Dr. Sebastian Schlenkrich
Institut für Mathematik
Bereich Stochastik

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Interest Rate Modelling and Derivative Pricing, WS 2020/21

Exercise 5

1. **Digital option pricing**
A digital call option on a zero coupon bond is specified via expiry time T_E, maturity time T_M with $T_M \geq T_E$, strike price K and payoff

$$V(T_E) = \mathbb{1}_{\{P(T_E,T_M)\geq K\}}.$$

In practice a digital call option price may be approximated via so-called a call-spread

$$CS(t,K) = \frac{ZBO(t,K-\delta) - ZBO(t,K+\delta)}{2\delta}.$$

Here, $ZBO(t,K)$ is a zero coupon bond call option (same expiry/maturity) with strike K and δ is a small spread width.

   (a) Derive the analytical price $V(t) = V(t,K)$ $(t < T_E)$ of the digital option in a Hull White model with constant mean reversion a and constant short rate volatility σ. (Hint: Use the lognormal distribution property of $P(T_E,T_M)$ and follow ZBO option pricing formula derivation.)

   (b) Show that for $K = 0$ the option price becomes $V(t,0) = P(t,T_E)$.

   (c) Why is the calls spread $CS(t,K)$ an approximation to the digital option $V(t,K)$? Implement the digital option pricing formula from (a) in Python. Analyse the approximation accuracy of the call spread formula depending on the option parameters and the choice of the spread width δ. Illustrate and discuss your results. (Hint: Extend the class `HullWhiteModel` for a digital option pricing formula and use `HullWhiteModel.zeroBondOption(...)` to calculate the call spread.)

2. **Coupon bond put option pricing**
The payoff of a put option on a future zero coupon bond is given by

$$V^{\text{CBOPut}}(T_E) = \left[\left(K - \left(\sum_{i=1}^{n} C_i P(T_E,T_i)\right)\right)^+\right].$$

   (a) Show that the option price $V^{\text{CBOPut}}(t)$ in the Hull White model can be derived as a sum of zero coupon bond put options

$$V^{\text{CBOPut}}(t) = \sum_{i=1}^{n} C_i \cdot V_i^{\text{ZBOPut}}(t).$$

   What are the underlying assumptions? How are the zero coupon bond options $V_i^{\text{ZBOPut}}(t)$ specified, i.e. which strike, expiry, maturity, ...?

(b) Consider a general cash flow option

$$V(T_E) = \left[\left(\sum_{i=1}^{n} C_i P(T_E, T_i) \right)^+ \right]$$

with expiry time T_E, payment times T_i and arbitrary (positive and/or negative) cash flows C_i. Under which constraints/assumptions can we apply Jamshidian's trick to derive a pricing formula as a sum of zero coupon bond options? Consider a representation

$$V(t) = \sum_{i=1}^{n} C_i \cdot V_i^{\text{ZBO}}(t)$$

How are the individual zero coupon bond options specified, i.e. which strike, call or put options?

3. **Bond options and European swaptions**
Consider a call option on a coupon bond with the following properties:

- strike price of 10.000 EUR paid in $10y$,
- coupons of 450 EUR received annually in $11y$, $12y, \ldots, 20y$,
- notional of 10.000 EUR received in $20y$.

Further assume a single curve setting with Libor projection curves equal to discount curve.

(a) What are the properties (notional, expiry, start, maturity, strike) of a European swaption equivalent to above bond option? Why are both instruments equivalent?

(b) Set up the bond option in Python and calculate the Hull White model price via `HullWhiteModel.couponBondOption(...)`. Also set up the equivalent European swaption in Python and calculate the Hull White model price via `Swaption.npvHullWhite(...)`. Confirm that both prices match up for different yield curve, short rate volatility and mean reversion scenarios. (Note: You might need to slightly adjust conventions in the `Swaption` class to exactly match all cash flows. See also the `Swaption.bondOptionDetails()` method.

(c) Suppose the strike in the bond option is changed to 9.000 EUR. How does this affect the equivalent European swaption?