{Swap}}}{dz}\left(z(R)\right)}_{\text{Pricing}} \cdot \underbrace{\frac{dz}{dR}\left(R\right)}_{\text{Calibration}}.$$

Derivative pricing usually involves model calibration (2/2)

$$\Delta_R = 1bp \cdot \underbrace{\frac{dV^{\text{Swap}}}{dz}\left(z(R)\right)}_{\text{Pricing}} \cdot \underbrace{\frac{dz}{dR}\left(R\right)}_{\text{Calibration}}.$$

- Suppose a large portfolio of swaps:
  - Calibration Jacobian $\frac{dz(R)}{dR}$ is the same for all swaps in portfolio.
  - Save computational effort by pre-calculating and storing Jacobian.
- Brute-force finite difference approximation of Jacobian may become inaccurate due to numerical scheme for calibration/optimisation.

Can we calculate calibration Jacobian more efficiently?

Theorem (Implicit Function Theorem)

Let $\mathcal{H} : \mathbb{R}^q \times \mathbb{R}^r \to \mathbb{R}^q$ be a continuously differentiable function with $\mathcal{H}(\bar{z}, \bar{R}) = 0$ for some pair $(\bar{z}, \bar{R})$. If the Jacobian

$$J_z = \frac{d\mathcal{H}}{dz}(\bar{z}, \bar{R})$$

is invertible, then there exists an open domain $\mathcal{U} \subset \mathbb{R}^r$ with $\bar{R} \in \mathcal{U}$ and a continuously differentiable function $g : \mathcal{U} \to \mathbb{R}^q$ with

$$\mathcal{H}\left(g(R), R\right) = 0 \quad \forall R \in \mathcal{U}.$$

Moreover, we get for the Jacobian of g that

$$\frac{dg(R)}{dR} = -\left[\frac{d\mathcal{H}}{dz}(g(R), R)\right]^{-1}\left[\frac{d\mathcal{H}}{dR}(g(R), R)\right].$$

Proof.

See Analysis. □

How does Implicit Function Theorem help for sensitivity calculation? (1/4)

- Consider $\mathcal{H}(z, R)$ the q-dimensional objective function of yield curve calibration problem:
  - $z = [z_1, \ldots, z_q]^\top$ yield curve parameters (e.g. zero rates or forward rates),
  - $R = [R_1, \ldots, R_q]^\top$ market quotes (par rates) for swaps and FRAs,
  - use same number of market quotes as model parameters, i.e. $r = q$.
- Reformulate calibration helpers slightly such that

$$\mathcal{H}_k(z, R) = \text{ModelRate}_k(z) - R_k.$$

- For example, for swap rate helpers, model-implied par swap rate becomes

$$\text{ModelRate}_k(z) = \frac{\sum_{j=1}^{m_k} L^\delta(0, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) \cdot \tilde{\tau}_j \cdot P(t, \tilde{T}_j)}{\sum_{i=1}^{n_k} \tau_i \cdot P(0, T_i)}.$$

How does Implicit Function Theorem help for sensitivity calculation? (2/4)

Suppose pair $(\bar{z}, \bar{R})$ solves calibration problem $\mathcal{H}(\bar{z}, \bar{R}) = 0$ and $\frac{d\mathcal{H}}{dz}(\bar{z}, \bar{R})$ is invertible.
Then, by Implicit Function Theorem, there exists a function

$$z = z(R)$$

in a vicinity of $\bar{R}$ and

$$\frac{dz}{dR}(R) = -\left[\frac{d\mathcal{H}}{dz}(g(R), R)\right]^{-1}\left[\frac{d\mathcal{H}}{dR}(g(R), R)\right].$$

How does Implicit Function Theorem help for sensitivity calculation? (3/4)

$$\frac{dz}{dR}(R) = -\left[\frac{d\mathcal{H}}{dz}(g(R),R)\right]^{-1}\left[\frac{d\mathcal{H}}{dR}(g(R),R)\right].$$

From reformulated calibration helpers we get

$$\frac{d\mathcal{H}}{dz}(g(R),R) = \begin{bmatrix} \frac{d}{dz}\text{ModelRate}_1(z) \\ \vdots \\ \frac{d}{dz}\text{ModelRate}_q(z) \end{bmatrix}, \quad \text{and}$$

$$\frac{d\mathcal{H}}{dR}(g(R),R) = \begin{bmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{bmatrix}.$$

Consequently

$$\frac{dz}{dR}(R) = \left[\frac{d\mathcal{H}}{dz}(g(R),R)\right]^{-1} = \begin{bmatrix} \frac{d}{dz}\text{ModelRate}_1(z) \\ \vdots \\ \frac{d}{dz}\text{ModelRate}_q(z) \end{bmatrix}^{-1}.$$

How does Implicit Function Theorem help for sensitivity calculation? (4/4)

We get Jacobian method for risk calculation

$$\Delta_R = 1bp \cdot \underbrace{\frac{dV^{\text{Swap}}}{dz}(z(R))}_{\text{Pricing}} \cdot \underbrace{\begin{bmatrix} \frac{d}{dz}\text{ModelRate}_1(z) \\ \vdots \\ \frac{d}{dz}\text{ModelRate}_q(z) \end{bmatrix}^{-1}}_{\text{Calibration}}.$$

- Requires only sensitivities w.r.t. model parameters.
- Reference market intruments/rates R_k can also be chosen independent of original calibration problem.
- Calibration Jacobian and matrix inversion can be pre-computed and stored.

We can also adapt Jacobian method to Vega calculation (1/3)

Bermudan swaption is determined via mapping

$$\underbrace{\left[\sigma_N^1, \ldots \sigma_N^{\bar{k}}\right]}_{\text{market-impl. normal vols}} \mapsto \underbrace{\left[\sigma^1, \ldots \sigma^{\bar{k}}\right]}_{\text{HW short rate vols}} \mapsto V^{\text{Berm}}.$$

Assign volatility calibration helpers

$$\mathcal{H}_k\left(\sigma, \sigma_N\right) = \underbrace{V_k^{\text{CBO}}(\sigma)}_{\text{Model}[\sigma]} - \underbrace{V_k^{\text{Swpt}}(\sigma_N^k)}_{\text{Market}\left(\sigma_N^k\right)}.$$

- $V_k^{\text{CBO}}(\sigma)$ Hull-White model price of kth co-terminal European swaption represented as coupon bond option.
- $V_k^{\text{Swpt}}(\sigma_N^k)$ Bachelier formula to calculate market price for kth co-terminal European swaption from given normal volatility σ_N^k.

We can also adapt Jacobian method to Vega calculation (2/3)

Implicit Function Theorem yields

$$\frac{d\sigma}{d\sigma_N} = -\left[\frac{d\mathcal{H}}{d\sigma}\left(\sigma\left(\sigma_N\right), \sigma_N\right)\right]^{-1}\left[\frac{d\mathcal{H}}{d\sigma_N}\left(\sigma\left(\sigma_N\right), \sigma_N\right)\right]$$

$$= \left[\frac{d}{d\sigma}\text{Model}[\sigma]\right]^{-1}\begin{bmatrix} \frac{d}{d\sigma_N} V_1^{\text{Swpt}}(\sigma_N^1) & & \\ & \ddots & \\ & & \frac{d}{d\sigma_N} V_{\bar{k}}^{\text{Swpt}}(\sigma_N^{\bar{k}}) \end{bmatrix}.$$

- $\frac{d}{d\sigma}\text{Model}[\sigma]$ are Hull-White model Vega(s) of co-terminal European swaptions.
- $\frac{d}{d\sigma_N} V_k^{\text{Swpt}}(\sigma_N^k)$ are Bachelier or market Vega(s) of co-terminal European swaptions.

We can also adapt Jacobian method to Vega calculation (3/3)

Bermudan Vega becomes

$$\frac{d}{d\sigma_N} V^{\text{Berm}} = \frac{d}{d\sigma} V^{\text{Berm}} \cdot \left[\frac{d}{d\sigma} \text{Model}[\sigma] \right]^{-1} \cdot \frac{d}{d\sigma_N} \text{Market}\left(\sigma_N^k\right).$$

Outline

Introduction to Sensitivity Calculation

Finite Difference Approximation for Sensitivities

Differentiation and Calibration

A brief Introduction to Algorithmic Differentiation

What is the idea behind Algorithmic Differentiation (AD)

- AD covers principles and techniques to augment computer models or programs.
- Calculate sensitivities of output variables with respect to inputs of a model.
- Compute numerical values rather than symbolic expressions.
- Sensitivities are exact up to machine precision (no rounding/cancellation errors as in FD).
- Apply chain rule of differentiation to operations like +, *, and intrinsic functions like `exp(.)`.

Functions are represented as Evaluation Procedures consisting of a sequence of elementary operations

Example: Black Formula

$$\text{Black}(\cdot) = \omega\left[F\Phi(\omega d_1) - K\Phi(\omega d_2)\right]$$

with $d_{1,2} = \frac{\log(F/K)}{\sigma\sqrt{\tau}} \pm \frac{\sigma\sqrt{\tau}}{2}$

- Inputs F, K, σ, τ
- Discrete parameter $\omega \in \{-1, 1\}$
- Output Black$(\cdot)$

$$
\begin{array}{lcllcl}
v_{-3} & = & x_1 = F & & \\
v_{-2} & = & x_2 = K & & \\
v_{-1} & = & x_3 = \sigma & & \\
v_0 & = & x_4 = \tau & & \\
\hline
v_1 & = & v_{-3}/v_{-2} & \equiv & f_1(v_{-3}, v_{-2}) \\
v_2 & = & \log(v_1) & \equiv & f_2(v_1) \\
v_3 & = & \sqrt{v_0} & \equiv & f_3(v_0) \\
v_4 & = & v_{-1} \cdot v_3 & \equiv & f_4(v_{-1}, v_3) \\
v_5 & = & v_2/v_4 & \equiv & f_5(v_2, v_4) \\
v_6 & = & 0.5 \cdot v_4 & \equiv & f_6(v_4) \\
v_7 & = & v_5 + v_6 & \equiv & f_7(v_5, v_6) \\
v_8 & = & v_7 - v_4 & \equiv & f_8(v_7, v_4) \\
v_9 & = & \omega \cdot v_7 & \equiv & f_9(v_7) \\
v_{10} & = & \omega \cdot v_8 & \equiv & f_{10}(v_8) \\
v_{11} & = & \Phi(v_9) & \equiv & f_{11}(v_9) \\
v_{12} & = & \Phi(v_{10}) & \equiv & f_{12}(v_{10}) \\
v_{13} & = & v_{-3} \cdot v_{11} & \equiv & f_{13}(v_{-3}, v_{11}) \\
v_{14} & = & v_{-2} \cdot v_{12} & \equiv & f_{14}(v_{-2}, v_{12}) \\
v_{15} & = & v_{13} - v_{14} & \equiv & f_{15}(v_{13}, v_{14}) \\
v_{16} & = & \omega \cdot v_{15} & \equiv & f_{16}(v_{15}) \\
\hline
y_1 & = & v_{16} & & \\
\end{array}
$$

Alternative representation is Directed Acyclic Graph (DAG)

$$
\begin{array}{lclcl}
v_{-3} & = & x_1 = F & & \\
v_{-2} & = & x_2 = K & & \\
v_{-1} & = & x_3 = \sigma & & \\
v_0 & = & x_4 = \tau & & \\
\hline
v_1 & = & v_{-3}/v_{-2} & \equiv & f_1(v_{-3}, v_{-2}) \\
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v_3 & = & \sqrt{v_0} & \equiv & f_3(v_0) \\
v_4 & = & v_{-1} \cdot v_3 & \equiv & f_4(v_{-1}, v_3) \\
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v_8 & = & v_7 - v_4 & \equiv & f_8(v_7, v_4) \\
v_9 & = & \omega \cdot v_7 & \equiv & f_9(v_7) \\
v_{10} & = & \omega \cdot v_8 & \equiv & f_{10}(v_8) \\
v_{11} & = & \Phi(v_9) & \equiv & f_{11}(v_9) \\
v_{12} & = & \Phi(v_{10}) & \equiv & f_{12}(v_{10}) \\
v_{13} & = & v_{-3} \cdot v_{11} & \equiv & f_{13}(v_{-3}, v_{11}) \\
v_{14} & = & v_{-2} \cdot v_{12} & \equiv & f_{14}(v_{-2}, v_{12}) \\
v_{15} & = & v_{13} - v_{14} & \equiv & f_{15}(v_{13}, v_{14}) \\
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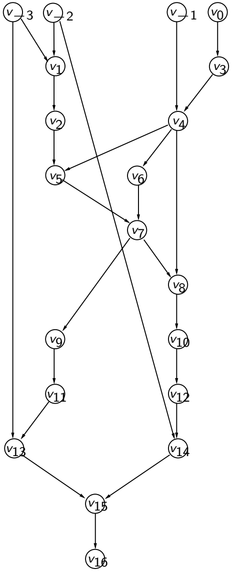

Evaluation Procedure can be formalized to make it more tractable

Definition (Evaluation Procedure)

Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ and $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{m_i}$. The relation $j \prec i$ denotes that $v_i \in \mathbb{R}$ depends directly on $v_j \in \mathbb{R}$. If for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ with $y = F(x)$ holds that

$$\begin{array}{rcll} v_{i-n} & = & x_i & i = 1, \ldots, n \\ v_i & = & f_i(v_j)_{j \prec i} & i = 1, \ldots, l \\ y_{m-i} & = & v_{l-i} & i = m-1, \ldots, 0, \end{array}$$

then we call this sequence of operations an evaluation procedure of F with elementary operations f_i. We assume differentiability of all elementary operations f_i $(i = 1, \ldots, l)$. Then the resulting function F is also differentiable.

- Abbreviate $u_i = (v_j)_{j \prec i} \in \mathbb{R}^{n_i}$ the collection of arguments of the operation f_i.
- Then we may also write
$$v_i = f_i(u_i).$$

Forward mode of AD calculates tangents (1/2)

- In addition to function evaluation $v_i = f_i(u_i)$ evaluate derivative

$$\dot{v}_i \; = \; \sum_{j \prec i} \frac{\partial}{\partial v_j} f_i(u_i) \cdot \dot{v}_j .$$

Forward Mode or Tangent Mode of AD

Use abbreviations $\dot{u}_i = (\dot{v}_j)_{j \prec i}$ and $\dot{f}_i(u_i, \dot{u}_i) = f_i'(u_i) \cdot \dot{u}_i$. The Forward Mode of AD is the augmented evaluation procedure

$$\begin{array}{rcll} [v_{i-n}, \dot{v}_{i-n}] & = & [x_i, \dot{x}_i] & i = 1, \ldots, n \\ [v_i, \dot{v}_i] & = & \left[f_i(u_i), \dot{f}_i(u_i, \dot{u}_i)\right] & i = 1, \ldots, l \\ [y_{m-i}, \dot{y}_{m-i}] & = & [v_{l-i}, \dot{v}_{l-i}] & i = m-1, \ldots, 0. \end{array}$$

Here, the initializing derivative values $\dot{x}_{i-n}$ for $i = 1 \ldots n$ are given and determine the direction of the tangent.

Forward mode of AD calculates tangents (2/2)

- With $\dot{x} = (\dot{x}_i) \in \mathbb{R}^n$ and $\dot{y} = (\dot{y}_i) \in \mathbb{R}^m$, the forward mode of AD evaluates

$$\dot{y} \; = \; F'(x)\dot{x}.$$

- Computational effort is approx. 2.5 function evaluations of F.

Black formula Forward Mode evaluation procedure...

$$
\begin{array}{lcl|lcl}
v_{-3} & = & x_1 = F & \dot{v}_{-3} & = & 0 \\
v_{-2} & = & x_2 = K & \dot{v}_{-2} & = & 0 \\
v_{-1} & = & x_3 = \sigma & \dot{v}_{-1} & = & 1 \\
v_0 & = & x_4 = \tau & \dot{v}_0 & = & 0 \\
\hline
v_1 & = & v_{-3}/v_{-2} & \dot{v}_1 & = & \dot{v}_{-3}/v_{-2} - v_1 \cdot \dot{v}_{-2}/v_{-2} \\
v_2 & = & \log(v_1) & \dot{v}_2 & = & \dot{v}_1/v_1 \\
v_3 & = & \sqrt{v_0} & \dot{v}_3 & = & 0.5 \cdot \dot{v}_0/v_3 \\
v_4 & = & v_{-1} \cdot v_3 & \dot{v}_4 & = & \dot{v}_{-1} \cdot v_3 + v_{-1} \cdot \dot{v}_3 \\
v_5 & = & v_2/v_4 & \dot{v}_5 & = & \dot{v}_2/v_4 - v_5 \cdot \dot{v}_4/v_4 \\
v_6 & = & 0.5 \cdot v_4 & \dot{v}_6 & = & 0.5 \cdot \dot{v}_4 \\
v_7 & = & v_5 + v_6 & \dot{v}_7 & = & \dot{v}_5 + \dot{v}_6 \\
v_8 & = & v_7 - v_4 & \dot{v}_8 & = & \dot{v}_7 - \dot{v}_4 \\
v_9 & = & \omega \cdot v_7 & \dot{v}_9 & = & \omega \cdot \dot{v}_7 \\
v_{10} & = & \omega \cdot v_8 & \dot{v}_{10} & = & \omega \cdot \dot{v}_8 \\
v_{11} & = & \Phi(v_9) & \dot{v}_{11} & = & \phi(v_9) \cdot \dot{v}_9 \\
v_{12} & = & \Phi(v_{10}) & \dot{v}_{12} & = & \phi(v_{10}) \cdot \dot{v}_{10} \\
v_{13} & = & v_{-3} \cdot v_{11} & \dot{v}_{13} & = & \dot{v}_{-3} \cdot v_{11} + v_{-3} \cdot \dot{v}_{11} \\
v_{14} & = & v_{-2} \cdot v_{12} & \dot{v}_{14} & = & \dot{v}_{-2} \cdot v_{12} + v_{-2} \cdot \dot{v}_{12} \\
v_{15} & = & v_{13} - v_{14} & \dot{v}_{15} & = & \dot{v}_{13} - \dot{v}_{14} \\
v_{16} & = & \omega \cdot v_{15} & \dot{v}_{16} & = & \omega \cdot \dot{v}_{15} \\
\hline
y_1 & = & v_{16} & \dot{y}_1 & = & \dot{v}_{16}
\end{array}
$$

Reverse Mode of AD calculates adjoints (1/3)

- Forward Mode calculates derivatives and applies chain rule in the same order as function evaluation.

- Reverse Mode of AD applies chain rule in reverse order of function evaluation.

- Define auxiliary derivative values $\bar{v}_j$ and assume initialisation $\bar{v}_j = 0$ before reverse mode evaluation.

- For each elementary operation f_i and all intermediate variables v_j with $j \prec i$, evaluate

$$\bar{v}_j \mathrel{+}= \bar{v}_i \cdot \frac{\partial}{\partial v_j} f_i(u_i).$$

- In other words, for each arguments of f_i the partial derivative is derived.

Reverse Mode of AD calculates adjoints (2/3)

Reverse Mode or Adjoint Mode of AD

Denoting $\bar{u}_i = (\bar{v}_j)_{j \prec i} \in \mathbb{R}^{n_i}$ and $\bar{f}_i(u_i, \bar{v}_i) = \bar{v}_i \cdot f_i'(u_i)$, the *incremental reverse mode of AD* is given by the evaluation procedure

$$
\begin{array}{rcll}
v_{i-n} & = & x_i & i = 1, \ldots, n \\
v_i & = & f_i(v_j)_{j \prec i} & i = 1, \ldots, l \\
y_{m-i} & = & v_{l-i} & i = m-1, \ldots, 0 \\
\hline
\bar{v}_i & = & \bar{y}_i & i = 0, \ldots, m-1 \\
\bar{u}_i & += & \bar{f}_i(u_i, \bar{v}_i) & i = l, \ldots, 1 \\
\bar{x}_i & = & \bar{v}_i & i = n, \ldots, 1.
\end{array}
$$

Here, all intermediate variables v_i are assigned only once. The initializing values $\bar{y}_i$ are given and represent a weighting of the dependent variables y_i.

Reverse Mode of AD calculates adjoints (3/3)

- Vector $\bar{y} = (\bar{y}_i)$ can also be interpreted as normal vector of a hyperplane in the range of F.
- With $\bar{y} = (\bar{y}_i)$ and $\bar{x} = (\bar{x}_i)$, reverse mode of AD yields

$$\bar{x}^T \;=\; \nabla \left[\bar{y}^T F(x)\right] \;=\; \bar{y}^T F'(x).$$

- Computational effort is approx. 4 function evaluations of F.

Black formula Reverse Mode evaluation procedure ... I

$$
\begin{array}{l}
v_{-3} = x_1 = F \\
v_{-2} = x_2 = K \\
v_{-1} = x_3 = \sigma \\
v_0 = x_4 = \tau \\
\hline
v_1 = v_{-3}/v_{-2} \\
v_2 = \log(v_1) \\
v_3 = \sqrt{v_0} \\
v_4 = v_{-1} \cdot v_3 \\
v_5 = v_2/v_4 \\
v_6 = 0.5 \cdot v_4 \\
v_7 = v_5 + v_6 \\
v_8 = v_7 - v_4 \\
v_9 = \omega \cdot v_7 \\
v_{10} = \omega \cdot v_8 \\
v_{11} = \Phi(v_9) \\
v_{12} = \Phi(v_{10}) \\
v_{13} = v_{-3} \cdot v_{11} \\
v_{14} = v_{-2} \cdot v_{12} \\
v_{15} = v_{13} - v_{14} \\
v_{16} = \omega \cdot v_{15} \\
\hline
y_1 = v_{16} \\
\bar{v}_{16} = \bar{y}_1 = 1 \\
\hline
\vdots
\end{array}
$$

Black formula Reverse Mode evaluation procedure ... II

$$\vdots$$

$$y_1 = v_{16}$$
$$\bar{v}_{16} = \bar{y}_1 = 1$$

$$\bar{v}_{15} \mathrel{+}= \omega \cdot \bar{v}_{16}$$
$$\bar{v}_{13} \mathrel{+}= \bar{v}_{15}; \quad \bar{v}_{14} \mathrel{+}= (-1) \cdot \bar{v}_{15}$$
$$\bar{v}_{-2} \mathrel{+}= v_{12} \cdot \bar{v}_{14}; \quad \bar{v}_{12} \mathrel{+}= v_{-2} \cdot \bar{v}_{14}$$
$$\bar{v}_{-3} \mathrel{+}= v_{11} \cdot \bar{v}_{13}: \quad \bar{v}_{11} \mathrel{+}= v_{-3} \cdot \bar{v}_{13}$$
$$\bar{v}_{10} \mathrel{+}= \phi(v_{10}) \cdot \bar{v}_{12}$$
$$\bar{v}_9 \mathrel{+}= \phi(v_9) \cdot \bar{v}_{11}$$
$$\bar{v}_8 \mathrel{+}= \omega \cdot \bar{v}_{10}$$
$$\bar{v}_7 \mathrel{+}= \omega \cdot \bar{v}_9$$
$$\bar{v}_7 \mathrel{+}= \bar{v}_8; \quad \bar{v}_4 \mathrel{+}= (-1) \cdot \bar{v}_8$$
$$\bar{v}_5 \mathrel{+}= \bar{v}_7; \quad \bar{v}_6 \mathrel{+}= \bar{v}_7$$
$$\bar{v}_4 \mathrel{+}= 0.5 \cdot \bar{v}_6$$
$$\bar{v}_2 \mathrel{+}= \bar{v}_5 / v_4; \quad \bar{v}_4 \mathrel{+}= (-1) \cdot v_5 \cdot \bar{v}_5 / v_4$$
$$\bar{v}_{-1} \mathrel{+}= v_3 \cdot \bar{v}_4; \quad \bar{v}_3 \mathrel{+}= v_{-1} \cdot \bar{v}_4$$
$$\bar{v}_0 \mathrel{+}= 0.5 \cdot \bar{v}_3 / v_3$$
$$\bar{v}_1 \mathrel{+}= \bar{v}_2 / v_1$$
$$\bar{v}_{-3} \mathrel{+}= \bar{v}_1 / v_{-2}; \quad \bar{v}_{-2} \mathrel{+}= (-1) \cdot v_1 \cdot \bar{v}_1 / v_{-2}$$

$$\bar{\tau} = \bar{x}_4 = \bar{v}_0$$
$$\bar{\sigma} = \bar{x}_3 = \bar{v}_{-1}$$
$$\bar{K} = \bar{x}_2 = \bar{v}_{-2}$$
$$\bar{F} = \bar{x}_1 = \bar{v}_{-3}$$

We summarise the properties of Forward and Reverse Mode

Forward Mode

$$\dot{y} = F'(x)\dot{x}$$

- Approx. 2.5 function evaluations.
- Computational effort independent of number of output variables (dimension of y).
- Chain rule in same order as computation.
- Memory consumption in order of function evaluation.

Reverse Mode

$$\bar{x}^T = \bar{y}^T F'(x)$$

- Approx. 4 function evaluations.
- Computational effort independent of number of input variables (dimension of x).
- Chain rule in reverse order of computation.
- Requires storage of all intermediate results (or re-computation).
- Memory consumption/management key challange for implementations.

- Computational effort can be improved by AD vector mode.
- Reverse Mode memory consumption can be managed via checkpointing techniques.

How is AD applied in practice?

- Typically, you don't want to differentiate all your source code by hand.
- Tools help augmenting existing programs for tangent and adjoint computations.

Source Code Transformation

- Applied to the model code in compiler fashion.
- Generate AD model as new source code.
- Original code may need to be adapted slightly to meet capabilities of AD tool.

Some example C++ tools:
ADIC2, dcc, TAPENADE

Operator Overloading

- provide new (active) data type.
- Overload all relevant operators/ functions with sensitivity aware arithmetic.
- AD model derived by changing intrinsic to active data type.

ADOL-C, dco/c++, ADMB/AUTODIF

- There are also tools for Python and other lamguages:

More details at `autodiff.org`

There is quite some literature on AD and its application in finance

Standard textbook on AD:

- A. Griewank and A. Walther. *Evaluating derivatives: principles and techniques of algorithmic differentiation - 2nd ed.* SIAM, 2008

Recent practitioner's textbook:

- U. Naumann. *The Art of Differentiating Computer Programs: An Introduction to Algorithmic Differentiation.* SIAM, 2012

One of the first and influencial papers for AD application in finance:

- M. Giles and P. Glasserman. Smoking adjoints: fast monte carlo greeks. *Risk*, January 2006

Part VIII

Wrap-up

Outline

What was this lecture about?

Interbank swap deal example

Trade details (fixed rate, notional, etc.)

Pays 3% on 100mm EUR

Start date: Oct 30, 2020

Date calculations

End date: Oct 30, 2040

Market conventions

(annually, 30/360 day count, modified following, Target calendar)

Stochastic interest rates

Pays 6-months Euribor floating rate on 100mm EUR

Start date: Oct 30, 2020

End date: Oct 30, 2040

(semi-annually, act/360 day count, modified following, Target calendar)

Optionalities

Bank A may decide to early terminate deal in 10, 11, 12,.. years

References I

F. Ametrano and M. Bianchetti.
Everything you always wanted to know about Multiple Interest Rate Curve Bootstrapping but were afraid to ask (April 2, 2013).
Available at SSRN: http://ssrn.com/abstract=2219548 or http://dx.doi.org/10.2139/ssrn.2219548, 2013.

L. Andersen and V. Piterbarg.
Interest rate modelling, volume I to III.
Atlantic Financial Press, 2010.

D. Bang.
Local-stochastic volatility for vanilla modeling.
`https://ssrn.com/abstract=3171877`, 2018.

M. Beinker and H. Plank.
New volatility conventions in negative interest environment.
d-fine Whitepaper, available at www.d-fine.de, December 2012.

D. Brigo and F. Mercurio.
Interest Rate Models - Theory and Practice.
Springer-Verlag, 2007.

References II

D. Duffy.
Finite Difference Methods in Financial Engineering.
Wiley Finance, 2006.

M. Fujii, Y. Shimada, and A. Takahashi.
Collateral posting and choice of collateral currency - implications for derivative pricing and risk management (may 8, 2010).
Available at SSRN: https://ssrn.com/abstract=1601866, May 2010.

M. Giles and P. Glasserman.
Smoking adjoints: fast monte carlo greeks.
Risk, January 2006.

P. Glasserman.
Monte Carlo Methods in Financial Engineering.
Springer, 2003.

A. Griewank and A. Walther.
Evaluating derivatives: principles and techniques of algorithmic differentiation - 2nd ed.
SIAM, 2008.

References III

P. Hagan, D. Kumar, A. Lesniewski, and D. Woodward.
Managing smile risk.
Wilmott magazine, September 2002.

P. Hagan and G. West.
Interpolation methods for curve construction.
Applied Mathematical Finance, 13(2):89–128, 2006.

M. Henrard.
Interest rate instruments and market conventions guide 2.0.
Open Gamma Quantitative Research, 2013.

M. Henrard.
A quant perspective on ibor fallback proposals.
https://ssrn.com/abstract=3226183, 2018.

M. Henrard.
A quant perspective on ibor fallback consultation results.
https://ssrn.com/abstract=3308766, 2019.

References IV

J. Hull and A. White.
Pricing interest-rate-derivative securities.
The Review of Financial Studies, 3:573–592, 1990.

Y. Iwashita.
Piecewise polynomial interpolations.
OpenGamma Quantitative Research, 2013.

A. Lyashenko and F. Mercurio.
Looking forward to backward-looking rates: A modeling framework for term rates replacing libor.
`https://ssrn.com/abstract=3330240`, 2019.

U. Naumann.
The Art of Differentiating Computer Programs: An Introduction to Algorithmic Differentiation.
SIAM, 2012.

V. Piterbarg.
Funding beyond discounting: collateral agreements and derivatives pricing.
Asia Risk, pages 97–102, February 2010.

References V

R. Rebonato.
Volatility and Correlation.
John Wiley & Sons, 2004.

S. Shreve.
Stochastic Calculus for Finance II - Continuous-Time Models.
Springer-Verlag, 2004.

Contact

Dr. Sebastian Schlenkrich
Office: RUD25, R 1.211
Mail: sebastian.schlenkrich@hu-berlin.de

d-fine GmbH
Mobile: +49-162-263-1525
Mail: sebastian.schlenkrich@d-fine.de