Mathematical Finance I

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Part V

Optimal Stopping and American Options

Outline

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We extend our concepts of options and payoffs

- D. Lamberton and B. Lapeyre. *Introduction to Stochastic Calculus Applied to Finance.*
  Chapman and Hall, 2008, sec. 2.
- H. Foellmer and A. Schied. *Stochastic Finance: An Introduction in Discrete Time.*
  de Gruyter, 2016, sec. 6.

Outline

We continue working in our multi-period model

- Consider discrete times $t \in \{0, 1, \ldots, T\}$.
- A filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\ldots,T}, \mathbb{P}\right)$.
- A risk-less asset with price process $\left(S_t^0\right)$ and $S_t^0 = (1+r)^t$.
- A risky asset with price prcess (S_t).

We will mostly work with discounted quantities, e.g. $S_t / (1+r)^t$.

We start with an American put option in our multi-period model

Consider model with model times $0, 1, \ldots T$, risky asset $(S_t)_{t=0,\ldots,T}$ and risk-free rate $r > -1$.

American put option with strike K is an option to sell the risky asset at price K at any time $t \in \{0, \ldots T\}$.

- Recall that a European put option can be exercised only at a single time T.
- The additional option to *choose the best time t* distinguishes American options from European options.

We will describe American options in terms of optimal stopping times.

We characterise the option from the buyer's point of view

At each time t the option buyer has two choices:

- Exercise the option. This gives immediate payoff

$$Z_t = (K - S_t) \quad \text{at } t.$$

- Hold the option. This leaves him with a derivative with hold value U_t.

Hold value U_t represents the price of the option if not exercised until t.

At final maturity T the hold value of the option equals the payoff

$$U_T = (K - S_T)^+ = \max\{Z_T, 0\}.$$

Assuming existance of a unique EMM allows a first characterisation of the option price

At $T-1$ the rational option buyer would only exercise the option if

$$Z_{T-1} = (K - S_{T-1}) > \mathbb{E}^* \left[\frac{U_T}{1+r} \,|\, \mathcal{F}_{T-1} \right].$$

This gives the hold value at $T-1$ via

$$U_{T-1} = \max \left\{ Z_{T-1}, \mathbb{E}^* \left[\frac{U_T}{1+r} \,|\, \mathcal{F}_{T-1} \right] \right\}.$$

Similarly, we can argue for all $t = 1, \ldots, T$ that the

$$U_{t-1} = \max \left\{ Z_{t-1}, \mathbb{E}^* \left[\frac{U_t}{1+r} \,|\, \mathcal{F}_{t-1} \right] \right\}.$$

Since there cannot be option exercise prior to $t = 0$ we have that the American put option price at $t = 0$ is U_0.

Recursion for hold value can be simplified in terms of discounted quantities

Set $\tilde{Z}_t = Z_t / (1+r)^t$ and $\tilde{U}_t = U_t / (1+r)^t$.

Then we have the terminal condition

$$\tilde{U}_T = \max\left\{\tilde{Z}_T, 0\right\}.$$

And the backward induction for the discounted hold value becomes

$$\tilde{U}_{t-1} = \frac{U_{t-1}}{(1+r)^{t-1}} = \max\left\{\tilde{Z}_{t-1}, \mathbb{E}^*\left[\tilde{U}_t \,|\, \mathcal{F}_{t-1}\right]\right\}.$$

This gives a general pricing method for American options:

1. Calculate the terminal condition $\tilde{U}_T$
2. Iterate from $T-1$ to 0 and calculate

$$\tilde{U}_{t-1} = \max\left\{\tilde{Z}_{t-1}, \mathbb{E}^*\left[\tilde{U}_t \,|\, \mathcal{F}_{t-1}\right]\right\}.$$

In principle, we could stop at the backward induction algorithm

- Backward induction algorithm gives a general procedure to price American options.
- This algorithm is implemented in most numerical methods for American option pricing.

So, what else is there to say?

- Can the option seller hedge its risks from selling the American option?
- When should the option buyer actually exercise the option?
- How can we characterise arbitrage-free prices in incomplete markets?

We give a first a first characterisation of the hold value (1/.)

Recall, a super-martingale M_t is characterised by $M_t \geq \mathbb{E}\left[M_s \mid \mathcal{F}_t\right]$ for $0 \leq t \leq s \leq T$.

Loosely speaking, a supermartingale is *decreasing on average*.

Lemma 89

The sequence $\left(\tilde{U}_t\right)_t$ *is the smallest super-martingale that dominates* $\left(\tilde{Z}_t^+\right)_t$*, where* $\tilde{Z}_t^+ = \max\left\{\tilde{Z}_t.0\right\}$.

We give a first a first characterisation of the hold value (2/.)

Proof:
From

$$\tilde{U}_T = \max\left\{\tilde{Z}_T, 0\right\} = \tilde{Z}_T^+,$$
$$\tilde{U}_{t-1} = \max\left\{\tilde{Z}_{t-1}, \mathbb{E}^*\left[\tilde{U}_t \,|\, \mathcal{F}_{t-1}\right]\right\} = \max\left\{\tilde{Z}_{t-1}^+, \mathbb{E}^*\left[\tilde{U}_t \,|\, \mathcal{F}_{t-1}\right]\right\}$$

follows

- $\tilde{U}_{t-1} \geq \mathbb{E}^*\left[\tilde{U}_t \,|\, \mathcal{F}_{t-1}\right]$ $(t = 1, \ldots, T)$, thus $\left(\tilde{U}_t\right)_t$ is a super-martingal, and
- $\tilde{U}_t \geq \tilde{Z}_t^+$ $(t = 0, \ldots, T)$, thus $\left(\tilde{U}_t\right)_t$ dominates $\left(\tilde{Z}_t^+\right)_t$.

We give a first a first characterisation of the hold value (3/.)

It remains to show that $\left(\tilde{U}_t\right)_t$ is the smallest super-martingale.
Suppose $\left(\tilde{V}_t\right)_t$ is another super-martingale that dominates $\left(\tilde{Z}_t^+\right)_t$. Then

$$\tilde{V}_T \geq \tilde{Z}_T^+ = \tilde{U}_T.$$

Now, we proceed by induction. Assume $\tilde{V}_t \geq \tilde{U}_t$. Then

$$\tilde{V}_{t-1} \geq \mathbb{E}^* \left[\tilde{V}_t \,|\, \mathcal{F}_{t-1}\right] \geq \mathbb{E}^* \left[\tilde{U}_t \,|\, \mathcal{F}_{t-1}\right]$$

and

$$\tilde{V}_{t-1} \geq \tilde{Z}_{t-1}^+.$$

Thus

$$\tilde{V}_{t-1} \geq \max\left\{\tilde{Z}_{t-1}^+, \mathbb{E}^* \left[\tilde{U}_t \,|\, \mathcal{F}_{t-1}\right]\right\} = \tilde{U}_{t-1}.$$

This shows that $\left(\tilde{V}_t\right)_t$ cannot be smaller then $\left(\tilde{U}_t\right)_t$.

Outline

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Stopping times can be considered strategies to specify relevant events in a model

- The buyer of an American option can choose the exercise time.
- She will naturally choose an exercise time that maximizes her return. This gives an *optimal* exercise time.
- However, decision whether to exercise at time t can only be based on information available at time t.

Definition 90 (Stopping time)

A function $\tau : \Omega \to \{0, 1, \ldots, T\} \cup \{+\infty\}$ is called a stopping time if for all $t \in \{0, 1, \ldots, T\}$

$$\{\tau = t\} = \{\omega \in \Omega : \tau(\omega) = t\} \in \mathcal{F}_t.$$

Stopping times can alternatively be characterised by similar events

If τ is a stopping time then $\{\tau = t\} \in \mathcal{F}_t$ but also

$$\{\tau = t - 1\} \in \mathcal{F}_{t-1} \subseteq \mathcal{F}_t.$$

Thus

$$\{\tau \leq t\} = \{\tau = 0\} \cup \ldots \cup \{\tau = t\} \in \mathcal{F}_t.$$

Alternatively, suppose $\{\tau \leq t\} \in \mathcal{F}_t$ for all t. Then

$$\{\tau = t\} = \underbrace{\{\tau \leq t\}}_{\in \mathcal{F}_t} \setminus \underbrace{\{\tau \leq t - 1\}}_{\in \mathcal{F}_{t-1} \subseteq \mathcal{F}_t}.$$

The random variable τ is a stopping time if and only if for all t

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

We give some examples of stopping times

1. The costant function $\tau \equiv t^*$ is a stopping time. We have

$$\{\tau = t\} = \begin{cases} \Omega, & t = t^* \\ \emptyset & t \neq t^* \end{cases} \quad \text{and} \quad \{\{\emptyset\}, \{\Omega\}\} = \mathcal{F}_0 \subseteq \mathcal{F}_t.$$

2. The first barrier hit time of an adapted process $(Y_t)_t$ is a stopping time. Consider an (upper) barrier level B.

$$\tau(\omega) = \inf\{t \geq 0 \mid Y_t \geq B\}.$$

We have

$$\{\tau \leq t\} = \left\{\omega \in \Omega : \max_{0 \leq s \leq t} Y_s \geq B\right\} \in \mathcal{F}_t.$$

Note, for this example $\{\tau = +\infty\}$ is the non-trivial event

$$\{\tau = +\infty\} = \{\omega \in \Omega : Y_t < B \,\forall\, t = 0, \ldots, T\}.$$

A natural application of stopping times are stopped processes

Definitions 91 (Stopped process)

Consider a stochastic process X and a stopping time τ. We denote X^τ the stopped process and define

$$X_t^\tau(\omega) = \begin{cases} X_t(\omega), & t \leq \tau(\omega) \\ X_{\tau(\omega)}(\omega) & t > \tau(\omega) \end{cases}.$$

- Note that if X is an adapted process then X^τ is also an adapted process.

It turns out that martingales are stable with respect to stopping (1/.)

Theorem 92 ((Super-)Martingale property for stopped processes)

Let (X_t) be an adapted stochastic process and τ a stopping time. Then the following properties hold:

1. *The stopped process X^τ is adapted.*
2. *If X is a (super-)martingale then X^τ is also a (super-)martingale.*

Result is important from two aspects:

- Under risk-neutral measure the martingale property of discounted assets (and absence of arbitrage) is preserved if early exercise of options is allows.
- A martingale cannot be turned into a favorable game by using a clever stopping strategy.

It turns out that martingales are stable with respect to stopping (2/.)

Proof:

The stopped process can be written as

$$X_t^\tau = X_0 + \sum_{j=1}^{t} \mathbb{1}_{\{j \leq \tau\}} \left(X_j - X_{j-1} \right).$$

The condition $\{j \leq \tau\}$ can be re-written as

$$\{j \leq \tau\} = \{\tau < j\}^c = \{\tau \leq j-1\}^c \in \mathcal{F}_{j-1}.$$

Consequently, $\left(\mathbb{1}_{\{j \leq \tau\}}\right)_j$ is a predictable process.

This verifies that X^τ is adapted.

It turns out that martingales are stable with respect to stopping (3/.)

If, in addition, X is a martingale with respect to the filtration $(\mathcal{F}_t)$ then

$$\begin{aligned}
\mathbb{E}\left[X_t^\tau \mid \mathcal{F}_{t-1}\right] &= \mathbb{E}\left[X_0 + \sum_{j=1}^{t} \mathbb{1}_{\{j \leq \tau\}} \left(X_j - X_{j-1}\right) \mid \mathcal{F}_{t-1}\right] \\
&= X_0 + \sum_{j=1}^{t} \mathbb{E}\left[\mathbb{1}_{\{j \leq \tau\}} \left(X_j - X_{j-1}\right) \mid \mathcal{F}_{t-1}\right] \\
&= X_0 + \sum_{j=1}^{t-1} \mathbb{1}_{\{j \leq \tau\}} \left(X_j - X_{j-1}\right) \\
&\quad + \mathbb{1}_{\{j \leq \tau\}} \underbrace{\mathbb{E}\left[\left(X_t - X_{t-1}\right) \mid \mathcal{F}_{t-1}\right]}_{0} \\
&= X_{t-1}^\tau
\end{aligned}$$

Thus, (X_t^τ) is a martingale as well. Super-martingale property follows analogously.

Outline

We return to the hold value characterisation for American options

- We already demonstrated that the hold value of an American option is the smallest super-martingale that dominates the immediate payoff function.
- We analyse such minimal super-martingales in more detail.

Definition 93 (Snell envelope)

Let (Z_t) be an adapted process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. The associated Snell envelope (U_t) is the smallest $\mathbb{P}$-super-martingale that dominates Z. That is

$$U_t = \begin{cases} Z_T, & t = T \\ \max\left\{Z_t, \mathbb{E}\left[U_{t+1} \,|\, \mathcal{F}_t\right]\right\}, & t < T \end{cases}.$$

The martingale property of the hold value can be preserved if stopped properly (1/.)

Note that if $U_t > Z_t$ then (by construction) $U_t = \mathbb{E}\left[U_{t+1} \mid \mathcal{F}_t\right]$. This situation means that early exercise is *not* beneficial.

Can we use this property to construct a martingale?

Theorem 94

Let Z be an adapted process and U the associated snell envelope. The random variable

$$\tau := \inf\left\{t \in \{0, 1, \ldots, T\} \;:\; U_t = Z_t\right\}$$

is a stopping time. And the stopped process U^τ is a $\mathbb{P}$-martingale.

- Super-martingale property for U^τ follows from Thm. 92.
- Martingale property for U^τ is a stronger result.

The martingale property of the hold value can be preserves if stopped properly (2/.)

Proof:

The random variable τ is well defined because $U_T = Z_T$ for all $\omega \in \Omega$.

We show the stopping time property.

For $t = 0$ we have

$$\{\tau = 0\} = \{U_0 = Z_0\} \in \{\{\emptyset\}, \{\Omega\}\} = \mathcal{F}_0.$$

For $t > 0$ we use that U dominates Z and get

$$\{\tau = t\} = \underbrace{\{U_0 > Z_0\}}_{\in \mathcal{F}_0} \cap \ldots \cap \underbrace{\{U_{t-1} > Z_{t-1}\}}_{\in \mathcal{F}_{t-1}} \cap \underbrace{\{U_t = Z_t\}}_{\in \mathcal{F}_t} \in \mathcal{F}_t.$$

Thus, τ is a stopping time.

The martingale property of the hold value can be preserves if stopped properly (3/.)

Next, we prove the martingale property.

We re-use the representation of the stopped proces in terms of returns

$$U_t^\tau = U_0 + \sum_{j=1}^{t} \mathbb{1}_{\{j \leq \tau\}} \left(U_j - U_{j-1} \right).$$

This yields

$$\begin{aligned} U_{t+1}^\tau - U_t^\tau &= \mathbb{1}_{\{t+1 \leq \tau\}} \left(U_{t+1} - U_t \right) \\ &= \begin{cases} 0, & \tau < t+1 \\ \left(U_{t+1} - U_t \right), & \tau \geq t+1 \end{cases}. \\ &= \begin{cases} 0, & \tau < t+1 \\ \left(U_{t+1} - \mathbb{E}\left[U_{t+1} \,\middle|\, \mathcal{F}_t \right] \right), & \tau \geq t+1 \end{cases}. \end{aligned}$$

The martingale property of the hold value can be preserves if stopped properly (4/.)

Taking conditional expectation yields

$$\mathbb{E}\left[U_{t+1}^{\tau} - U_t^{\tau} \,|\, \mathcal{F}_t\right] = \mathbb{1}_{\{t+1\leq\tau\}}\mathbb{E}\left[\left(U_{t+1} - \mathbb{E}\left[U_{t+1} \,|\, \mathcal{F}_t\right]\right) \,|\, \mathcal{F}_t\right] = 0.$$

Consequently,

$$U_t^{\tau} = \mathbb{E}\left[U_{t+1}^{\tau} \,|\, \mathcal{F}_t\right]$$

which is the martingale property for the process U^{τ} and concludes the proof.

- Stopping time $\tau := \inf\left\{t \in \{0, 1, \ldots, T\} \;:\; U_t = Z_t\right\}$ represents the buyer's strategy to manage the option.
- Next, we will show that the buyer's stopping time is also optimal.

We show that the buyer's strategy maximizes expected payoffs (1/.)

Denote $\mathcal{T}_{t,T}$ the set of all stopping times taking values in

$$\{t, t+1, \dots T\}.$$

Theorem 95 (Optimal stopping for τ)

Let Z be an adapted process and U the associated snell envelope. The stopping time τ satisfies

$$U_0 = \mathbb{E}[Z_\tau] = \sup_{\tau' \in \mathcal{T}_{0,T}} \mathbb{E}[Z_{\tau'}].$$

- The proof combines the martingale property for U^τ and super-martingale property of the snell envelope U.

We show that the buyer's strategy maximizes expected payoffs (2/.)

Proof:

1. We have from martingale property

$$U_0 = U_0^{\tau} = \mathbb{E}\left[U_T^{\tau}\right] = \mathbb{E}\left[U_{\tau}\right] = \mathbb{E}\left[Z_{\tau}\right].$$

2. Consider another stopping time τ'. The process $U^{\tau'}$ is a super-martingale, see Thm. 92. Then

$$U_0 \geq \mathbb{E}\left[U_T^{\tau'}\right] = \mathbb{E}\left[U_{\tau'}\right] \geq \mathbb{E}\left[Z_{\tau'}\right].$$

   This shows the supremum property.

We introduce the concept of optimal stopping time (1/.)

Definition 96 (Optimal stopping time)

A stopping time τ^* is called optimal for the adapted process Z if

$$\mathbb{E}\left[Z_{\tau^*}\right] = \sup_{\tau' \in \mathcal{T}_{0,T}} \mathbb{E}\left[Z_{\tau'}\right].$$

- We already have that $\tau = \inf\left\{t \in \{0, 1, \ldots, T\} \;:\; U_t = Z_t\right\}$ is optimal.
- Next, we show that τ is also smallest optimal time.

We introduce the concept of optimal stopping time (2/.)

Theorem 97 (Optimal stopping time characterisation)

A stopping time τ^* is optimal if and only if

$$Z_{\tau^*} = U_{\tau^*} \quad \textit{and} \quad \left(U_t^{\tau^*}\right) \textit{ is a martingale.}$$

We introduce the concept of optimal stopping time (3/.)

Suppose τ^* is optimal. Then via Thm. 95,

$$U_0 = \sup_{\tau' \in \mathcal{T}_{0,T}} \mathbb{E}\left[Z_{\tau'}\right] = \mathbb{E}\left[Z_{\tau^*}\right] \leq \mathbb{E}\left[U_{\tau^*}\right].$$

In addition, the super-martingale property of U gives

$$\mathbb{E}\left[Z_{\tau^*}\right] = U_0 \geq \mathbb{E}\left[U_{\tau^*}\right].$$

Consequently

$$\mathbb{E}\left[Z_{\tau^*}\right] = \mathbb{E}\left[U_{\tau^*}\right].$$

Since also $U_{\tau^*} \geq Z_{\tau^*}$ we get $U_{\tau^*} = Z_{\tau^*}$.

We introduce the concept of optimal stopping time (4/.)

Next, we show that the stopped process is a martingale.

The super-martingale property for U yields

$$\mathbb{E}\left[U_{\tau^*}\right] = U_0 \geq \mathbb{E}\left[U_t^{\tau^*}\right] \geq \mathbb{E}\left[\mathbb{E}\left[U_{\tau^*} \,|\, \mathcal{F}_t\right]\right] = \mathbb{E}\left[U_{\tau^*}\right].$$

Consequently, with tower law we get

$$\mathbb{E}\left[U_t^{\tau^*}\right] = \mathbb{E}\left[U_{\tau^*}\right] = \mathbb{E}\left[\mathbb{E}\left[U_{\tau^*} \,|\, \mathcal{F}_t\right]\right].$$

With $U_t^{\tau^*} \geq \mathbb{E}\left[U_{\tau^*} \,|\, \mathcal{F}_t\right]$ follows now

$$U_t^{\tau^*} = \mathbb{E}\left[U_{\tau^*} \,|\, \mathcal{F}_t\right]$$

and we get the martingale property.

We introduce the concept of optimal stopping time (5/.)

For the converse implication assume $Z_{\tau^*} = U_{\tau^*}$ and U^{τ^*} is a martingale.

Then

$$U_0 = \mathbb{E}\left[U_{\tau^*}\right] = \mathbb{E}\left[Z_{\tau^*}\right].$$

The result follows then via Thm. 95.

This concludes the proof.

Outline

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We want to characterise the option from the seller's point of view who wants to hedge against all possible claims of the buyer

- So far, we looked at the American option from the buyer's point of view.
- Now, we turn to the seller's point of view.

We want to split the martingale part from the Snell envelope (1/.)

- We use the Doob decomposition of a super-martingale and apply it to the Snell envelope.

Theorem 98 (Doob decomposition for super-martingales)

Let (U_t) be a super-martingale. There exists a unique decomposition

$$U_t = M_t - A_t$$

such that

1. *(M_t) is a martingale and*
2. *(A_t) is a non-decreasing predictable process with $A_0 = 0$.*

We want to split the martingale part from the Snell envelope (2/.)

Proof:

We set $A_0 = 0$. Then $M_0 = U_0$.

Suppose there exists a decomposition with above properties, then we get

$$U_{t+1} - U_t = (M_{t+1} - M_t) - (A_{t+1} - A_t)\,.$$

With $\mathbb{E}\left[M_{t+1} - M_t \,|\, \mathcal{F}_t\right] = 0$ and A predictable we find that the process A is uniquely defined via

$$-(A_{t+1} - A_t) = \mathbb{E}\left[U_{t+1} \,|\, \mathcal{F}_t\right] - U_t. \tag{14}$$

From the super-martingal property of U follows $A_{t+1} - A_t \geq 0$.

With the process A specified we find that also M is uniquely defind via

$$M_t = U_t + A_t. \tag{15}$$

By construction, M is a martingal.

We want to split the martingale part from the Snell envelope (3/.)

We have uniqueness of M and A.

Existence follows from the fact that $A_0 = 0$, equations (14) and (15) fully specify processes with the desired properties.

This concludes the proof.

We characterise the largest optimal stopping time (1/.)

- We use the Doob decomposition to characterise the largest optimal stopping time.

Theorem 99 (Largest optimal stopping time)

Let Z be an adapted process, U the associated snell envelope with Doob decomposition $M - A$. The largest optimal stopping time $\hat{\tau}$ for Z is given by

$$\hat{\tau} = \begin{cases} T, & A_T = 0, \\ \inf\left\{t \in \{0, \dots, T-1\} : A_{t+1} > 0\right\} & A_T > 0 \end{cases}.$$

We characterise the largest optimal stopping time (2/.)

Proof:

First we use Thm. 97 to show that $\hat{\tau}$ is optimal.

Then, we show that $\hat{\tau}$ is indeed maximal.

From A predictable we may conclude that $\hat{\tau}$ is a stopping time.

Moreover, by construction of $\hat{\tau}$ we get $U^{\hat{\tau}} = M^{\hat{\tau}}$. Consequently, the stopped process $U^{\hat{\tau}}$ is a martingale.

It remains to show that

$$U_{\hat{\tau}} = Z_{\hat{\tau}}.$$

We characterise the largest optimal stopping time (3/.)

We write

$$U_{\hat{\tau}} = \sum_{j=0}^{T-1} \mathbb{1}_{\{\hat{\tau}=j\}} U_j + \mathbb{1}_{\{\hat{\tau}=T\}} U_T$$
$$= \sum_{j=0}^{T-1} \mathbb{1}_{\{\hat{\tau}=j\}} \max\left\{Z_j, \mathbb{E}\left[U_{j+1} \,|\, \mathcal{F}_j\right]\right\} + \mathbb{1}_{\{\hat{\tau}=T\}} U_T.$$

If $\hat{\tau}(\omega) = T$ then $U_T = Z_T$.

Now assume $\hat{\tau}(\omega) = j < T$, then

$$U_j = \max\left\{Z_j, \mathbb{E}\left[U_{j+1} \,|\, \mathcal{F}_j\right]\right\}.$$

We characterise the largest optimal stopping time (4/.)

We analyse the case $\hat{\tau} = j < T$ and $U_j = \max\{Z_j, \mathbb{E}[U_{j+1} \mid \mathcal{F}_j]\}$.

From construction of $\hat{\tau}$ we get

$$A_i = 0 \; (i \leq j) \quad \text{and} \quad A_{j+1} > 0.$$

With Doob decomposition we now get

$$\mathbb{E}[U_{j+1} \mid \mathcal{F}_j] = U_j - A_{j+1}.$$

Consequently, $U_j > \mathbb{E}[U_{j+1} \mid \mathcal{F}_j]$ and hence

$$U_j = \max\{Z_j, \mathbb{E}[U_{j+1} \mid \mathcal{F}_j]\} = Z_j.$$

This gives

$$U_{\hat{\tau}} = Z_{\hat{\tau}}$$

and we get that $\hat{\tau}$ is an optimal stopping time.

We characterise the largest optimal stopping time (5/.)

It remains to show that $\hat{\tau}$ is the *largest* optimal stopping time.

Assume there is another optimal stopping time τ' with $\tau' \geq \hat{\tau}$ and $\mathbb{P}[\tau' > \hat{\tau}] > 0$.

Then

$$\mathbb{E}[U_{\tau'}] = \mathbb{E}[M_{\tau'}] - \mathbb{E}[A_{\tau'}] = U_0 - \underbrace{\mathbb{E}[A_{\tau'}]}_{>0} < U_0.$$

Consequently, $U^{\tau'}$ is not a martingal.

This is a contradiction to the assumptions.

As a result, we find that $\hat{\tau}$ is indeed the largest optimal stopping time.

Outline

We illustrate how theory of optimal stopping applies to American options

- We apply the theory to discounted payoffs and value processes.
- The adapted process $\left(\tilde{Z}_t\right)$ represents the discounted underlying payoff in which the option holder may exercise.
- The discounted value procsses $\left(\tilde{U}_t\right)$ of an American option is given by the associated Snell envelope.

Our theory gives

$$\tilde{U}_t = \sup_{\tau' \in \mathcal{T}_{t,T}} \mathbb{E}^* \left[\tilde{Z}_{\tau'} \,|\, \mathcal{F}_t \right].$$

Undiscounted value process is

$$U_t = (1+r)^t \sup_{\tau' \in \mathcal{T}_{t,T}} \mathbb{E}^* \left[\frac{Z_{\tau'}}{(1+r)^{\tau'}} \,|\, \mathcal{F}_t \right].$$

Value process of the replicating trading strategy uses the martingale part of the American option

We have

$$\tilde{U}_t = \tilde{M}_t - \tilde{A}_t$$

with

- $\mathbb{P}^*$-martingale $\tilde{M}$ and
- non-decreasing predictable process $\tilde{A}$.

We assumed that the market is complete.

Thus, there exists a self-financing trading strategy ξ and discounted value process $\tilde{V}$ for the martingale part $\tilde{M}$.

That is,

$$\tilde{V}_T = \tilde{M}_T \quad \text{and} \quad \tilde{V}_t = \tilde{V}_0 + \sum_{k=1}^{t} \xi_k \left(X_k - X_{k-1}\right).$$

The American option can be characterised from the sellers and buyers point of view

Since $\tilde{V}$ is (also) a $\mathbb{P}^*$-martingale, we must have

$$\tilde{V}_t = \tilde{M}_t.$$

From the claim, the option holder can get

$$\tilde{U}_t = \tilde{V}_t - \tilde{A}_t.$$

If the option holder acts rational (i.e. maximises gains) then she exercises at the optimal stopping time

$$\hat{\tau} = \begin{cases} T, & A_T = 0, \\ \inf\{t \in \{0, \ldots, T-1\} : A_{t+1} > 0\} & A_T > 0 \end{cases}$$

There are some important consequences for the buyer and the seller of the American option

- The option seller can hedge the American option perfectly by a portfolio ξ and value process $\tilde{V}$.
- For the option buyer the Snell envelope $\tilde{U}$ decribes the price of the option.
- Also, for the buyer there is no point in exercising the option after $\hat{\tau}$.
- If $\tilde{A}_t > 0$ for some scenario ω and time t, the option seller may even withdraw funds for consumption.
  - In such a situation the option buyer missed an optimal exercise time.

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Time value of an option is linked to early exercise

Recall the time value of a European call option as difference between option price and intrinsic value.

Time value in terms of discounted quantities becomes

$$\underbrace{\mathbb{E}^* \left[\left[X_T - \frac{K}{(1+r)^T} \right]^+ \mid \mathcal{F}_t \right]}_{\text{price at } t} - \underbrace{\left[X_t - \frac{K}{(1+r)^t} \right]^+}_{\text{intrinsic value at } t}.$$

- If $r \geq 0$ then time value is non-negative.
- Suppose the option holder has an American exercise right.
  - The American option must me worth at least as much as the European option.
  - It would always be more beneficial to sell the call at t instead of exercising the option.

We compare American and European option in general

Consider a American exercise process (Z_t).

- Denote (U_t) the discounted value process associated with American exercise right (Z_t).
- Denote (u_t) the discounted value process of a corresponding European option with payoff Z_T at final maturity T.

Because final payoffs coincide,

$$U_T = u_T = Z_T.$$

Super martingale property of U and martingale property of u yield for all t,

$$U_t \geq \mathbb{E}^* \left[U_T \mid \mathcal{F}_t \right] = \mathbb{E}^* \left[u_T \mid \mathcal{F}_t \right] = u_t.$$

Time value is formally characterised by comparing European price with American exercise

Define time value as

$$\text{TimeValue}_t = u_t - Z_t.$$

Suppose European option has a non-negative time value. That is

$$u_t \geq Z_t.$$

Then we get

$$U_t \geq Z_t.$$

- Now, (U_t) is a $\mathbb{P}^*$-martingale.
- This follows because we already know that (U_t) is the *smallest* super-martingale that dominates (Z_t).

We verify positive time value again for a call option

We have with $r \geq 0$

$$
\begin{aligned}
u_t &= \mathbb{E}^* \left[\left[X_T - \frac{K}{(1+r)^T} \right]^+ \mid \mathcal{F}_t \right] \\
&\geq \left[\underbrace{\mathbb{E}^* \left[X_T \mid \mathcal{F}_t \right]}_{X_t} - \frac{K}{(1+r)^T} \right]^+ \\
&\geq \left[X_t - \frac{K}{(1+r)^t} \right]^+ \\
&= Z_t.
\end{aligned}
$$

This shows that it is indeed not beneficial to exercise an American call option before final maturity T.

Outline

Now we check the time value for a put option

We have

$$Z_t = \left[\frac{K}{(1+r)^t} - X_t\right]^+ \quad \text{and} \quad u_t = \mathbb{E}^*\left[\left[\frac{K}{(1+r)^T} - X_T\right]^+ \mid \mathcal{F}_t\right].$$

We can use Jensen's inequality

$$\mathbb{E}^*\left[\left[\frac{K}{(1+r)^T} - X_T\right]^+ \mid \mathcal{F}_t\right] \geq \left[\frac{K}{(1+r)^T} - X_t\right]^+.$$

But, if $r > 0$ then

$$\left[\frac{K}{(1+r)^T} - X_t\right]^+ < \left[\frac{K}{(1+r)^t} - X_t\right]^+ = Z_t.$$

Put options exhibit an early exercise premium

- For put options we have scenarios where

$$u_t < Z_t.$$

- Then American option is indeed more expensive than a European option.
- Difference

$$U_t - u_t$$

is considered the early exercise premium of the American option.

We analyse the put option price function in CRR model (1/.)

Recall the CRR model with $S_0 > 0$,

$$S_t = S_0 \Lambda_t, \quad \Lambda_t = \prod_{k=1}^{t} (1 + R_k), \quad R_k \in \{a, b\}.$$

Model parameters satisfy

$$-1 < a < 0 < r < b.$$

Thus, CRR model is arbitrage-free and complete.

Risk-neutral measure is given by

$$\mathbb{P}^* [R_t = b] = p^* = \frac{r - a}{b - a}.$$

We analyse the put option price function in CRR model (2/.)

Consider put option price $\pi(x)$ as a function of the initial asset price $x := S_0$.

Write $\pi(x)$ using optimal stopping time and

$$\pi(x) = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}^* \left[\frac{[K - x\,\Lambda_\tau]^+}{(1+r)^\tau} \right].$$

- Price function $\pi(x)$ is convex and decreasing.
- We analyse $\pi(x)$ for small and large values of the initial asset price x.

Put option price vanishes for high initial values

We have

$$(1+a)^t \leq \Lambda_t \leq (1+b)^t .$$

Suppose option is *far out of the money*, $x \geq (1+a)^{-T}\ K$. Then

$$S_t = x\,\Lambda_t \geq (1+a)^{-T}\ K\ (1+a)^t \geq K.$$

Consequently, $[K - S_t]^+ \equiv 0$ and thus

$$\pi(x) = 0.$$

Put option price equals intrinsic value for low initial values

Suppose put option is far in the money, $x \le (1+b)^{-T} K$. Then for all t,

$$S_t = x\,\Lambda_t \le (1+b)^{-T}\, K\, (1+b)^t \le K$$

Consequently, $[K - S_t]^+ \equiv K - S_t$ and we get

$$\begin{aligned}\pi(x) &= \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}^* \left[\frac{[K - x\,\Lambda_\tau]^+}{(1+r)^\tau} \right] = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}^* \left[\frac{K - S_\tau}{(1+r)^\tau} \right] \\ &= \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}^* \left[\frac{K}{(1+r)^\tau} \right] - S_0 = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}^* \left[\frac{K}{(1+r)^\tau} \right] - x.\end{aligned}$$

Note that $\tau \equiv 0$ is a stopping time. Thus

$$\pi(x) = K - x = [K - x]^+\,.$$

- Option would be exercied immediately; there is no point in selling such an option.

For at the money options the option price is higher then the intrinsic value

Consider the case

$$K \leq x < \frac{K}{(1+a)^T}.$$

Option is at the money or not too far out of the money. Intrinsic value still vanishes, i.e.

$$[K - x]^+ = 0.$$

However, there is a $t > 0$ such that

$$\mathbb{P}^* \left[[K - S_t]^+ > 0 \right] > 0.$$

Thus option price is positive, $\pi(x) > 0$.

- Option price is strictly higher then intrinsic value.
- It is not optimal for the buyer to exercise option immediately.

We summarise the price function for an American put option

We get that there exits a break-even strike x^*,

$$\frac{K}{(1+b)^T} \leq x^* < K$$

such that

$$\begin{cases} \pi(x) = [K - x], & x \leq x^* \\ \pi(x) > [K - x], & x^* < x < (1+a)^{-T} K \\ \pi(x) = 0 & x \geq (1+a)^{-T} K \end{cases}.$$

We give a final remark on Snell envelopes in a Markovian setting (1/.)

In CRR model setting (discounted) asset prices follow a time-homogenous Markov chain.

Furthermore, the value from immediate exercise is given as a (time-dependent) function of asset price,

$$Z_t = h\left(t, S_t\right).$$

General theory of Snell envelopes for Markov chains implies that the value price is also given as

$$U_t = u\left(t, S_t\right).$$

We give a final remark on Snell envelopes in a Markovian setting (2/.)

The function $u(t, \cdot)$ can be determined recursively via

$$\begin{aligned} u(T, x) &= h(T, x) \quad \text{(terminal condition)}, \\ v(t, x) &= p^* u(t+1, x(1+b)) + (1-p^*) u(t+1, x(1+a)), \\ u(t, x) &= \max\{h(t, x), v(t, x)\}. \end{aligned}$$

It turns out that the state space $[0, T] \times [0, \infty)$ can be decomposed into

- stopping region

$$\mathcal{R}_s = \{(t, x) : u(t, x) = h(t, x)\},$$

- continuation region

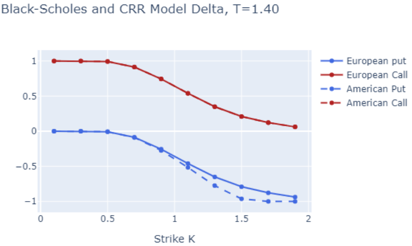
$$\mathcal{R}_c = \{(t, x) : u(t, x) > h(t, x)\}.$$

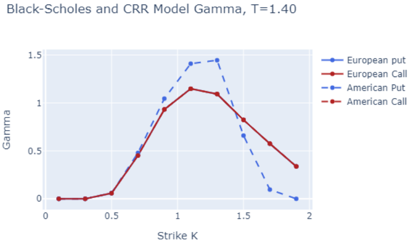
Minimal optimal stopping time can be viewed as first exit time of the process (t, S_t) from R_c.

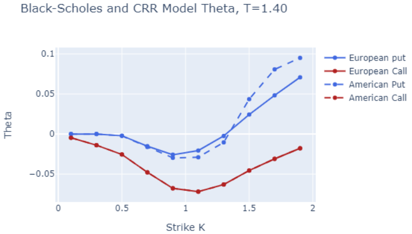
Outline

We price European and American options and compare results for various strikes

Set $r = 5\%$, $\sigma = 30\%$, $S_0 = 1.0$. CRR model uses $N = 10$ steps here

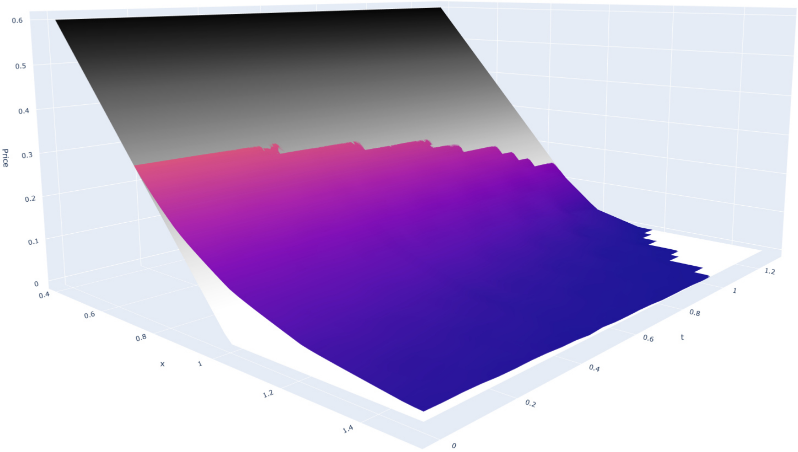






And we plot American put option price and intrinsic value to estimate continuation and stopping region

We plot $u(t,x)$ and $h(t,x)$.



Outline

Outline

What changes if we drop the condition of a complete market model?

- So far, we assumed a complete market model with unique EMM $\mathbb{P}^*$ for American option pricing.
- Now, we drop the condition of completeness. Still, we want to characterise prices of American options.
- For European options we elaborated that arbitrage-free prices form an intervall

$$\Pi(H) = (\pi_{\inf}(H), \pi_{\sup}(H))\,.$$

Can we establish a similar result for American options?

We establish a link between European options and American options with *fixed* exercise strategy (1/.)

Consider an American option with discounted payoff process (H_t).

- Depending on when H is exercised the buyer receives the payoff

Also, assume we fix an exercise strategy τ.

We link American option H and fixed exercise strategy τ to form the random variable

$$H_\tau : \Omega \to \mathbb{R}.$$

Payoff H_τ is still received at a random time.

But we can invest the complete amount received at τ into the numeraire asset S^0. This gives an equivalent payoff at $T \geq \tau$ as

$$H^E = H_\tau \frac{S_T^0}{S_\tau^0}.$$

H^E is equivalent (un-discounted) European option for a fixed strategy τ.

We establish a link between European options and American options with *fixed* exercise strategy (2/.)

Arbitrage-free prices of H^E become

$$\pi\left(H^E\right) = S_0^0\,\mathbb{E}^*\left[\frac{H^E}{S_0^T}\right] = S_0^0\,\mathbb{E}^*\left[\frac{H_\tau}{S_\tau^0}\right].$$

for EMMs $\mathbb{P}^* \in \mathcal{P}$.

Consequently, the set of arbitrage-free discounted prices for a discounted American claim with fixed exercise strategy H_τ can be represented as

$$\Pi\left(H_\tau\right) = \left\{\mathbb{E}^*\left[H_\tau\right] \mid \mathbb{P}^* \in \mathcal{P},\ \mathbb{E}^*\left[H_\tau\right] < \infty\right\}.$$

What is important for arbirage-free prices from the buyer's and the seller's perspective?

Suppose that an American claim H is offered at time $t = 0$ for a price $\pi > 0$.

Buyer's view:

- The price should not be *too high*.
- There should be *an* (or at least one) exercise strategy τ such that

$$\pi \leq \pi' \quad \text{for some } \pi' \in \Pi(H_\tau).$$

Seller's view:

- The price should not be *too low*.
- There should be *no* exercise strategy τ' such that

$$\pi < \pi' \quad \text{for all } \pi' \in \Pi(H_{\tau'}).$$

Combining the buyer's and seller's view leads to the definition of arbitrage-free prices

Definition 100 (Arbitrage-free price for American option)

A real number π is called an arbitrage-free price of a discounted American claim H if the following two conditions are satisfied:

1. There exists a stopping time τ and a $\pi' \in \Pi(H_\tau)$ (arbitrage-free European price) such that
$$\pi \leq \pi'.$$
2. There exists no stopping time τ' such that
$$\pi < \pi' \quad \text{for all } \pi' \in \Pi(H_{\tau'}).$$

The set of all arbitrage-free prices of H is denoted $\Pi(H)$. Moreover, we define

$$\pi_{\inf}(H) := \inf \Pi(H) \quad \text{and} \quad \pi_{\sup}(H) := \sup \Pi(H).$$

Arbitrage-free American option prices are consistent to arbitrage-free European prices

Recall that every discounted European claim H^E can be regarded as discounted American claim H^A with

$$H_t^A = \begin{cases} 0, & t < T \\ H^E & t = T \end{cases}.$$

- Note that stopping time $\tau \equiv T$ can be used to specify the American price $\pi\left(H^A\right)$.
- Also, any other stopping time τ' yields $H_{\tau'}^A \leq H_T^A$.

Consequently,

$$\pi\left(H^A\right) = \pi\left(H_\tau^A\right) = \pi\left(H_T^A\right) = \pi\left(H^E\right)$$

and

$$\Pi\left(H^A\right) = \Pi\left(H^E\right).$$

American option prices can be bounded by European option prices and stopping times

- Any arbitrage-free price π of H is also an arbitrage-free European price $\pi = \mathbb{E}^* [H_\tau]$ for some $\mathbb{P}^*$and τ.

- Condition 2 in Def. 100 also yields that for all stopping times $\tau' \in \mathcal{T}$,
$$\pi \geq \inf_{P^* \in \mathcal{P}} \mathbb{E}^* [H_{\tau'}] .$$

Consequently, we can enclose any $\pi \in \Pi (H)$ via

$$\sup_{\tau \in \mathcal{T}} \inf_{P^* \in \mathcal{P}} \mathbb{E}^* [H_\tau] \leq \pi \leq \sup_{\tau \in \mathcal{T}} \sup_{P^* \in \mathcal{P}} \mathbb{E}^* [H_\tau] . \tag{16}$$

- If the market model is complete and $\mathcal{P} = \{P^*\}$ then (as expected)
$$\pi = \sup_{\tau \in \mathcal{T}} \mathbb{E}^* [H_\tau] .$$

We introduce some additional assumpions and notation

We assume that the American option is suitably bounded. that is

$$H_t \in \mathcal{L}^1(P^*) \text{ for all } t \text{ and each } P^* \in \mathcal{P}. \qquad (17)$$

- That is $\mathbb{E}^*[|H_t|] < \infty$.

For each $P^* \in \mathcal{P}$, we denote U^{P^*} the corresponding Snell envelope of H, i.e.,

$$U_t^{P^*} = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}^*[H_\tau \mid \mathcal{F}_t].$$

With the notation for U^{P^*} we construct the basic building blocks for arbitrage-free price bounds

We get for the upper bound in Definition 100

$$\sup_{\tau\in\mathcal{T}} \sup_{P^*\in\mathcal{P}} \mathbb{E}^*\left[H_\tau\right] = \sup_{P^*\in\mathcal{P}} \sup_{\tau\in\mathcal{T}} \mathbb{E}^*\left[H_\tau\right] = \sup_{P^*\in\mathcal{P}} U_0^{P^*}.$$

We will also show later that for the lower bounds

$$\sup_{\tau\in\mathcal{T}} \inf_{P^*\in\mathcal{P}} \mathbb{E}^*\left[H_\tau\right] = \inf_{P^*\in\mathcal{P}} \sup_{\tau\in\mathcal{T}} \mathbb{E}^*\left[H_\tau\right] = \inf_{P^*\in\mathcal{P}} U_0^{P^*}.$$

- Proof of swap of inf and sup given $H_t \in \mathcal{L}^1\left(P^*\right)$ is postponed.
- We will *apply* swap of inf and sup in the following theorem which characterises American option prices.

We characterise the arbitrage-free prices of American options (1/.)

Theorem 101 (Intervall representation of American option prices)

Under the assumption in (17), the set of arbitrage-free prices for the discounted American claim H is a real intervall with endpoints

$$\pi_{\inf}(H) = \sup_{\tau \in \mathcal{T}} \inf_{P^* \in \mathcal{P}} \mathbb{E}^*[H_\tau] = \inf_{P^* \in \mathcal{P}} \sup_{\tau \in \mathcal{T}} \mathbb{E}^*[H_\tau] = \inf_{P^* \in \mathcal{P}} U_0^{P^*}$$

and

$$\pi_{\sup}(H) = \sup_{\tau \in \mathcal{T}} \sup_{P^* \in \mathcal{P}} \mathbb{E}^*[H_\tau] = \sup_{P^* \in \mathcal{P}} \sup_{\tau \in \mathcal{T}} \mathbb{E}^*[H_\tau] = \sup_{P^* \in \mathcal{P}} U_0^{P^*}. \qquad (18)$$

Moreover, $\Pi(H)$ consists of a single point or does not contain the upper endpoint $\pi_{\sup}(H)$.

We characterise the arbitrage-free prices of American options (2/.)

Proof:

We proof the result in three steps:

1. We show that $\Pi(H)$ is indeed an intervall.
2. We establish the lower and upper bound.
3. We prove that the upper intervall bound $\pi_{\sup}(H)$ does not belong to $\Pi(H)$.

We characterise the arbitrage-free prices of American options (3/.)

Step 1:
We show that $\left\{ U_0^{P^*} \mid P^* \in \mathcal{P} \right\}$ is an intervall.
Recall that $\mathcal{P}$ is convex. We can select measures $P_1, P_2 \in \mathcal{P}$ and $\alpha \in [0,1]$ and construct a measure

$$P_\alpha = \alpha \, P_1 + (1-\alpha) \; P_2 \in \mathcal{P}.$$

Moreover,

$$U_0^{P_\alpha} = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}} \left\{ \alpha \, \mathbb{E}_1 \left[H_\tau \right] + (1-\alpha) \; \mathbb{E}_2 \left[H_\tau \right] \right\}.$$

We consider the affine function $f(\alpha) := U_0^{P_\alpha}$. The function f can be considered the point-wise supremum over τ of affine functions

$$\alpha \mapsto \alpha \, \mathbb{E}_1 \left[H_\tau \right] + (1-\alpha) \; \mathbb{E}_2 \left[H_\tau \right], \quad \tau \in \mathcal{T}.$$

We characterise the arbitrage-free prices of American options (4/.)

From the construction of $f(\alpha)$ we conclude that f is convex and lower-semicontinuous on $[0, 1]$.

We note that convex functions are always upper-semicontinuous on the interior of their effective domain.

Thus, f is continuous in α.

We summarize that for any $P_1, P_2 \in \mathcal{P}$, $\left\{ U_0^{P_\alpha} \mid \alpha \in [0,1] \right\}$ forms an intervall. Hence, we conclude that also

$$\left\{ U_0^{P^*} \mid P^* \in \mathcal{P} \right\}$$

is an intervall.

We characterise the arbitrage-free prices of American options (5/.)

Step 2:

We derive the intervall bounds.

We consider $\tau^* = \tau^*(P^*)$ the stopping time that is optimal for a given $P^* \in \mathcal{P}$.

Then

$$U_0^{P^*} = \mathbb{E}^*[H_{\tau^*}] = \sup_{\tau \in \mathcal{T}} \mathbb{E}^*[H_\tau].$$

Consequently, $U_0^{P^*} \in \Pi(H)$. Together with the a priori bounds in (16) we get

$$\left\{ U_0^{P^*} \mid P^* \in \mathcal{P} \right\} \subset \Pi(H) \subset [a, b]$$

with

$$a := \sup_{\tau \in \mathcal{T}} \inf_{P^* \in \mathcal{P}} \mathbb{E}^*[H_\tau], \quad b := \sup_{\tau \in \mathcal{T}} \sup_{P^* \in \mathcal{P}} \mathbb{E}^*[H_\tau].$$

We characterise the arbitrage-free prices of American options (5/.)

The minimax identity (18) gives also

$$a := \inf_{P^* \in \mathcal{P}} \sup_{\tau \in \mathcal{T}} \mathbb{E}^* [H_\tau] = \inf_{P^* \in \mathcal{P}} U_0^{P^*}, \quad b := \sup_{P^* \in \mathcal{P}} U_0^{P^*}.$$

This gives the bounds.

We characterise the arbitrage-free prices of American options (6/.)

Step 3:

It remains to show that the upper bound $b \notin \Pi(H)$ if $a < b$. We show by contradiction.

Assume $b \in \Pi(H)$. Then, condition 1 in Def. 100 implies the existence of a stopping time $\hat{\tau} \in \mathcal{T}$ and measure $\hat{P} \in \mathcal{P}$ such that $b \leq \hat{\mathbb{E}}[H_{\hat{\tau}}]$.

We fix that $\hat{\tau}$. It follows that

$$\hat{\mathbb{E}}[H_{\hat{\tau}}] \geq b = \sup_{\tau \in \mathcal{T}} \sup_{P^* \in \mathcal{P}} \mathbb{E}^*[H_\tau].$$

In particular, b attains the supremum of $\sup_{P^* \in \mathcal{P}} \mathbb{E}^*[H_{\hat{\tau}}]$.

We characterise the arbitrage-free prices of American options (7/.)

Note that $H_{\hat{\tau}}$ is a European claim and Thm. 68 implies that $H_{\hat{\tau}}$ is attainable.

Moreover, $\mathbb{E}^* [H_{\hat{\tau}}]$ is independent of the measure $P^* \in \mathcal{P}$.

Hence,

$$b = \hat{\mathbb{E}} [H_{\hat{\tau}}] = \inf_{P^* \in \mathcal{P}} \mathbb{E}^* [H_{\hat{\tau}}] \leq \sup_{\tau \in \mathcal{T}} \inf_{P^* \in \mathcal{P}} \mathbb{E}^* [H_{\hat{\tau}}] \leq a.$$

Obviously, $b \leq a$ is a contradiction to the assumption $a < b$.

This gives $b \notin \Pi(H)$ and completes the proof.

We close with a brief discussion of attainability of American claims (1/.)

Definition 102

A discounted American claim H is called attainable if there is a stopping time τ and a self-financing trading strategy ξ such that the value process (V_t) with

$$V_t = V_0 + \sum_{k=1}^{t} \xi \left(X_k - X_{k-1}\right)$$

satisfies almost surely

$$V_t \geq H_t \quad \text{for all } t \text{ and } V_\tau = H_\tau .$$

We close with a brief discussion of attainability of American claims (2/.)

We state the following result characterising attainable American claims.

Theorem 103
For a discounted American claim H the following statements are equivalent:

1. *H is attainable.*
2. $|\Pi(H)| = 1$.
3. $\pi_{\text{sup}}(H) \in \Pi(H)$.

Equivalence of 2. and 3. is an immediate consequence of Thm. 101. For the remainder of the proof we refer to [6, Thm. 6.34 and Remark 7.10].

We see that this is a similar characterisation as for European options.

Outline

Now we return to the open proof of the minimax identity in (18)

In order to proof (18) we use the following steps:

- We define a pasting of measures and a stopping time.
- Then we conclude that the set of EMMs is stable under pasting
- This allows to develop the minimax identity for stable sets (see next Sec. **??**).

We start with some preparations

We need the events that are observable up to some stopping time τ.

Definition 104

Let τ be a stopping time. The σ-algebra of events which are observable up to time τ is defined as

$$\mathcal{F}_\tau := \{A \in \mathcal{F} \mid A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t\}.$$

- It turns out that

$$\mathcal{F}_\tau = \{A \in \mathcal{F} \mid A \cap \{\tau = t\} \in \mathcal{F}_t \text{ for all } t\}.$$

- For deterministic stopping times $\tau \equiv t$, we get

$$\mathcal{F}_\tau = \mathcal{F}_t.$$

We give an additional representation of the Snell envelope

The following theorem provides a solution to the optimal stopping problem at any stopping time $\tau \leq T$.

Theorem 105 (Snell envelope at stopping time)

Let H be an adapted process in $\mathcal{L}^1(\Omega, \mathcal{F}, Q)$. And define for $\tau \in \mathcal{T}$ (all stopping times)

$$\mathcal{T}_\tau := \{\sigma \in \mathcal{T} \mid \sigma \geq \tau\}.$$

Then the Snell envelope U^Q of H satisfies Q-a.s.

$$U_\tau^Q = \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}_\tau} \mathbb{E}_Q [H_\sigma \mid \mathcal{F}_\sigma].$$

For reference and guidance for the proof, see [6, Thm. 6.38].

Pasting represents an operation to *switch* probability measures at a given stopping time (1/.)

A pasting is represented by a new probability measure.

Definition 106 (Pasting of measures in a stopping time)

Let Q_1 and Q_2 be two equivalent probability measures and let $\sigma \in \mathcal{T}$ be a stopping time. The pasting of Q_1 and Q_2 in σ is the probability measure $\tilde{Q}$ which is specified via

$$\tilde{Q}\left[A\right] := \mathbb{E}_{Q_1}\left[Q_2\left[A \,|\, \mathcal{F}_\sigma\right]\right] = \mathbb{E}_{Q_1}\left[\mathbb{E}_{Q_2}\left[\mathbb{1}_A \,|\, \mathcal{F}_\sigma\right]\right].$$

Pasting represents an operation to *switch* probability measures at a given stopping time (2/.)

- Monotone convergence for conditional expectations guaranties that $\tilde{Q}$ is a probability measure and

$$\mathbb{E}_{\tilde{Q}}[Y] = \mathbb{E}_{Q_1}[\mathbb{E}_{Q_2}[Y \mid \mathcal{F}_\sigma]]$$

  for all $\mathcal{F}_T$-measurable $Y \geq 0$.

- Also $\tilde{Q}$ coincides with Q_1 on $\mathcal{F}_\sigma$, i.e.,

$$\mathbb{E}_{\tilde{Q}}[Y] = \mathbb{E}_{Q_1}[Y] \quad \text{for all } \mathcal{F}_\sigma\text{-measurable } Y \geq 0.$$

We formulate some properties of pastings

Lemma 107 (Properties of pastings)

The measure $\tilde{Q}$ satisfies the following conditions:

1. $\tilde{Q} = Q_1$ *on* $\mathcal{F}_\sigma$.
2. $\tilde{Q}$ *is equivalent to* Q_1 *and satisfies*

$$\frac{d\tilde{Q}}{dQ_1} = \frac{Z_T}{Z_\sigma}$$

   where Z *is the density process of* Q_2 *with respect to* Q_1, *i.e.* $Z_t = \frac{dQ_2}{dQ_1}|\mathcal{F}_t$.
3. *For all stopping times* $\tau \in \mathcal{T}$ *and* $\mathcal{F}_T$*-measurable* $Y \geq 0$,

$$\mathbb{E}_{\tilde{Q}}\left[Y \mid \mathcal{F}_\tau\right] = \mathbb{E}_{Q_1}\left[\mathbb{E}_{Q_2}\left[Y \mid \mathcal{F}_{\sigma\vee\tau}\right] \mid \mathcal{F}_\tau\right].$$

For a proof see [6, Lem. 6.40/41].

Now we can specify sets of measures that are stable under pasting

Definition 108 (Stable sets of measures)

A set $\mathcal{Q}$ of equivalent probability measures on $(\Omega, \mathcal{F})$ is called stable if, for any $Q_1, Q_2 \in \mathcal{Q}$ and $\sigma \in \mathcal{T}$, the pasting in σ is also contained in $\mathcal{Q}$, i.e. $\tilde{Q} \in \mathcal{Q}$.

Our most important application of stable sets is given in the following proposition:

Lemma 109

$\mathcal{P}$ *is stable.*

See [6, Prop. 6.43].

Outline

We analyse the minimax identity for stable sets

We still want to proof the minimax identity in (18).

We fix a set $\mathcal{Q}$ of equivalent probability measures; think of $\mathcal{Q} = \mathcal{P}$.

Also we fix an adapted process H; think of H as discounted American option payoff.

We assume

$$H \in \mathcal{L}^1(Q) \quad \forall t, \ \forall Q \in \mathcal{Q}.$$

- Assumption implies

$$\inf_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathbb{E}_Q[H_\tau] = \inf_{Q \in \mathcal{Q}} U_0^Q < \infty.$$

We exted the notion of Snell envelope

Definition 110 (Lower Snell envelope)

The *lower* Snell envelope of H is defined as

$$U_t^{\downarrow} := \operatorname*{ess\,inf}_{Q\in\mathcal{Q}} U_t^Q = \operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}_Q\left[H_\tau \mid \mathcal{F}_t\right]. \quad t = 0,1,\ldots,T.$$

We aim at showing that

$$U_t^{\downarrow} = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \mathbb{E}_Q\left[H_\tau \mid \mathcal{F}_t\right].$$

Desired representation for lower Snell envelope can be viewed as optimisation under model uncertainty

$U_0^{\downarrow} = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}} \operatorname*{ess\,inf}_{Q \in \mathcal{Q}} \mathbb{E}_Q [H_\tau]$ can be calculated by solving the following robust stopping problem:

$$\text{maximize} \operatorname*{ess\,inf}_{Q \in \mathcal{Q}} \mathbb{E}_Q [H_\tau] \quad \text{over all stopping times.} \tag{19}$$

- Stopping problem (19) can be viewed as an optimal stopping game under model uncertainty.
- Here, $\mathcal{Q}$ represents the set of possible financial market models.

We state the critical result to show the minimax identity (1/.)

Theorem 111 (Lower Snell envelope for optimally stopped strategy)

Define a stopping time $\tau_t \in \mathcal{T}_t$ by

$$\tau_t := \min\left\{ u \geq t \mid U_u^{\downarrow} = H_u \right\}.$$

Then, P-a.s.,

$$U_t^{\downarrow} = \operatorname*{ess\,inf}_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[H_{\tau_t} \mid \mathcal{F}_t \right].$$

We state the critical result to show the minimax identity (2/.)

With Thm. 111 we can enclose $\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \operatorname*{ess\,inf}_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[H_\tau \mid \mathcal{F}_t \right]$ via

$$
\begin{aligned}
U_t^{\downarrow} &= \operatorname*{ess\,inf}_{Q \in \mathcal{Q}} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}_Q \left[H_\tau \mid \mathcal{F}_t \right] \\
&\geq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \operatorname*{ess\,inf}_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[H_\tau \mid \mathcal{F}_t \right] \\
&\geq \operatorname*{ess\,inf}_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[H_{\tau_i} \mid \mathcal{F}_t \right] \\
&= U_t^{\downarrow}.
\end{aligned}
$$

This is the critical piece for the minimax identity (18).

- In order to proof Thm. 111 we need two additional auxilliary results.

We specify the Snell envelope for a pasting (1/.)

Lemma 112 (Snell envelope for pasting)

Suppose we have measures $Q_1, Q_2 \in \mathcal{Q}$ *and stopping time* $\tau \in \mathcal{T}$*, and a set* $B \in \mathcal{F}_\tau$*. Let* $\tilde{Q}$ *be the pasting of* Q_1 *and* Q_2 *in the stopping time*

$$\sigma := \tau \, \mathbb{1}_B + T \, \mathbb{1}_{B^c}.$$

Then the Snell envelopes associated with these three measures are related as follows:

$$U_\tau^{\tilde{Q}} = U_\tau^{Q_2} \, \mathbb{1}_B + U_\tau^{Q_1} \, \mathbb{1}_{B^c}.$$

We specify the Snell envelope for a pasting (2/.)

Proof:
With Thm. 105 we have

$$U_\tau^{\tilde{Q}} = \operatorname*{ess\,sup}_{\rho \in \mathcal{T}_\tau} \mathbb{E}_{\tilde{Q}} \left[H_\rho \mid \mathcal{F}_\tau \right].$$

We need to calculate $\mathbb{E}_{\tilde{Q}} \left[H_\rho \mid \mathcal{F}_\tau \right]$.

As an intermediate step, note that

$$\mathbb{E}_{Q_2} \left[H_\rho \mid \mathcal{F}_{\sigma \vee \tau} \right] = \mathbb{E}_{Q_2} \left[H_\rho \mid \mathcal{F}_\tau \right] \mathbb{1}_B + H_\rho \, \mathbb{1}_{B^c}.$$

Now, we apply Lem. 107,

$$\mathbb{E}_{\tilde{Q}} \left[H_\rho \mid \mathcal{F}_\tau \right] = \mathbb{E}_{Q_1} \left[\mathbb{E}_{Q_2} \left[H_\rho \,|\, \mathcal{F}_{\sigma \vee \tau} \right] \,|\, \mathcal{F}_\tau \right].$$

We specify the Snell envelope for a pasting (3/.)

We continue with

$$\begin{aligned}\mathbb{E}_{\tilde{Q}}\left[H_\rho \mid \mathcal{F}_\tau\right] &= \mathbb{E}_{Q_1}\left[\mathbb{E}_{Q_2}\left[H_\rho \,|\, \mathcal{F}_{\sigma\vee\tau}\right] \,|\, \mathcal{F}_\tau\right] \\ &= \mathbb{E}_{Q_1}\left[\mathbb{E}_{Q_2}\left[H_\rho \mid \mathcal{F}_\tau\right] \mathbb{1}_B + H_\rho\, \mathbb{1}_{B^c} \,|\, \mathcal{F}_\tau\right] \\ &= \mathbb{E}_{Q_2}\left[H_\rho \mid \mathcal{F}_\tau\right] \mathbb{1}_B + \mathbb{E}_{Q_1}\left[H_\rho \,|\, \mathcal{F}_\tau\right] \mathbb{1}_{B^c}.\end{aligned}$$

Next, note that whenever $\rho_1, \rho_2 \in \mathcal{T}_\tau$ then

$$\tilde{\rho} := \rho_1\, \mathbb{1}_B + \rho_2\, \mathbb{1}_{B^c}$$

is also a stopping time in $\mathcal{T}_\tau$.

We specify the Snell envelope for a pasting (4/.)

This gives the result

$$
\begin{aligned}
U_\tau^{\tilde{Q}} &= \operatorname*{ess\,sup}_{\rho \in \mathcal{T}_\tau} \mathbb{E}_{\tilde{Q}} \left[H_\rho \mid \mathcal{F}_\tau \right] \\
&= \operatorname*{ess\,sup}_{\rho \in \mathcal{T}_\tau} \left\{ \mathbb{E}_{Q_2} \left[H_\rho \mid \mathcal{F}_\tau \right] \mathbb{1}_B + \mathbb{E}_{Q_1} \left[H_\rho \mid \mathcal{F}_\tau \right] \mathbb{1}_{B^c} \right\} \\
&= \operatorname*{ess\,sup}_{\rho_1 \in \mathcal{T}_\tau} \mathbb{E}_{Q_2} \left[H_{\rho_1} \mid \mathcal{F}_\tau \right] \mathbb{1}_B + \operatorname*{ess\,sup}_{\rho_2 \in \mathcal{T}_\tau} \mathbb{E}_{Q_1} \left[H_{\rho_2} \mid \mathcal{F}_\tau \right] \mathbb{1}_{B^c} \\
&= U_\tau^{Q_2} \mathbb{1}_B + U_\tau^{Q_1} \mathbb{1}_{B^c}.
\end{aligned}
$$

This completes the proof.

We also need a convergence result

Lemma 113 (Convergence to lower Snell envelope)

For any $Q \in \mathcal{Q}$ and $\tau \in \mathcal{T}$ there exists a sequence $(Q_k) \subset \mathcal{Q}$ such that $Q_k = Q$ on $\mathcal{F}_\tau$ and

$$U_\tau^{Q_k} \searrow \operatorname*{ess\,inf}_{\hat{Q} \in \mathcal{Q}} U_\tau^{\hat{Q}} = U_t^{\downarrow}.$$

The proof uses Lem. 112. For details, see [6, Lem. 6.49].

Finally, we can use the result to prove the last piece for the minimax identity (1/.)

Proof of Thm. 111:

The super-martingale property of $\left(U_t^Q\right)$ yields for each $Q \in \mathcal{Q}$,

$$U_t^Q \geq \mathbb{E}_Q\left[H_{\tau_t} \mid \mathcal{F}_t\right].$$

Consequently,

$$U_t^{\downarrow} \geq \operatorname*{ess\,inf}_{Q \in \mathcal{Q}} \mathbb{E}_Q\left[H_{\tau_t} \mid \mathcal{F}_t\right].$$

And it only remains to show

$$U_t^{\downarrow} \leq \operatorname*{ess\,inf}_{Q \in \mathcal{Q}} \mathbb{E}_Q\left[H_{\tau_t} \mid \mathcal{F}_t\right].$$

Finally, we can use the result to prove the last piece for the minimax identity (2/.)

We fix some $Q \in \mathcal{Q}$ and consider $(Q_k) \subset \mathcal{Q}$ from Lem. 113 with $Q_k = Q$ on $\mathcal{F}_{\tau_t}$ and $U_{\tau_t}^{Q_k} \searrow U_{\tau_t}^{\downarrow}$.

Then

$$
\begin{aligned}
\mathbb{E}_Q\left[H_{\tau_t} \mid \mathcal{F}_t\right] &= \mathbb{E}_Q\left[U_{\tau_t}^Q \mid \mathcal{F}_t\right] \quad \text{(construction of } \tau_t\text{)} \\
&= \mathbb{E}_Q\left[\lim_{k\uparrow\infty} U_{\tau_t}^{Q_k} \mid \mathcal{F}_t\right] \quad \text{(Lem. 113)} \\
&= \lim_{k\uparrow\infty} \mathbb{E}_Q\left[U_{\tau_t}^{Q_k} \mid \mathcal{F}_t\right] \quad \text{(monotone convergence)} \\
&= \lim_{k\uparrow\infty} \mathbb{E}_{Q_k}\left[U_{\tau_t}^{Q_k} \mid \mathcal{F}_t\right] \quad \text{(because } Q_k = Q \text{ on } \mathcal{F}_{\tau_t} \supset \mathcal{F}_t\text{)} \\
&= \lim_{k\uparrow\infty} U_t^{Q_k} \quad \text{(by martingale property)} \\
&\geq U_t^{\downarrow}.
\end{aligned}
$$

Outline

References I

- L. Ballabio.
  Implementing QuantLib.
  leanpub.com/implementingquantlib, 2021.
- L. Ballabio and G. Balaraman.
  QuantLib Python Cookbook.
  leanpub.com/quantlibpythoncookbook, 2021.
- J. Cox, S. Ross, and M. Rubinstein.
  Option pricing: a simple aproach.
  J. Financial Economics, 7, 1979.
- D. J. Duffy.
  Finite Difference Methods in Financial Engineering.
  Wiley, 2006.
- E. F. Fama.
  Efficient capital markets: A review of theory and empirical work.
  Journal of Finance, 25:383–417, 1970.

References II

H. Foellmer and A. Schied.
Stochastic Finance: An Introduction in Discrete Time.
de Gruyter, 2016.

Espen Gaarder Haug.
The complete guide to option pricing formulas.
McGraw-Hill, 2007.

D. Lamberton and B. Lapeyre.
Introduction to Stochastic Calculus Applied to Finance.
Chapman and Hall, 2008.

R. Rebonato.
Volatility and Correlation, 2nd Edition.
Wiley, 2004.

S. Shreve.
Stochastic Calculus for Finance II - Continuous-Time Models.
Springer-Verlag, 2004.

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