Mathematical Finance I

Sebastian Schlenkrich

HU Berlin, Department of Mathematics

WS, 2021/22

Part VI

Introduction to Continuous Time Finance

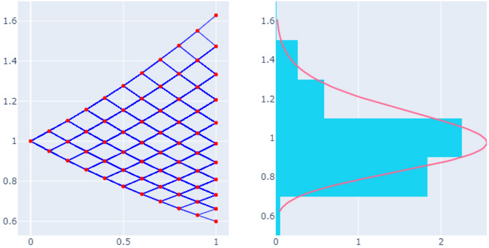
Outline

Outline

Outline

Why do we consider continuous models?

- We found that in CRR model the distribution of asset prices $S_N^{(N)}$ converges to a lognormal random variable S_T:



- Lognormal distribution approximation can also be established for earlier model times $0 < t < T$.

We know the (lognormal) distribution of S_t. But how would a continuous process evolve from 0 to T such that it complies with the (lognormal) terminal distributions for all t?

What do we cover in this part of the lecture?

- We specify what we mean by stochastic process and continuous model.
- We introduce Brownian motion as the stochastic driver for our model.
- We introduce the Riemann-Stieltjes integral as a mean to integrate over stochastic processes.
- Then we can derive the Ito formula.
  - Ito formula is the basis for option pricing and hedging.
- Finally, we discuss the Bachelier and Black-Scholes model.

There are a couple of important aspects that we will not cover in the first part of the course

The following related topics will be covered (among others) in the second part of the course:

- Stochastic differential equations and Girsanov's theorem.
- Risk-neutral pricing, equivalent martingale measure and absennce of arbitrage.
- Fundamental theorem of asset pricing in continuous time.

Outline

We work on a filtered probability space with continuous observation time

Probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right)$

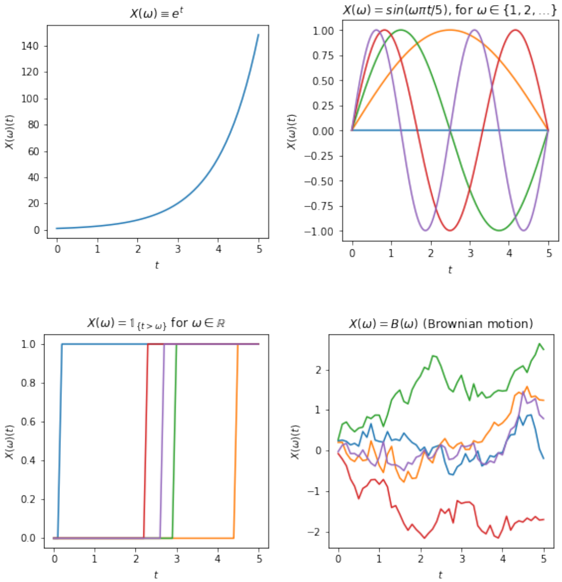
- Ω a (infinite) set of potential outcomes,
- $\mathcal{F}$ a σ-algebra describing *all* the events which we can assign a probability using the measure $\mathbb{P}$,
- $(\mathcal{F}_t)_{t \geq 0}$ a filtration describes the flow of information, and $\mathcal{F}_t \subseteq \mathcal{F}$ the events which are observable up to time $t \in [0, \infty)$.

Stochastic process $X = (X_t)_{t \geq 0}$

- X is a family of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $\mathbb{R}$.
- X is adapted to the filtration $(\mathcal{F}_t)$, if each random variable X_t is $\mathcal{F}_t$-measurable.
- A trajectory or sample path of X is the realisation $X(\omega)$ for $\omega \in \Omega$, that is,

$$X(\omega) : [0, \infty) \to \mathbb{R}.$$

Stochastic processes can be of very different kind



When are two processes *almost equal*?

Definition 114 (Modification and indistinguishable processes)

Suppose X and Y are stochastic processes on the same probability space.

1. X and Y are called a modification of each other if, for all $t \geq 0$, $\mathbb{P}[X_t = Y_t] = 1$.
2. X and Y are called indistinguishable if $\mathbb{P}[X_t = Y_t \quad \forall t \geq 0] = 1$.

We get directly:

- If X and Y are indistinguishable then X and Y are a modification of each other.

Lemma 115

If X and Y are a modification of each other and have almost surely right-continuous sample paths then X and Y are indistinguishable.

- For example, Brownian motions have continuous sample paths.

We will mainly work with adapted processes with some martingale property

Definition 116 (Martingale properties)

Let X be an adapted process with $X_t \in L^1(\mathbb{P})$ for all t. X is called

- a submartingale if $\mathbb{E}[X_t \mid \mathcal{F}_s] \geq X_s$ for all $t \geq s$,
- a supermartingale if $\mathbb{E}[X_t \mid \mathcal{F}_s] \leq X_s$ for all $t \geq s$,
- a martingale if $\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$ for all $t \geq s$.

The Markov property of a process can be linked to a efficient market hypothesis

Definition 117 (Markov process)

A process X is called a Markov process if for all $0 < t \leq s$ and every bounded real-valued function f on $\mathbb{R}$ we have

$$\mathbb{E}\left[f\left(X_s\right) \mid \mathcal{F}_t\right] = \mathbb{E}\left[f\left(X_s\right) \mid \sigma\left(X_t\right)\right].$$

Here, $\sigma\left(X_t\right)$ is the σ-algebra generated by the random variable X_t.

- Intuitively, for a Markov process, distributions of future states X_s are completely determined by the current state X_t (and model).
- This is equivalent to saying: all available information is contained in current price; a weak form of efficient market hypothesis (EMH).
  - E. F. Fama. Efficient capital markets: A review of theory and empirical work.
    Journal of Finance, 25:383–417, 1970

Outline

Brownian motion (or Wiener process) is the most important building block of continuous time finance

Definition 118 (Brownian motion)

A stochastic process $B = (B_t)_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a standard one-dimensional Brownian motion if the following conditions are satisfied:

1. $B_0 = 0$ $\mathbb{P}$-a.s.
2. B has independent increments. That is, for all $0 < t \leq s$ we have that
$$B_s - B_t \quad \text{is independent of} \quad (B_u)_{0 \leq u \leq t} .$$
3. The increments are stationary and normally distributed; for all $0 < t \leq s$,
$$B_s - B_t \sim \mathcal{N}(0, s - t) .$$
4. B has almost surely continuous paths.

We see directly that each B_t is normally distributed with

$$B_t \sim \mathcal{N}(0, t) .$$

We can calculate the covariance of Brownian motion at two time points

Suppose $t \leq s$,

$$\begin{aligned}
\mathrm{Cov}\left[B_t, B_s\right] &= \mathbb{E}\left[B_t\, B_s\right] - \underbrace{\mathbb{E}\left[B_t\right]}_{0}\,\underbrace{\mathbb{E}\left[B_s\right]}_{0} \\
&= \mathbb{E}\left[B_t\,\left(B_s - B_t + B_t\right)\right] \\
&= \underbrace{\mathbb{E}\left[B_t\right]}_{0}\,\underbrace{\mathbb{E}\left[\left(B_s - B_t\right)\right]}_{0} + \underbrace{\mathbb{E}\left[B_t^2\right]}_{\mathrm{Var}[B_t]} \\
&= t.
\end{aligned}$$

Consequently,

$$\mathrm{Cov}\left[B_t, B_s\right] = \min\left\{t, s\right\}.$$

Proving existence of Brownian motion is not trivial

Theorem 119

A standard Brownian motion exists.

- Proof is beyond the scope of this course.
- It uses Kolmogorov's existence theorem:
  - Specifies conditions under which a collection of finite-dimensional distributions allows to define a stochastic process.
- For details see e.g.
  - `https://www.uni-ulm.de/fileadmin/website_uni_ulm/mawi.inst.110/lehre/ws13/Stochastik_II/Skript_4.pdf`

Brownian motion satisfies two imprtant martingale properties (1/.)

- Definition of Brownian a priori does not require any specification of a filtration $(\mathcal{F}_t)$.
- We use the canonical filtration specified via

$$\mathcal{F}_t = \sigma\left(B_s, s \leq t\right).$$

Theorem 120 (Martingale property of Brownian motion)

Let B be a standard Brownian motion. Then (B_t) and $\left(B_t^2 - t\right)$ are martingals with respect to the canonical filtration.

Brownian motion satisfies two imprtant martingale properties (2/.)

Proof:
Martingale property of (B_t) follows from

$$\mathbb{E}\left[B_s \,|\, \mathcal{F}_t\right] = \mathbb{E}\left[B_s - B_t + B_t \,|\, \mathcal{F}_t\right] = \underbrace{\mathbb{E}\left[B_s - B_t \,|\, \mathcal{F}_t\right]}_{0} + \underbrace{\mathbb{E}\left[B_t \,|\, \mathcal{F}_t\right]}_{B_t}.$$

For the second assertion we need to show that

$$\mathbb{E}\left[B_s^2 - s \,|\, \mathcal{F}_t\right] = B_t^2 - t.$$

Since B is adapted, this is equivalent to

$$\mathbb{E}\left[B_s^2 - B_t^2 \,|\, \mathcal{F}_t\right] = s - t.$$

Brownian motion satisfies two imprtant martingale properties (3/.)

We calculate

$$
\begin{aligned}
\mathbb{E}\left[B_s^2 - B_t^2 \,|\, \mathcal{F}_t\right] &= \mathbb{E}\left[B_s^2 \,|\, \mathcal{F}_t\right] - B_t^2 - 2\underbrace{\mathbb{E}\left[B_t\left(B_s - B_t\right) \,|\, \mathcal{F}_t\right]}_{0} \\
&= \mathbb{E}\left[B_s^2 \,|\, \mathcal{F}_t\right] - 2\mathbb{E}\left[B_s B_t \,|\, \mathcal{F}_t\right] + B_t^2 \\
&= \mathbb{E}\left[\left(B_s - B_t\right)^2 \,|\, \mathcal{F}_t\right] \\
&= \mathbb{E}\left[B_{s-t}^2\right] \\
&= s - t.
\end{aligned}
$$

This concludes the proof.

The famous characterisation result of Brownian motion by Paul Levy also gives the converse result

Theorem 121 (Levy characterisation of Brownian motion)

A real-valued continuous stochastic process (X_t) *defined on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and with* $X_0 = 0$ *is a Brownian motion if and only if* (X_t) *and* $\left(X_t^2 - t\right)$ *are (local) martingales.*

We formulate another useful property of Brownian motion

Corollary 122

Let B be a Browian motion and $t \leq s$. Then $\mathbb{E}\left[B_s\left(B_s - B_t\right)\right] = s - t$.

Proof:

$$\begin{aligned}
\mathbb{E}\left[B_s\left(B_s - B_t\right)\right] &= \mathbb{E}\left[B_s\left(B_s - B_t\right)\right] + \underbrace{\mathbb{E}\left[B_t\left(B_s - B_t\right)\right]}_{0} \\
&= \mathbb{E}\left[B_s^2 - B_t^2\right] \\
&= \mathbb{E}\left[\underbrace{\mathbb{E}\left[B_s^2 - B_t^2 \mid \mathcal{F}_t\right]}_{s-t}\right] \\
&= s - t.
\end{aligned}$$

Outline

Integrating over stochastic processes is relevant for pricing and hedging

Recall from discrete time models the discounted value process of an option

$$V_t = V_0 + \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1}) .$$

- Replication strategy (ξ_k) and asset prices (X_k) are discrete stochastic processes.
- Sum $\sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1})$ can be viewed as a discrete version of an integral

$$\int_0^t \xi_u \, dX_u .$$

For continuous processes we are interested in integrals of the form

$$\int_0^T B_t(\omega) \, dB_t(\omega) .$$

Let us consider two approaches for integration based on partitioning

We take a partition $0 = t_0 < t_1 < \ldots < t_n = T$ of $[0, T]$.

Apply two piece-wise constant discretisations/approximations to $B_t(\omega)$:

1. Use left function value

$$\phi_1(t, \omega) = \sum_{j=0}^{n-1} B_{t_j}(\omega)\ \mathbb{1}_{(t_j, t_{j+1}]}(t).$$

2. Use right function value

$$\phi_2(t, \omega) = \sum_{j=0}^{n-1} B_{t_{j+1}}(\omega)\ \mathbb{1}_{(t_j, t_{j+1}]}(t).$$

We can calculate the integral for step functions

Suppose, we define

$$\int_s^u dB_t = B_u - B_t.$$

Then

$$\int_0^T \phi_1(t,\omega)\, dB_t = \sum_{j=0}^{n-1} B_{t_j}(\omega)\left(B_{t_{j+1}}(\omega) - B_{t_j}(\omega)\right)$$

and

$$\int_0^T \phi_2(t,\omega)\, dB_t = \sum_{j=0}^{n-1} B_{t_{j+1}}(\omega)\left(B_{t_{j+1}}(\omega) - B_{t_j}(\omega)\right).$$

In principle, both discretisations could be considered reasonable approximations.

However, the construction of the integral impacts its distribution

1. We get for ϕ_1

$$\mathbb{E}\left[\int_0^T \phi_1\left(t,\omega\right) dB_t\right] = \sum_{j=0}^{n-1} \mathbb{E}\left[B_{t_j}\left(B_{t_{j+1}} - B_{t_j}\right)\right] = 0.$$

2. And we get for ϕ_2

$$\begin{aligned}\mathbb{E}\left[\int_0^T \phi_2\left(t,\omega\right) dB_t\right] &= \sum_{j=0}^{n-1} \mathbb{E}\left[B_{t_{j+1}}\left(B_{t_{j+1}} - B_{t_j}\right)\right] \\ &= \sum_{j=0}^{n-1}\left(t_{j+1} - t_j\right) \\ &= T.\end{aligned}$$

We will analyse an alternative approach to define integrals for stochastic processes.

Outline

Outline

We consider one- and d-dimensional processes on partitions

We usually denote $X, Y : [0, T] \to \mathbb{R}$ as one-dimensional continuous functions.

In some situations we will use the notation of a vector-valued function

$$X : [0, T] \to \mathbb{R}^d.$$

Then, components of X are $X^j : [0, T] \to \mathbb{R}$ for $j = 1, \ldots, d$.

Process values at time $t \in [0, T]$ will be denoted X_t.

Our discussion will be based on sequences of partitions:

Definition 123 (Partition and mesh)

A partition Π^n of an intervall $[0, T]$ into n segments is a set of points $0 = t_0 < t_1 < \ldots < t_n = T$. the mesh of the partition is

$$|\Pi^n| = \max_{i=1,\ldots n} (t_i - t_{i-1}).$$

First we define total the variation of a function and functions of finite variation

Definition 124 (Total variation, finite variation)

Consider functions $X : [0, T] \to \mathbb{R}$. The (total) variation of X along a patition Π^n of the intervall $[0, T]$ is

$$V_T(X, \Pi^n) := \sum_{i=1}^{n} \left| X_{t_i} - X_{t_{i-1}} \right| .$$

The (total) variation on $[0, T]$ is

$$V_T(X) := \sup \{ V_T(X, \Pi) : \Pi \text{ is a partition of } [0, T] \} .$$

If $V_T(X) < \infty$, then X is of bounded/finite variation.

We give some immediate consequences for total variation

- Every monotone function is of bounded variation, because

$$V_T\left(X,\Pi^n\right)=\sum_{i=1}^{n} X_{t_i}-X_{t_{i-1}}=X_T-X_0.$$

- Every function with bounded variation defined in $[0,T]$ is bounded.
- If $X\in\mathcal{C}^1$ with $X_0=0$ then

$$X_t=\int_0^t X_s' ds \quad \text{and} \quad V_t\left(X\right)=\int_0^t \left|X_s'\right| ds.$$

- A function is of bounded variation if and only if it can be represented as the difference of two non-decreasing functions. (Exercise.)

For stocastic processes the corresponding covariance process will become important

Definition 125 (Covariance process and quadratic variation)

The covariation process of functions $X, Y : [0, T] \to \mathbb{R}$ along a partition Π^n of the intervall $[0, T]$ is defined as

$$V_t^2(X, Y, \Pi^n) := \sum_{i=1}^{n} \mathbb{1}_{\{t_i \leq t\}} \left(X_{t_i} - X_{t_{i-1}}\right)\left(Y_{t_i} - Y_{t_{i-1}}\right).$$

The quadratic variation of X along a partition Π^n is

$$V_t^2(X, \Pi^n) := V_t^2(X, X, \Pi^n).$$

Moreover, we define the covariation process of X and Y on $[0, T]$ as

$$\langle X, Y\rangle_t := \lim_{n\to\infty} V_t^2(X, Y, \Pi^n).$$

Here, $\{\Pi^n\}$ is a sequence of partitions with $|\Pi^n| \to 0$ for $n \to \infty$.

We give some further remarks on covariation process

- Strictly speaking, we should have defined covariation as

$$V_t^2(X, Y, \Pi^n) := \sum_{i=1}^{n} \left(X_{t_i \wedge t} - X_{t_{i-1} \wedge t}\right)\left(Y_{t_i \wedge t} - Y_{t_{i-1} \wedge t}\right).$$

  - However, for continuous X and Y this leads to the same limit as $|\Pi^n| \to 0$.
  - For our analysis, we assume $t \in \Pi^n$ when computing V_t^2.

- Notice, that for $\langle X, Y \rangle_t$ we did not further specify the sequence of partitions. In particular, for Brownian motions we will see that the choice of sequence is irrelevant as long as $|\Pi^n| \to 0$.

Continuous covariation processes will be particularly relevant

Definition 126 (Continuous quadratic variation)

Let $X : [0, T] \to \mathbb{R}^d$ be a vector-values process with components X^i for $i = 1, \ldots, d$. X is of continuous quadratic variation if for all combinations $i, j = 1, \ldots, d$ and a sequence $\{\Pi^n\}$ with $|\Pi^n| \to 0$

$$\left\langle X^i, X^j \right\rangle_t = \lim_{n \to \infty} V_t^2 \left(X^i, X^j, \Pi^n \right)$$

exists and is continuous in t.

For a scalar function $X : [0, T] \to \mathbb{R}$ the quadratic variation process is $\langle X \rangle$ with

$$\langle X \rangle_t := \langle X, X \rangle_t \, .$$

- For a continuous scalar function X the quadratic variation $\langle X \rangle$ is increasing with $\langle X \rangle_0 = 0$.

There is an important relation between bounded variation and quadratic variation

Corollary 127

If $X : [0, T] \to \mathbb{R}$ is continuous and of bounded variation, then $\langle X \rangle \equiv 0$.

Proof: For any partition Π^n we have

$$
\begin{aligned}
V_t^2(X, \Pi^n) &= \sum_{i=1}^{n} \mathbb{1}_{\{t_i \leq t\}} \left(X_{t_i} - X_{t_{i-1}}\right)^2 \\
&\leq \sup_{i=1,\ldots,n} \left|X_{t_i} - X_{t_{i-1}}\right| \sum_{i=1}^{n} \mathbb{1}_{\{t_i \leq t\}} \left|X_{t_i} - X_{t_{i-1}}\right| \\
&\leq \sup_{i=1,\ldots,n} \left|X_{t_i} - X_{t_{i-1}}\right| V_T(X, \Pi^n).
\end{aligned}
$$

Since X is continuous and hence uniformly continuous on compact sets, we get

$$
\lim_{n \to \infty} \sup_{i=1,\ldots,n} \left|X_{t_i} - X_{t_{i-1}}\right| = 0 \quad \text{for } |\Pi^n| \to 0.
$$

We can derive various properties for covariance process and quadratic variation

Lemma 128

Let $X, Y : [0, T] \to \mathbb{R}$ be continuous and of continuous quadratic variation with respect to a common sequence of partitions $\{\Pi^n\}$.

1. *The covariation process $\langle X, Y \rangle$ exists and is continuous if and only if $\langle X + Y \rangle$ exists and is continuous.*
2. *$\langle \cdot, \cdot \rangle$ defines a bilinear form (not necessarily positive semi-definite)*
3. *The polarization identity holds*

$$\langle X, Y \rangle = \frac{1}{2} \left(\langle X + Y \rangle - \langle X \rangle - \langle Y \rangle \right).$$

   In particular, covariation processes are differences of two non-decreasing processes and hence of bounded variation.

4. *The Cauchy-Schwarz inequality holds*

$$|\langle Y, Y \rangle| \leq \sqrt{\langle X \rangle \langle Y \rangle}.$$

Polarization inequality and Cauchy-Schwarz yield that quadratic variation is invariant w.r.t. shifts

Corollary 129

Let X be a continuous stochastic process with continuous quadratic variation and A a process with bounded variation. Then

$$\langle X + A \rangle = \langle X \rangle .$$

Proof:

$$\begin{aligned}\langle X + A \rangle &= \langle X \rangle + \underbrace{\langle A \rangle}_{0} + 2 \underbrace{\langle X, A \rangle}_{|\cdot| \le \sqrt{\langle X \rangle \langle A \rangle} = 0} \\ &= \langle X \rangle .\end{aligned}$$

Outline

First we recall some facts from measure theory (1/.)

Let $\mathcal{F}$ be a σ-algebra and $S \in \mathcal{F}$ a set of *observable events*.

- A mapping $\mu : \mathcal{F} \to \mathbb{R}$ is called a measure if it is countable additive.
- A (signed) measure can be decomposed as

  $$\mu = \mu^+ - \mu^- .$$

  Here, μ^+ and μ^- are (non-negtive) measures concentrated on disjoint subsets of S (Hahn-Jordan decomposition).
- The total variation is given by

  $$|\mu| := \mu^+ + \mu^-$$

  and the total variation norm is given by

  $$\|\mu\| := |\mu|(S) .$$

First we recall some facts from measure theory (2/.)

- A measure $\mu : \mathcal{B}(S) \to \mathbb{R}$ defined on the Borel field $\mathcal{B}(S)$ of a set S is called a Borel measure.
- For $S = \mathbb{R}$ (random variables) and a bounded measure on $\mathcal{B}(\mathbb{R})$ we can define a distribution function

$$F_\mu(s) := \mu((-\infty, s)).$$

If μ is non-negtive then F_μ is increasing. Since μ is bounded, F_μ is of bounded variation.

First we recall some facts from measure theory (3/.)

Now, think of S as our time intervall $[0, T]$.

For every function $X : [0, T] \to \mathbb{R}$ of bounded variation, we can assign a bounded measure $\mu : \mathcal{B}([0, T]) \to \mathbb{R}$:

- Use bounded variation and decompose $X = X^+ - X^-$ where $X^\pm$ are non-decreasing.
- Define non-negative measures $\mu^\pm$ via

$$\mu^\pm([a, b)) := X_b^\pm - X_a^\pm.$$

- Define the singend measure as $\mu := \mu^+ - \mu^-$.

This allows to define integrals

$$\int_0^T Y_t \, dX_t = \int_{[0,T)} Y \, d\mu$$

for continuous process Y and processes X with bounded variation.

We will use the Riemann-Stieltjes integral to discuss integrals with respect to processes of bounded variation

We consider

- scalar processes $X, Y : [0, T] \to \mathbb{R}$,
- a sequence of partitions $\{\Pi^n\}$ with $\Pi^n = (t_k^n)_{k=0,\dots,n}$, and
- a sequence of vectors $\{\xi^n\}$ such that $\xi^n = (\xi_k^n)_{k=1,\dots,n}$ and $t_{k-1}^n \leq \xi_k^n \leq t_k^n$.

Definition 130 (Riemann-Stieltjes sum)

We define the Riemann-Stieltjes (RS) sum as

$$S(X, Y, \Pi^n, \xi^n) := \sum_{k=1}^{n} Y_{\xi_k} \left[X_{t_k} - X_{t_{k-1}}\right].$$

A sequence $\{S(X, Y, \Pi^n, \xi^n)\}_{n=1,2,\dots}$ is called a RS sequence if $|\Pi^n| \to 0$ as $n \to \infty$.

A Riemann-Stieltjes integral is the limit of Riemann-Stieltjes sums

Definition 131 (Riemann-Stieltjes integral)

Consider a RS sequence $\{S(X, Y, \Pi^n, \xi^n)\}_{n=1,2,\ldots}$. If the RS sequence converges we define the RS integral as

$$\int_0^T Y_s \, dX_s := \lim_{n \to \infty} S(X, Y, \Pi^n, \xi^n).$$

- If RS sequence converges then the limit is unique; same argument as for Riemann sums
- RS integral satisfies same properties as Riemann integral (linearity, etc.)
- If Y is bounded, then

$$\left| \int Y dX \right| \leq \|Y\|_\infty \, V_T(X).$$

The RS integral is linear in the integrator

Lemma 132 (Linearity of RS integral)

Let $X_1, X_2, Y : [0, T] \to \mathbb{R}$*. If* $\int Y dX_1$ *and* $\int Y dX_2$ *exist then* $\int Y d(X_1 + X_2)$ *exists and*

$$\int Y d(X_1 + X_2) = \int Y dX_1 + \int Y dX_2.$$

The RS integral exists for continuous integrand and bounded variation integrator

Theorem 133 (Existence of RS integral)

Let $Y : [0, T] \to \mathbb{R}$ be continuous and $X : [0, T] \to \mathbb{R}$ be of bounded variation. Then the RS integral $\int YdX$ exists.

Proof:

Since X is of bounded variation we can decompose $X = X^+ - X^-$ into non-decreasing functions.

Existence of $\int YdX^+$ and $\int YdX^-$ can then be established exactly as for Riemann integrals.

Existance of

$$\int YdX = \int YdX^+ - \int YdX^-$$

then follows from Lem. 132.

RS integrals depend continuously on the integration boundaries

Theorem 134 (Continuity and bounded variation of RS integral)

Let $Y : [0, T] \to \mathbb{R}$ *be continuous and* $X : [0, T] \to \mathbb{R}$ *be of bounded variation. Denote*

$$F(t) := \int_0^t Y_s dX_s.$$

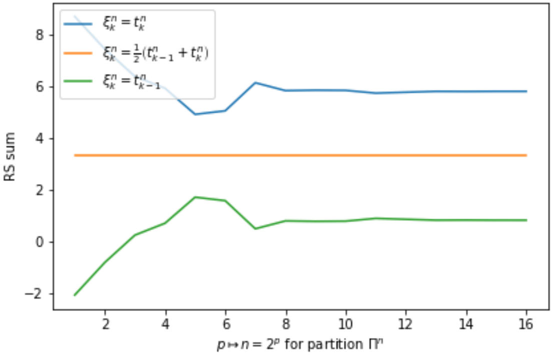
Then F *is continuous and of bounded variation on* $[0, T]$.

- The analysis of this subsection suggests that a Brownian sample path $B(\omega)$ is typically *not* of bounded variation.
- Forthcomming we analyse the *quadratic variation* of Brownian motion.

Unfortunately, RS integrals on its own are not enough for integrators with unbounded variation

We simulate B_t for $t \in [0,5]$ and try to calculate the inegral

$$\int_0^5 B_t(\omega)\, dB_t(\omega) \approx S\left(B_t(\omega), B_t(\omega), \Pi^n, \xi^n\right), \qquad n = 2^p,\ p = 1, 2, \dots$$



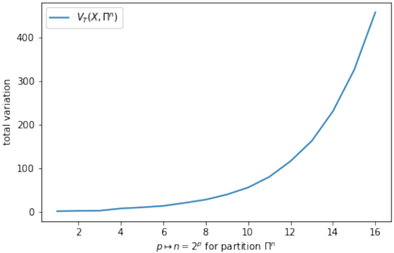
Integral depends on $\{\xi^n\}$. Which one should we take?

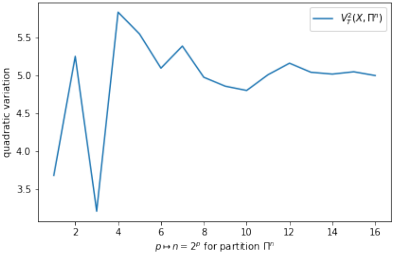
Outline

We see that Brownian motion causes some difficulties with RS integrals - what are its total and quadratic variation?

We simulate B_t for $t \in [0,5]$ and calculate total/quadratic variation

$$V_T\left(B,\Pi^n\right) = \sum_{i=1}^{n} \left| B_{t_i^n} - B_{t_{i-1}^n} \right|, \quad V_T^2\left(B,\Pi^n\right) = \sum_{i=1}^{n} \left(B_{t_i^n} - B_{t_{i-1}^n} \right)^2.$$





Numerical experiment indicates that total varation is not bounded but quadratic variation converges to positive value.

Quadratic variation of Brownian motion is calculated as a deterministic limit of almost every path

Theorem 135 (Quadratic variation of Brownian motion)

Let $\{\Pi^n\}$ *be a sequence of partitions of* $[0, T]$ *such that* $\sum_{n=1}^{\infty} |\Pi^n| < \infty$. *Then*

$$\lim_{n\to\infty} V_t^2\left(B\left(\omega\right), \Pi^n\right) = t, \quad \textit{for all } t \in \left[0, T\right], \quad \mathbb{P} - \textit{a.s.}$$

- Note, condition $\sum_{n=1}^{\infty} |\Pi^n| < \infty$ implies

$$\lim_{n\to\infty} |\Pi^n| = 0,$$

but it is a stronger condition.

First we scetch the proof and state an additional result

The proof of Thm. 135 will follow the following steps:

1. We show convergence as an L^2 limit. This gives convergence for a subsequence.
2. Use Borel-Cantelli lemma to conclude that probability of not converging to limit *infinitely often* is zero.
3. Construct converging sequences that enclose the limit for almost all paths.

Before going into the details we state the Borel-Cantelli lemma.

Borell-Cantelli lemma is about sequences of events and when events occur infinetely often

- For a sequence of events $\{A_n\} \subset \mathcal{F}$ we define

$$\limsup_{n\to} A_n := \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \right).$$

- Probabilistically, the event $\limsup_{n\to} A_n$ is the subset of outcomes that occur infinitely often.

Lemma 136 (Borel-Cantelli)

Let $\{A_n\}$ be a sequence of events in $(\Omega, \mathcal{F}, \mathbb{P})$. If $\sum_{n=1}^{\infty} \mathbb{P}[A_n] < \infty$, then

$$\mathbb{P}\left[\limsup_{n\to} A_n\right] = 0.$$

We proof convergence of quadratic variation of Brownian motion in L^2 (1/.)

For the proof we assume $t \in \Pi^n$ to simplify notation. To make the reasoning fully precise for all t we could use a standard ε-argument.

We denote $\bar{n}$ such that $t = t_{\bar{n}} \in \Pi^n$.

Recall that variance of Brownian motion directly gives the expectation

$$\mathbb{E}\left[V_t^2\left(B_t, \Pi^n\right)\right] = \sum_{i=1}^{\bar{n}} \mathbb{E}\left[\left(B_{t_i} - B_{t_{i-1}}\right)^2\right] = \sum_{i=1}^{\bar{n}} \left(t_i - t_{i-1}\right) = t.$$

As a first step want to show that also

$$\mathbb{E}\left[\left(V_t^2\left(B_t, \Pi^n\right) - t\right)^2\right] \to 0.$$

We proof convergence of quadratic variation of Brownian motion in L^2 (2/.)

We re-write

$$\begin{aligned} V_t^2\left(B_t, \Pi^n\right) - t &= \sum_{i=1}^{\bar{n}} \left(B_{t_i} - B_{t_{i-1}}\right)^2 - \sum_{i=1}^{\bar{n}} \left(t_i - t_{i-1}\right) \\ &= \sum_{i=1}^{\bar{n}} \underbrace{\left(B_{t_i} - B_{t_{i-1}}\right)^2 - \left(t_i - t_{i-1}\right)}_{\delta B_i^2}. \end{aligned}$$

We work with δB_i^2 to slightly simplify notation.

We proof convergence of quadratic variation of Brownian motion in L^2 (3/.)

Variance of Brownian motion incremnts gives

$$\mathbb{E}\left[\delta B_i^2\right] = \mathbb{E}\left[\left(B_{t_i} - B_{t_{i-1}}\right)^2\right] - (t_i - t_{i-1}) = 0.$$

Moreover, indepencence of Brownian motion incremnts yields that also δB_i and δB_j are independent for $i \neq j$.

Then we get

$$\mathbb{E}\left[\left(V_t^2\left(B_t, \Pi^n\right) - t\right)^2\right] = \mathbb{E}\left[\left(\sum_{i=1}^{\bar{n}} \delta B_i^2\right)^2\right] = \sum_{i=1}^{\bar{n}} \sum_{j=1}^{\bar{n}} \mathbb{E}\left[\delta B_i^2 \, \delta B_j^2\right]$$

$$= \sum_{i=1}^{\bar{n}} \mathbb{E}\left[\left(\delta B_i^2\right)^2\right] + \sum_{i \neq j}^{\bar{n}} \underbrace{\mathbb{E}\left[\delta B_i^2\right]}_{0} \underbrace{\mathbb{E}\left[\delta B_j^2\right]}_{0}.$$

We proof convergence of quadratic variation of Brownian motion in L^2 (4/.)

As an intermediate result, we get

$$\mathbb{E}\left[\left(V_t^2\left(B_t,\Pi^n\right)-t\right)^2\right]=\sum_{i=1}^{\bar{n}}\mathbb{E}\left[\left\{\left(B_{t_i}-B_{t_{i-1}}\right)^2-\left(t_i-t_{i-1}\right)\right\}^2\right].$$

Using $\mathbb{E}\left[\delta B_i^2\right]=0$ and $\mathrm{Var}\left[X\right]=\mathbb{E}\left[X^2\right]-\mathbb{E}\left[X\right]^2$ gives

$$\begin{aligned}\mathbb{E}\left[\left\{\left(B_{t_i}-B_{t_{i-1}}\right)^2-\left(t_i-t_{i-1}\right)\right\}^2\right]&=\mathrm{Var}\left[\left(B_{t_i}-B_{t_{i-1}}\right)^2-\left(t_i-t_{i-1}\right)\right]\\&=\mathrm{Var}\left[\left(B_{t_i}-B_{t_{i-1}}\right)^2\right].\end{aligned}$$

We proof convergence of quadratic variation of Brownian motion in L^2 (5/.)

Next, we can use the fourth moment of a standard normal random variable.
Then, $\left(B_{t_i} - B_{t_{i-1}}\right) \sim \mathcal{N}\left(0, t_i - t_{i-1}\right)$ yields

$$\operatorname{Var}\left[\left(B_{t_i} - B_{t_{i-1}}\right)^2\right] = \underbrace{\mathbb{E}\left[\left(B_{t_i} - B_{t_{i-1}}\right)^4\right]}_{3(t_i - t_{i-1})^2} - \underbrace{\mathbb{E}\left[\left(B_{t_i} - B_{t_{i-1}}\right)^2\right]^2}_{(t_i - t_{i-1})^2}$$
$$= 2\left(t_i - t_{i-1}\right)^2.$$

Finally, we get L^2-convergence via the estimation

$$\mathbb{E}\left[\left(V_t^2\left(B_t, \Pi^n\right) - t\right)^2\right] = 2\sum_{i=1}^{\bar{n}}\left(t_i - t_{i-1}\right)^2 \leq 2\,|\Pi^n|\,t \to 0 \text{ for } n \to \infty.$$

We find that the variance of the quadratic variation vanishes if the mesh of the partitions goes to zero.

We proof path-wise convergence of quadratic variation of Brownian motion in (1/.)

L^2 convergence implied $\mathbb{P}$-a.s. convergence for a subsequence of $\left\{ V_t^2 \left(B_t, \Pi^n \right) \right\}_n$.

But we aim to prove convergence for the entire sequence.

First, we check assumptions for Borel-Cantelli using Chebyshev's inequality.

We get from Chebyshev's inequality for *all* $\epsilon > 0$,

$$\mathbb{P}\left[\left| V_t^2 \left(B_t, \Pi^n \right) - t \right| > \epsilon \right] \leq \frac{\text{Var}\left[V_t^2 \left(B_t, \Pi^n \right) \right]}{\epsilon^2}.$$

We proof path-wise convergence of quadratic variation of Brownian motion in (2/.)

From L^2-convergence we have

$$\mathrm{Var}\left[V_t^2\left(B_t, \Pi^n\right)\right] = \mathbb{E}\left[\left(V_t^2\left(B_t, \Pi^n\right) - \underbrace{\mathbb{E}\left[V_t^2\left(B_t, \Pi^n\right)\right]}_{t}\right)^2\right] \leq 2\,|\Pi^n|\;T.$$

Consequently,

$$\mathbb{P}\left[\left|V_t^2\left(B_t, \Pi^n\right) - t\right| > \epsilon\right] \leq \frac{2\,|\Pi^n|\;T}{\epsilon^2}$$

and also

$$\sum_{n=1}^{\infty} \mathbb{P}\left[\left|V_t^2\left(B_t, \Pi^n\right) - t\right| > \epsilon\right] \leq \frac{2\,T}{\epsilon^2} \quad \underbrace{\sum_{n=1}^{\infty} |\Pi^n|}_{<\infty \text{ by assumption}} \quad < \infty.$$

We proof path-wise convergence of quadratic variation of Brownian motion in (3/.)

Borel-Cantelli lemma now gives

$$\mathbb{P}\left[\left|V_t^2\left(B_t, \Pi^n\right) - t\right| > \epsilon \text{ infinitely often}\right] = 0.$$

We deduce that for all $t \in [0, T]$ there exists a set N_t with $\mathbb{P}\left[N_t\right] = 0$ and such that for all $\omega \notin N_t$

$$\lim_{n\to\infty} V_t^2\left(B_t\left(\omega\right), \Pi^n\right) = t.$$

It remains to show that $\lim_{n\to\infty} V_t^2\left(B_t\left(\omega\right), \Pi^n\right) = t$ holds $\mathbb{P}$-a.s. independent of t (in N_t).

We proof path-wise convergence of quadratic variation of Brownian motion in (4/.)

We set

$$N := \bigcup_{t \in [0,T] \cap \mathbb{Q}} N_t .$$

Then $[0, T] \cap \mathbb{Q}$ is countable and each N_t has zero probablity. Thus, also

$$\mathbb{P}[N] = 0.$$

By construction, we now get for each $t \in [0, T] \cap \mathbb{Q}$ and $\omega \notin N$ the desired result

$$\lim_{n \to \infty} V_t^2 (B_t(\omega), \Pi^n) = t.$$

It remains to show that the limit also holds for $t \notin [0, T] \cap \mathbb{Q}$.

We proof path-wise convergence of quadratic variation of Brownian motion in (5/.)

Select any $t \notin [0, T] \cap \mathbb{Q}$. Then there exist sequences

- $\{t_k\} \subset [0, T] \cap \mathbb{Q}$ with $t_k \uparrow t$ and
- $\{s_k\} \subset [0, T] \cap \mathbb{Q}$ with $s_k \downarrow t$.

Then for all $\omega \notin N$

$$\lim_{n\to\infty} V^2_{t_k}\left(B_{t_k}(\omega), \Pi^n\right) = t_k \le t \le s_k = \lim_{n\to\infty} V^2_{s_k}\left(B_{s_k}(\omega), \Pi^n\right).$$

From the last inequalities we can conclude that for all $\varepsilon > 0$,

- there exist (a random variable) $n_\varepsilon(\omega)$,
- such that for all $n \ge n_\varepsilon(\omega)$

$$t_k - \varepsilon \le V^2_{t_k}\left(B_{t_k}(\omega), \Pi^n\right) \le V^2_{t}\left(B_{t}(\omega), \Pi^n\right) \le V^2_{s_k}\left(B_{s_k}(\omega), \Pi^n\right) \le s_k + \varepsilon.$$

We proof path-wise convergence of quadratic variation of Brownian motion in (6/.)

Now, we let $k \to \infty$:

For all $\varepsilon > 0$, there exist $n_\varepsilon(\omega)$, such that for all $n \geq n_\varepsilon(\omega)$

$$t - \varepsilon \leq V_t^2\left(B_t(\omega), \Pi^n\right) \leq t + \varepsilon.$$

Consequently,

$$\lim_{n \to \infty} V_t^2\left(B_t(\omega), \Pi^n\right) = t.$$

This concludes the proof.

Outline

We aim at calculating integrals with respect to Brownian motion

Consider a process X with bounded variation and a differentiable function f.

Then, we can calculate the Riemann-Stieltjes integral and get

$$\int_0^t f'(X_s)\, dX_s = f(X_t) - f(X_0)\,.$$

Consider $\langle X \rangle > 0$ (i.e. X is of unbounded variation):

- We still aim at calculating $\int_0^t f'(X_s)\, dX_s$.
- Generalisation is to account for *many small fluctuations generating the quadratic variation*.
- Ito formula with Ito integral provides that generalisation.

Outline

We prepare the scene with a result on the weak convergence of signed measures (1/.)

- Recall, a signed measure μ can be decomposed into non-negative measures $\mu = \mu^+ - \mu^-$ (Hahn-Jordan decomposition)

Definition 137 (Weak convergence of measure)

A sequence of signed measures $\{\mu_n\}$ is said to converge weakly to a signed measure μ if for all bounded, continuous functions f

$$\int f d\mu_n \to \int f d\mu.$$

We prepare the scene with a result on the weak convergence of signed measures (2/.)

Theorem 138

Let $\{\mu_n\}$ be a sequence of signed measures on $[a, b]$. Denote $\{F_{\mu_n}\}$ the corresponding sequence of distribution functions.

The sequence $\{\mu_n\}$ converges weakly to a signed measure μ if and only if

- $\sup_n \|\mu_n\| < \infty$ *and*
- *every subsequence of $\{F_{\mu_n}\}$ contains another subsequence which converges to F_μ (distribution function of μ) for all $t \in [a, b]$ except points of an (at most) countable set.*

If μ is a probability measure then above result simplifies to Portemonteau theorem:

- A sequence of probability measures converges weakly if and only if the associated sequence of distribution functions converges at any point of continuity (of the limit distribution function).

We can calculate the integral against a covariance process (1/.)

Now, we consider a vector-valued process $X : [0, T] \to \mathbb{R}^d$ of continuous quadratic variation:

- Process elements are $X = \left(X^1, \ldots, X^d\right)$.
- Increments of the j-th element are $\Delta_i X^j = X^j_{t_i} - X^j_{t_{i-1}}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, d$.

Corollary 139 (RS integral for covariance processes)

Let $\{\Pi^n\}$ be a sequence of partitions of $[0, T]$ with $|\Pi^n| \to 0$. Let $g : [0, T] \to \mathbb{R}$ be a continuous function. Then

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{1}_{t_i \le t}\, g(t_i) \Delta_i X^j \Delta_i X^k = \int_0^t g(s) d \left\langle X^j, X^k \right\rangle .$$

We can calculate the integral against a covariance process (2/.)

Proof:

We can define signed measues μ_n and μ on $[0, T]$ via the distribution functions

$$\mu_n([0,t)) := \sum_{i=1}^{n} \mathbb{1}_{t_i \leq t} \Delta_i X^j \Delta_i X^k \quad \text{and} \quad \mu([0,t)) := \left\langle X^j, X^k \right\rangle_t .$$

We note that $\mu([0,t))$ is well-defined by the continuity of the

co-variance process.

By definition of the co-variance process we have for all $t \in [0, T]$,

$$\mu_n([0,t)) \to \mu([0,t)) .$$

We can calculate the integral against a covariance process (3/.)

Now, we can also set

$$\int_0^t g(s) d\mu_n(s) = \sum_{i=1}^{n} \mathbb{1}_{t_i \le t}\, g(t_i) \Delta_i X^j \Delta_i X^k$$

and

$$\int_0^t g(s) d\mu(s) = \int_0^t g(s) d \left\langle X^j, X^k \right\rangle .$$

Then, the previous corollary on weak convergence yields

$$\int_0^t g(s) d\mu_n(s) \to \int_0^t g(s) d\mu(s).$$

This gives the result.

Itô formula will be formulated for a sufficiently smooth function taking vector-valued arguments

Denote a function F,

$$F : \mathbb{R}^c \times \mathbb{R}^d \to \mathbb{R}, \quad F(A, X),\ A \in \mathbb{R}^c,\ X \in \mathbb{R}^d.$$

- $F \in C^{1,2}\left(\mathbb{R}^c \times \mathbb{R}^d\right)$, which means F is continuous with
  - first continuous partial derivatives on $\mathbb{R}^c$, and
  - first and second continuous partial derivatives on $\mathbb{R}^d$.
- For $c = 1$, we also consider $F \in C^{1,2}\left([0, T) \times \mathbb{R}^d\right)$. This means
  - $F \in C^{1,2}\left((0, T) \times \mathbb{R}^d\right)$ (use open intervall), and
  - all partial derivatives extend continuously to $[0, T) \times \mathbb{R}^d$.
- We may drop the domain of the function if it is clear from the context.

We will use first and second partial derivatives of F

- Gradient of F with respect to $A = \left(A^1, \dots, A^c\right)$ is denoted as

$$\nabla_A F = \left(\frac{\partial F}{\partial A^1}, \dots, \frac{\partial F}{\partial A^c}\right).$$

- Gradient of F with respect to $X = \left(X^1, \dots, X^d\right)$ is denoted as

$$\nabla_X F = \left(\frac{\partial F}{\partial X^1}, \dots, \frac{\partial F}{\partial X^d}\right).$$

- Second derivatives of F with respect to X are

$$\frac{\partial^2 F}{\partial X^j \, \partial X^k}, \quad j, k = 1, \dots, d.$$

We define the Itô integral similarly as the RS integral but observe integrand at the left boundaries of the sub-intervals (1/.)

- Recall that RS integral is limit of RS sequence $\{S(X, Y, \Pi^n, \xi^n)\}_{n=1,2,\ldots}$ for arbitrary $\{\xi^n\}$ with $t_{i-1}^n \leq \xi_i^n \leq t_i^n$.
- For Itô integral, we restrict $\{\xi^n\}$ to use the left boundary of the sub-intervals of the partition $\xi_i^n = t_{i-1}^n$.
- Using the left boundary also has an ecoomic justification:
  - Integrand will typically be a trading strategy representing portfolio weights over a small period $[t_{i-1}^n, t_i^n]$.
  - In practice, we need to decide on the strategy at the beginning of the period; we cannot look into the future.
  - This is related to the trading strategy beeing a predictable process.

We define the Itô integral similarly as the RS integral but observe integrand at the left boundaries of the sub-intervals (2/.)

Definition 140 (Itô integral)

Let $X : [0, T] \to \mathbb{R}^d$ be continuous with continuous quadratic variation and $g : \mathbb{R}^d \to \mathbb{R}^d$ continuously differentiable. Moreover, let $\{\Pi^n\}$ be a sequence of partitions with $|\Pi^n| \to 0$. We define the Itô integral for $t \in [0, T]$ as

$$\int_0^t g\left(X_s\right) dX_s := \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{1}_{\{t_i \le t\}} g\left(X_{t_{i-1}}\right)\left(X_{t_i} - X_{t_{i-1}}\right).$$

- Condition of g being continuously differentiable is a technical restriction related to our formulation of the basic Itô formula.
- Itô integral can be defined more generally e.g. by replacing $g\left(X_s\right)$ by a locally square-integrable process adapted to the filtration generated by X.

Itô formula can be viewed as a generalisation of the fundamental theorem of calculus

Theorem 141 (Basic Itô formula)

Let $A : [0, T] \to \mathbb{R}^c$ *be continuous and of bounded variation,* $X : [0, T] \to \mathbb{R}^d$ *be continuous with continuous quadratic variation, and* $F \in C^{1,2}\left(\mathbb{R}^c \times \mathbb{R}^d\right)$*. Then*

$$
\begin{aligned}
F(A_t, X_t) = & F(A_0, X_0) \\
& + \int_0^t \nabla_A F(A_s, X_s)\, dA_s \qquad \text{(RS integral)} \\
& + \int_0^t \nabla_X F(A_s, X_s)\, dX_s \qquad \text{(Itô integral)} \\
& + \frac{1}{2} \sum_{j,k=1}^{d} \int_0^t \frac{\partial^2}{\partial X^j\, \partial X^k} F(A_s, X_s)\, d\left\langle X_s^j, X_s^k \right\rangle \qquad \text{(RS integral).}
\end{aligned}
$$

We prove basic Itô formula (1/.)

The proof is structured along the following steps:

- We prove statement for $c = d = 1$. General case follows by straight forward modifications along the several dimensions.
- We split $F(A_t, X_t) - F(A_0, X_0)$ into seperate differences along A_t and X_t.
- We analyse $F(A_t, \cdot) - F(A_0, \cdot)$ using mean value theorem and properties of RS integral.
- We analyse $F(\cdot, X_t) - F(\cdot, X_0)$ using second order Taylor expansion.

We prove basic Itô formula (2/.)

Consider partition $\{\Pi^n\}$ with $|\Pi^n| \to 0$. Assume for simplicity $t \in \Pi^n$ such that $t = t_{\bar{n}}$.
We can write decompose

$$
\begin{aligned}
F(A_t, X_t) - F(A_0, X_0) &= \sum_{i=1}^{\bar{n}} \left[F(A_{t_i}, X_{t_i}) - F\left(A_{t_{i-1}}, X_{t_{i-1}}\right) \right] \\
&= \sum_{i=1}^{\bar{n}} \left[F(A_{t_i}, X_{t_i}) - F\left(A_{t_{i-1}}, X_{t_i}\right) \right] + \\
&\quad \sum_{i=1}^{\bar{n}} \left[F\left(A_{t_{i-1}}, X_{t_i}\right) - F\left(A_{t_{i-1}}, X_{t_{i-1}}\right) \right].
\end{aligned}
$$

We prove basic Itô formula (3/.)

We analyse

$$\sum_{i=1}^{\bar{n}} \left[F\left(A_{t_i}, X_{t_i}\right) - F\left(A_{t_{i-1}}, X_{t_i}\right) \right].$$

Since F is continuously differentiable in A, mean value theorem yields a sequence $\{a_i\}$ such that $a_i \in \left(\min\left\{A_{t_{i-1}}, A_{t_i}\right\}, \max\left\{A_{t_{i-1}}, A_{t_i}\right\}\right)$ and

$$F\left(A_{t_i}, X_{t_i}\right) - F\left(A_{t_{i-1}}, X_{t_i}\right) = \nabla_A F\left(a_i, X_{t_i}\right) \left[A_{t_i} - A_{t_{i-1}}\right].$$

Moreover, since A is continuous, we also get a sequence $\{\xi_i\}$ with $\xi_i \in [t_{i-1}, t_i]$ and

$$a_i = A_{\xi_i}.$$

We prove basic Itô formula (4/.)

This gives

$$\begin{aligned}
&\sum_{i=1}^{\bar{n}} \left[F\left(A_{t_i}, X_{t_i}\right) - F\left(A_{t_{i-1}}, X_{t_i}\right)\right] \\
=&\sum_{i=1}^{\bar{n}} \nabla_A F\left(A_{\xi_i}, X_{t_i}\right) \left[A_{t_i} - A_{t_{i-1}}\right] \\
=&\sum_{i=1}^{\bar{n}} \nabla_A F\left(A_{\xi_i}, X_{\xi_i}\right) \left[A_{t_i} - A_{t_{i-1}}\right] + \\
&\underbrace{\sum_{i=1}^{\bar{n}} \left[\nabla_A F\left(A_{\xi_i}, X_{t_i}\right) - \nabla_A F\left(A_{\xi_i}, X_{\xi_i}\right)\right] \left[A_{t_i} - A_{t_{i-1}}\right]}_{R^A_{\bar{n}}}.
\end{aligned}$$

We prove basic Itô formula (5/.)

For the residual part $R_{\bar{n}}^A$ we estimate

$$\left|R_{\bar{n}}^A\right| \leq \sup_{i=1,\ldots,\bar{n}} \left|\nabla_A F\left(A_{\xi_i}, X_{t_i}\right) - \nabla_A F\left(A_{\xi_i}, X_{\xi_i}\right)\right| \sum_{i=1}^{\bar{n}} \left|A_{t_i} - A_{t_{i-1}}\right|$$

- $\sum_{i=1}^{\bar{n}} \left|A_{t_i} - A_{t_{i-1}}\right| \to C < +\infty$ due to bounded variation of A.
- $\max_{i=1,\ldots,\bar{n}} \left|\nabla_A F\left(A_{\xi_i}, X_{t_i}\right) - \nabla_A F\left(A_{\xi_i}, X_{\xi_i}\right)\right| \to 0$ due to $\xi_i \uparrow t_i$ and F continuously differentiable and X uniformly continuous on $[0, T]$.

Consequently, $\left|R_{\bar{n}}^A\right|$ vanishes for $\bar{n} \to \infty$ and we can concentrate on

$$\sum_{i=1}^{\bar{n}} \nabla_A F\left(A_{\xi_i}, X_{\xi_i}\right) \left[A_{t_i} - A_{t_{i-1}}\right].$$

We prove basic Itô formula (6/.)

We can now define $Y_t = \nabla_A F(A_t, X_t)$. Then

$$\begin{aligned}
&\lim_{\bar{n}\to\infty} \sum_{i=1}^{\bar{n}} \left[F(A_{t_i}, X_{t_i}) - F\left(A_{t_{i-1}}, X_{t_i}\right)\right] \\
&= \lim_{\bar{n}\to\infty} \sum_{i=1}^{\bar{n}} Y_{\xi_i} \left[A_{t_i} - A_{t_{i-1}}\right] + \underbrace{\lim_{\bar{n}\to\infty} R^A_{\bar{n}}}_{0} \\
&= \int_0^t Y_s \, dA_s \qquad \text{(definition of RS integral)} \\
&= \int_0^t \nabla_A F(A_s, X_s) \, dA_s .
\end{aligned}$$

This proves the deterministic part.

We prove basic Itô formula (7/.)

Next, we analyse

$$\sum_{i=1}^{\bar{n}} \left[F\left(A_{t_{i-1}}, X_{t_i}\right) - F\left(A_{t_{i-1}}, X_{t_{i-1}}\right) \right].$$

Second order Taylor expansion in t_{i-1} yields

$$\begin{aligned} & F\left(A_{t_{i-1}}, X_{t_i}\right) - F\left(A_{t_{i-1}}, X_{t_{i-1}}\right) \\ = & \nabla_X F\left(A_{t_{i-1}}, X_{t_{i-1}}\right) \left[X_{t_i} - X_{t_{i-1}}\right] \\ & + \frac{1}{2} \frac{d^2}{dX^2} F\left(A_{t_{i-1}}, X_{t_{i-1}}\right) \left[X_{t_i} - X_{t_{i-1}}\right]^2 + R_i^{\bar{n}}. \end{aligned}$$

The Lagrange form of the remainder $R_i^{\bar{n}}$ is

$$R_i^{\bar{n}} = \frac{1}{2} \left(\frac{d^2}{dX^2} F\left(A_{t_{i-1}}, x_i\right) - \frac{d^2}{dX^2} F\left(A_{t_{i-1}}, X_{t_{i-1}}\right) \right) \left[X_{t_i} - X_{t_{i-1}}\right]^2$$

for a sequence $\{x_i\}$ with $x_i \in \left(\min\left\{X_{t_{i-1}}, X_{t_i}\right\}, \max\left\{X_{t_{i-1}}, X_{t_i}\right\}\right)$.

We prove basic Itô formula (7/.)

We have

$$\lim_{\bar{n}\to\infty}\sum_{i=1}^{\bar{n}} \nabla_X F\left(A_{t_{i-1}}, X_{t_{i-1}}\right)\,\left[X_{t_i} - X_{t_{i-1}}\right] = \int_0^t \nabla_X F\left(A_s, X_s\right)\, dX_s$$

by definition of the Itô integral.

Moreover, we can set $g(t) = \frac{d^2}{dX^2} F\left(A_t, X_t\right)$ and apply Cor. 139 to deduce

$$\lim_{\bar{n}\to\infty}\sum_{i=1}^{\bar{n}} \frac{d^2}{dX^2} F\left(A_{t_{i-1}}, X_{t_{i-1}}\right)\,\left[X_{t_i} - X_{t_{i-1}}\right]^2 = \int_0^t \frac{d^2}{dX^2} F\left(A_s, X_s\right)\, d\left\langle X, X\right\rangle_s .$$

Thus, it remains to show that

$$\lim_{\bar{n}\to\infty}\sum_{i=1}^{\bar{n}} R_i^{\bar{n}} = 0.$$

We prove basic Itô formula (8/.)

We note that X is uniformly continuous on $[0, T]$, so

$$\max_{i=1,\dots,\bar{n}} \left| X_{t_i} - X_{t_{i-1}} \right| \le \delta_{\bar{n}} \to 0.$$

Consequently, we can estimate

$$\left| R_i^{\bar{n}} \right| \le \underbrace{\max_{i=1,\dots,\bar{n}} \left\{ \frac{d^2}{dX^2} F\left(A_{t_{i-1}}, x\right) - \frac{d^2}{dX^2} F\left(A_{t_{i-1}}, y\right) \ : \ |x - y| \le \delta_{\bar{n}} \right\}}_{\hat{R}^{\bar{n}}} \cdot \left[X_{t_i} - X_{t_{i-1}} \right]^2 .$$

Continuity of $\frac{d^2}{dX^2} F$ and A yields that $\hat{R}^{\bar{n}} \to 0$ for $\bar{n} \to \infty$.

We prove basic Itô formula (9/.)

Continuous quadratic variation of X finally yields

$$\sum_{i=1}^{\bar{n}} R_i^{\bar{n}} \leq \sum_{i=1}^{\bar{n}} \left| R_i^{\bar{n}} \right| \leq \underbrace{\hat{R}^{\bar{n}}}_{\to 0} \underbrace{\sum_{i=1}^{\bar{n}} \left[X_{t_i} - X_{t_{i-1}} \right]^2}_{\to \langle X, X \rangle_t \leq \langle X, X \rangle_T < \infty} \to 0.$$

This concludes the proof of the basic Itô formula.

Outline

We discuss some consequences of Itô formula relevant for continuous time finace models

- Continuity of Itô integral,
- Itô formula for time- and state-dependent functions,
- Itô product rule,
- Itô integral for some spcific state-dependent functions, and
- Geometric Brownian motion as foundation of Black-Scholes model.

Itô integrals are continuous

- Recall that RS integral depends continuously on the boundaries of integration (Thm. 134).

Corollary 142 (Continuity of Itô integral)

Under the assumptions of the basic Itô formula (Thm. 141), the Itô integral $\int_0^t \nabla_X F(A_s, X_s)\, dX_s$ *is well defined and*

$$t \mapsto \int_0^t \nabla_X F(A_s, X_s)\, dX_s$$

is continuous.

We can specialise Itô formula for time- and state-dependent function

- Set $A_t = t$ and $X : [0, T] \to \mathbb{R}$.

Corollary 143

Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be of class $C^{1,2}$ and $X : [0, T] \to \mathbb{R}$ be continuous with continuous quadratic variation. Then

$$
\begin{aligned}
f(t, X_t) - f(0, X_0) = &\int_0^t f_t(s, X_s)\, ds \qquad \textit{(Riemann integral)} \\
&+ \int_0^t f_x(s, X_s)\, dX_s \qquad \textit{(Itô integral)} \\
&+ \frac{1}{2} \int_0^t f_{xx}(s, X_s)\, d\langle X, X\rangle_s \qquad \textit{(RS integral).}
\end{aligned}
$$

Itô product rule is a useful tool for covariance calculation

- Set $X = \left(X^1, X^2\right)$ and $F\left(A, X\right) = F(X) = X^1\,X^2$.

Corollary 144

Let $X = \left(X^1, X^2\right)$ be continuous with continuous quadratic variation. Then

$$X_t^1\,X_t^2 - X_0^1\,X_0^2 = \int_0^t X_s^1\,dX_s^2 + \int_0^t X_s^2\,dX_s^1 + \left\langle X^1, X^2\right\rangle_t .$$

Most relevant application for us is Itô formula for Brownian motion

- Set $X = B$, $F(A, X) = F(B)$.

Examples 145

Let B be a standard Brownian motion and $F : \mathbb{R} \to \mathbb{R}$ twice continuously differentiable. Then (with $\langle B_t, B_t \rangle = t$)

$$F(B_t) - F(B_0) = \int_0^t F'(B_s)\, dB_s + \frac{1}{2}\int_0^t F''(B_s)\, ds.$$

The special case $F(B_t) = B_t^2$ yields

$$B_t^2 = 2\int_0^t B_s\, dB_s + t.$$

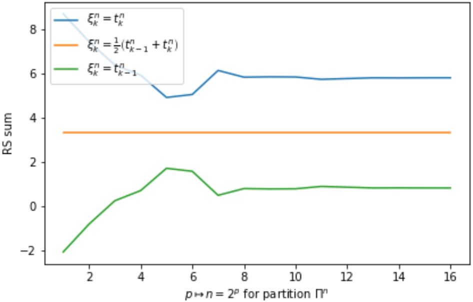
Previous example gives analytic expression for integral of Browninan motion with itself

We get

$$\int_0^t B_s \, dB_s = \left(B_t^2 - t\right)/2.$$

Compare with

$$\int_0^s B_t(\omega) \; dB_t(\omega) \approx S\left(B_t(\omega), B_t(\omega), \Pi^n, \xi^n\right), \qquad n = 2^p, \; p = 1, 2, \ldots$$



We introduce Geometric Brownian motion as basis of Black-Scholes model

Definition 146 (Geometric Brownian motion)

Let B be a standard Brownian motion, $\mu \in \mathbb{R}$, $\sigma > 0$, and $S_0 > 0$. The geometric Brownian motion is defined as

$$S_t := S_0 \exp\left\{\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right\}.$$

- μ is denoted the drift parameter,
- σ is denoted the volatility parameter, and
- S_0 is the initial underlying value.

We want to describe the evolution from S_0 to S_t (1/.)

Substitute $X_t := \sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right) t$ and use $F(X) = S_0 e^X$.

This gives the identities

$$S_t = F(X_t) = F'(X_t) = F''(X_t)$$

Then Itô formula gives

$$\begin{aligned} S_t =& F(X_0) + \int_0^t F'(X_u)\, dX_u + \frac{1}{2}\int_0^t F''(X_u)\, d\langle X, X\rangle_u \\ =& S_0 + \int_0^t S_u\, dX_u + \frac{1}{2}\int_0^t S_u\, d\langle X, X\rangle_u . \end{aligned}$$

Linearity of integrals gives

$$dX_u = \sigma\, dB_u + \left(\mu - \frac{1}{2}\sigma^2\right) du.$$

We want to describe the evolution from S_0 to S_t (2/.)

Moreover, we will see that

$$\langle X, X \rangle_t = \sigma^2 t.$$

Then

$$\begin{aligned} S_t =& S_0 + \int_0^t S_u \left[\sigma \, dB_u + \left(\mu - \frac{1}{2}\sigma^2 \right) du \right] + \frac{1}{2} \int_0^t S_u \sigma^2 \, du \\ =& S_0 + \int_0^t S_u \, \mu \, du + \int_0^t S_u \, \sigma \, dB_u. \end{aligned}$$

In differential form we arrive at

$$\frac{dS_t}{S_t} = \mu \, dt + \sigma \, dB_t.$$

We want to describe the evolution from S_0 to S_t (3/.)

Assume dynamics of a stock price S_t given as geometric Brownian motion,

$$\frac{dS_t}{S_t} = \mu \, dt + \sigma \, dB_t.$$

What does this mean econoically?

- Average (relative) growth rate of the stock is μ.
- Relative returns fluctuate (around average rate) according to $\sigma \, dB_t$.

Above properties form the assumptions underlying the Black-Scholes model.

Outline

How does quadratic variation of a process extend to functions of that process and integrals based on that process?

- We already know quadratic variation $\langle B, B\rangle_t = \langle B\rangle_t = t$ for Brownian motion B.
- What is the quadratic variation
  - $\langle g(B)\rangle_t$ for a function g or
  - $\langle Y\rangle_t$ for an Itô integral $Y_t = \int_0^t g(B_s)\, dB_s$?

Quadratic variation does extend to functions of a process (1/.)

Theorem 147 (Quadratic variation of function of a process)

Let $X : [0, T] \to \mathbb{R}$ be continuous and of continuous quadratic variation. Let $g \in C^1$ (continuously differentiable). Then the process $(g(X_t))$ is of continuous quadratic variation given by

$$\langle g(X) \rangle_t = \int_0^t g'(X_s)^2 \, d\langle X \rangle_s \,.$$

Quadratic variation does extend to functions of a process (2/.)

Proof:
Consider partition $\{\Pi^n\}$ with $|\Pi^n| \to 0$. Assume for simplicity $t \in \Pi^n$ such that $t = t_{\bar{n}}$.
Quadratic variation (for the partition) is

$$V_t^2\left(g\left(X\right), \Pi^n\right) = \sum_{i=1}^{\bar{n}} \left[g\left(X_{t_i}\right) - g\left(X_{t_{i-1}}\right)\right]^2.$$

Apply Taylor expansion gives

$$\begin{aligned} g\left(X_{t_i}\right) - g\left(X_{t_{i-1}}\right) =& g'\left(X_{t_{i-1}}\right)\left[X_{t_i} - X_{t_{i-1}}\right] \\ &+ \left[g'\left(x_i^{\bar{n}}\right) - g'\left(X_{t_{i-1}}\right)\right]\left[X_{t_i} - X_{t_{i-1}}\right]. \end{aligned}$$

Quadratic variation does extend to functions of a process (3/.)

Substitution into $V_t^2(g(X), \Pi^n)$ yields

$$V_t^2(g(X), \Pi^n) = \sum_{i=1}^{\bar{n}} g'\left(X_{t_{i-1}}\right)^2 \left[X_{t_i} - X_{t_{i-1}}\right]^2 + \sum_{i=1}^{\bar{n}} R_i^{\bar{n}}.$$

For the remainder $R_i^{\bar{n}}$ we have

$$\begin{aligned} R_i^{\bar{n}} =& 2g'\left(X_{t_{i-1}}\right)\left[g'\left(x_i^{\bar{n}}\right) - g'\left(X_{t_{i-1}}\right)\right]\left[X_{t_i} - X_{t_{i-1}}\right]^2 \\ &+ \left[g'\left(x_i^{\bar{n}}\right) - g'\left(X_{t_{i-1}}\right)\right]^2\left[X_{t_i} - X_{t_{i-1}}\right]^2 \\ =& \underbrace{\left\{2g'\left(X_{t_{i-1}}\right) + \left[g'\left(x_i^{\bar{n}}\right) - g'\left(X_{t_{i-1}}\right)\right]\right\}}_{|\cdot| \to 2\left|g'\left(X_{t_{i-1}}\right)\right| \le C < \infty} \\ &\cdot \underbrace{\left[g'\left(x_i^{\bar{n}}\right) - g'\left(X_{t_{i-1}}\right)\right]}_{\to 0}\left[X_{t_i} - X_{t_{i-1}}\right]^2. \end{aligned}$$

Quadratic variation does extend to functions of a process (4/.)

We find that

$$\sum_{i=1}^{\bar{n}} \left| R_i^{\bar{n}} \right| \leq \underbrace{C_{\bar{n}}}_{\to 0} \underbrace{\sum_{i=1}^{\bar{n}} \left[X_{t_i} - X_{t_{i-1}} \right]^2}_{\to \langle X \rangle_t \leq \langle X \rangle_T < \infty} \to 0.$$

As a result we get

$$\begin{aligned} \langle g(X) \rangle_t &= \lim_{\bar{n} \to \infty} V_t^2 \left(g(X), \Pi^n \right) \\ &= \underbrace{\lim_{\bar{n} \to \infty} \sum_{i=1}^{\bar{n}} g'\left(X_{t_{i-1}} \right)^2 \left[X_{t_i} - X_{t_{i-1}} \right]^2}_{\text{RS integral}} + \underbrace{\lim_{\bar{n} \to \infty} \sum_{i=1}^{\bar{n}} R_i^{\bar{n}}}_{\to 0} \\ &= \int_0^t g'(X_s)^2 \, d \langle X \rangle_s . \end{aligned}$$

Quadratic variation does extend to Itô integrals (1/.)

Corollary 148 (Quadratic variation of Itô integrals)

Let $X : [0, T] \to \mathbb{R}$ be continuous and of continuous quadratic variation. Let $g \in C^1$ (continuously differentiable). Then the process

$$Y_t = \int_0^t g\left(X_s\right)\, dX_s$$

has continuus quadratic variation given by

$$\langle Y \rangle_t = \int_0^t g^2\left(X_s\right)\, d\,\langle X \rangle_s\,.$$

- Corollary can be viewed as a special case of Itô isometry.

Quadratic variation does extend to Itô integrals (2/.)

Proof:
Denote

$$G(x) = G(X_0) + \int_{X_0}^{x} g(u)\,du.$$

Then $G'(x) = g(x)$.
Itô formula yields

$$\int_0^t g(X_s)\ dX_s = G(X_t) - \underbrace{G(X_0)}_{\text{const.}} - \frac{1}{2} \underbrace{\int_0^t g'(X_s)\ d\langle X\rangle_s}_{\text{bounded variation of RS integral}} .$$

Consequently, with Thm. 147, we get

$$\langle Y\rangle_t = \langle G(X)\rangle_t = \int_0^t g^2(X_s)\ d\langle X\rangle_s .$$

Now we can also calculate the quadratic variation of a geometric Brownian motion

Example 149

Recall the geometric Brownian motion process

$$S_t = S_0 + \int_0^t \mu\, S_u\, du + \int_0^t \sigma\, S_u\, dB_u.$$

Drift $\int_0^t \mu\, S_u\, du$ is of bounded variation. Consequently,

$$\langle S \rangle_t = \int_0^t (\sigma\, S_u)^2\, d\,\langle B \rangle_u = \int_0^t (\sigma\, S_u)^2\, du.$$

- Example has some implications on paramter estimation for μ and σ from time series.
- See lecture notes, Example 7.6.3 and proceeding discussion.

Results on quadratic variation can also be extended to smooth time-dependent functions

Let $X : [0, T] \to \mathbb{R}$ be continuous and of continuous quadratic variation and $g \in C^{1,1}([0, T] \times \mathbb{R})$.

- The process $(g(t, X_t))$ is of continuous quadratic variation given by

$$\langle g(\cdot, X) \rangle_t = \int_0^t g_x(s, X_s)^2 \, d\langle X \rangle_s .$$

- The process $Y_t = \int_0^t g(s, X_s)\, dX_s$ is of continuous quadratic variation given by

$$\langle Y \rangle_t = \int_0^t g(s, X_s)^2 \, d\langle X \rangle_s .$$

Outline

Outline

We consider a continuous time model with a risk-free rate and a risky asset

Continuous time model

- Consider time horizon $[0, T]$.
- Risk-free rate $r \in \mathbb{R}$.
- Risky asset process $S = (S_t)$ for $t \in [0, T]$.

European option with payoff at maturity

- Denote H_S a derivative with payoff paid at T.
- Assume payoff depends on the risky asset S only at T, i.e.

$$H_S = f_S(S_T).$$

- For example, a European call option has payoff function $f_S(S_T) = [S_T - K]^+$.

Setting can be made more general. But then we need more (probabilistic) theory on continuous time processes.

Risk-free rate allows us to design a numeraire asset

Continuously compounded bank account

- Denote $S^0 = \left(S^0_t\right)$ for $t \in [0, T]$ the price process of a savings account.
- W.l.o.g. assume initial price $S^0_0 = 1$, i.e. we consider the price of one unit of money.
- Assume savings account accrues interest at the continuous rate r; this can be generalised to deterministic or random process (r_t).

Price process S^0 becomes

$$S^0_t = e^{\int_0^t r\,dt} = e^{r\,t}.$$

We have $S^0_t > 0$ and thus S^0 is a valid numeraire asset in our model.

Discounted assets and derivative prices allow us to link continuous time models to results from discrete time models

Discounted risky asset price process

Define discounted price process $X = (X_t)$ for $t \in [0, T]$ via

$$X_t = \frac{S_t}{S_t^0}.$$

Discounted option and payoff function

Define discounted option H at maturity via

$$H = H_S \,/\, S_T^0.$$

And corresponding (discounted) payoff function

$$H = f\left(X_T\right) := \frac{f_S\left(S_T^0\, X_T\right)}{S_T^0}.$$

What do we know from discrete time models?

Assume S (and X) are defined on a filtered space $(\Omega, (\mathcal{F}_{t_i}), \mathbb{P})$.

- Absence of arbitrage implies existence of EMM $\mathbb{P}^*$ such that (X_{t_i}) is a martingale (under $\mathbb{P}^*$).
- For any self-financing strategy (ξ_{t_i}) the discounted value process

$$V_{t_i} = V_0 + \sum_{i=1}^{n} \xi_{t_i} \left(X_{t_i} - X_{t_{i-1}} \right) \quad \text{is a martingale.}$$

- If discounted option H is attainable (i.e. $\exists\, \xi$ s.t. $V_T = H$), then

$$V_{t_i} = \mathbb{E}^* \left[H \,|\, \mathcal{F}_{t_i} \right].$$

Discounted option price process is also a $\mathbb{P}^*$-martingale.

What else do we know from the Binomial model?

- Discounted value process (V_{t_i}) of H can be represented as a sequence of value functions (v_{t_i}),

$$V_{t_i} = v_{t_i}\left(X_{t_i}\right).$$

- The functions v_{t_i} are the pricing formulas in the Binomial model.
- They can be evaluated once we know the discounted underlying X_{t_i}.
- The (Delta-)hedging or replication strategy becomes

$$\xi_{t_i} = e^{r\,(t_i - t_{i-1})}\,\frac{v_{t_i}\left(X_{t_{i-1}}\hat{b}\right) - v_{t_i}\left(X_{t_{i-1}}\hat{a}\right)}{X_{t_{i-1}}\left(\hat{b} - \hat{a}\right)} \approx \frac{dv_{t_i}}{dx}\left(X_{t_{i-1}}\right).$$

We aim at extending the results for value functions and hedging strategy to continuous time models.

In continuous time models we can expect the value process to become an integral

Consider a continuous process (X_t) and an option with payoff function

$$H = f(X_T).$$

We want to find a value process (V_t) such that for all ω

$$V_T(\omega) = H(\omega) = f(X_T(\omega))$$

and

$$V_t(\omega) = V_0 + \int_0^t \xi_u(\omega)\, dX_u$$

for an initial investment V_0 and a self-financing trading strategy (ξ_t).

We formulate a general local volatility model for the discounted asset process

Assume discounted asset X is defined on a filtered space $(\Omega, (\mathcal{F}_t), \mathbb{P})$ on a horizon $[0, T]$.

- Measure $\mathbb{P}$ is assumed the real-world probability measure.
- We assume a ($\mathbb{P}$-)Brownian motion (B_t) as risk factor generating the canonical filtration $\mathcal{F}_t = \sigma(B_s, s \leq t)$.

Local volatility model

$$X_t = X_0 + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s.$$

- Initial price $X_0 > 0$.
- Continuous $\mu : [0, T] \times \mathbb{R} \to \mathbb{R}$ representing the drift.
- Smooth $\sigma \in C^{1,1}([0, T] \times \mathbb{R})$ representing the local volatility.

Price process falls within the scope of Itô calculus theory

For each $\omega \in \Omega$, the process $(X_t(\omega))$ is of continuous quadratic variation

$$\langle X(\omega) \rangle_t = \int_0^t \sigma(s, X_s(\omega))^2 \, ds.$$

We will later discuss two important instances:

- Bachelier model with $\sigma(t, x) = \sigma$ (constant), and
- Black-Scholes model with $\sigma(t, x) = \sigma x$.

Outline

We state the PDE pricing result in terms of discounted prices

Theorem 150 (Option pricing PDE)

Suppose (X_t) *follows a local volatility model with local volatility function* $\sigma(t,x)$ *and* $V \in C^{1,2}([0,T] \times \mathbb{R})$ *solves the terminal value problem*

$$\frac{1}{2}\sigma(t,x)\, V_{xx}(t,x) + V_t(t,x) = 0, \quad (t,x) \in [0,T) \times \mathbb{R},$$
$$V(T,x) = f(x).$$

Then we have for all $\omega \in \Omega$ *that*

$$H(\omega) := f(X_T(\omega)) = V(0,X_0) + \int_0^T V_x(u, X_u(\omega))\, dX_u(\omega).$$

- PDE can also be fomulated in terms of undiscounted prices $S_t = S_t^0 X_t = e^{r\,t} X_t$.
- Then, we arrive at the famous Black-Scholes pricing PDE.

PDE pricing is established via Itô formula (1/.)

Proof:
We need to be slightly careful about differentiability of V w.r.t. t at the boundaries of $[0, T]$.
Consider $\epsilon > 0$. Itô formula gives

$$\begin{aligned} V\left(T-\epsilon, X_{T-\epsilon}(\omega)\right) = & V\left(\epsilon, X_{\epsilon}(\omega)\right) \\ & + \int_{\epsilon}^{T-\epsilon} V_t\left(u, X_u(\omega)\right) du \\ & + \int_{\epsilon}^{T-\epsilon} V_x\left(u, X_u(\omega)\right) dX_u(\omega) \\ & + \frac{1}{2}\int_{\epsilon}^{T-\epsilon} V_{xx}\left(u, X_u(\omega)\right) d\left\langle X(\omega)\right\rangle_u . \end{aligned}$$

PDE pricing is established via Itô formula (2/.)

Recall that X is continuous and of continuous quadratic variation

$$\langle X(\omega)\rangle_t = \int_0^t \sigma(u, X_u(\omega))^2 \, du.$$

In particular, $t \mapsto \langle X(\omega)\rangle_t$ is differentiable such that

$$d\langle X(\omega)\rangle_u = \sigma(u, X_u(\omega))^2 \, du.$$

This gives

$$\begin{aligned}
& V(T-\epsilon, X_{T-\epsilon}(\omega)) \\
=& V(\epsilon, X_\epsilon(\omega)) + \int_\epsilon^{T-\epsilon} V_x(u, X_u(\omega)) \, dX_u(\omega) \\
& + \int_\epsilon^{T-\epsilon} \underbrace{\left[V_t(u, X_u(\omega)) + \frac{1}{2} V_{xx}(u, X_u(\omega)) \sigma(u, X_u(\omega))^2\right]}_{0 \text{ (PDE solution)}} du.
\end{aligned}$$

PDE pricing is established via Itô formula (3/.)

Taking limit $\epsilon \downarrow 0$ yields with continuity in V and X, and terminal condition that

$$\lim_{\epsilon \downarrow 0} V\left(T-\epsilon, X_{T-\epsilon}(\omega)\right) = V\left(T, X_T(\omega)\right) = f\left(X_T(\omega)\right).$$

Moreover, we get

$$\lim_{\epsilon \downarrow 0} V\left(T-\epsilon, X_{T-\epsilon}(\omega)\right) = \lim_{\epsilon \downarrow 0} \Big\{ V\left(\epsilon, X_\epsilon(\omega)\right) + \underbrace{\int_0^{T-\epsilon} V_x\left(u, X_u(\omega)\right) \, dX_u(\omega)}_{\to \int_0^T \cdot} - \underbrace{\int_0^{\epsilon} V_x\left(u, X_u(\omega)\right) \, dX_u(\omega)}_{\to 0} \Big\}.$$

PDE pricing is established via Itô formula (4/.)

With $V(\epsilon, X_\epsilon(\omega)) \to V(0, X_0)$ we get the desired result

$$f(X_T(\omega)) = V(0, X_0) + \int_0^T V_x(u, X_u(\omega))\, dX_u(\omega).$$

This concludes the proof.

We can state two important consequences from the PDE solution

Replication strategy from option delta

From the solution V we get the option's replicating strategy by

$$\xi_t(\omega) = V_x(t, X_t(\omega)).$$

We can hedge the option with Delta of the (discounted) value function.

Independence of (real-world) drift

The solution V does not depend on the model's real-world drift $\mu(t, x)$.

Consequently, for option pricing, drift estimation is not required.

However, the option price V and the hedging strategy $\xi(\omega) = V_x(\cdot, X(\omega))$ do depend on the volatility $\sigma(t, x)$.

A key aspect of option pricing and hedging is appropriate modelling and calibration of volatility function.

The PDE solution represents the unique arbitrage-free price for the option (1/.)

- Suppose $V(t,x)$ solves the pricing PDE in Thm. 150.
- An option writer can use the trading strategy $\xi = V_x$ to eliminate all risk associated with issuing the option H.
- Implementing the strategy ξ requires the initial investment $V(0, X_0)$.

Fact 151

$V(0, X_0)$ is the unique arbitrage-free price of the option H.

We can construct a portfolio of option and trading strategy to verify the result.

The PDE solution represents the unique arbitrage-free price for the option (2/.)

Suppose $\pi(H)$ is a tradeable price of the option. Then

- Sell option at $t = 0$ for $\pi(H)$.
- Invest proceeds $\pi(H)$ (from selling the option) into numeraire asset (i.e. savings account).
- Trade the ξ with $\xi = V_x$.

At $t = T$, we have (in terms of discounted quantities)

$$\underbrace{-H(\omega)}_{\text{deliver payoff}} + \underbrace{\int_0^T V_x(u, X_u(\omega))\, dX_u(\omega)}_{\text{returns from strategy}} + \underbrace{\pi(H)}_{\substack{\text{discounted}\\ \text{initial}\\ \text{price}}} = \underbrace{\pi(H) - V(0, X_0)}_{\text{indepenent of } \omega}.$$

If $\pi(H) > V(0, X_0)$, we have an arbitrage opportunity. Arbitrage opportunity can be constructed similarly if $\pi(H) < V(0, X_0)$.

Can PDE pricing be applied to exotic options as well?

- So far, we established value functions V for Vanilla options with payoff $H = f(X_T)$ determined only from the asset value at maturity T.
  - Important representatives are call and put options with
$$f_{\text{Call}}(X_T) = [X_T - K]^+ ,$$
$$f_{\text{Put}}(X_T) = [K - X_T]^+ .$$
- PDE pricing can also be extended to more exotic options, e.g.
  - barrier options and
  - American options.
- In fact, PDE methods often are the *method of choice* for exotic options in practice. For details, see e.g. [4].

We briefly discuss two examples to illustrate how PDE methods can be used for exotic options.

For a Down-and-Out put option we need to add Dirichlet boundary conditions at the lower boundary

We have the terminal value problem for the put option:

$$\frac{1}{2}\sigma(t,x)\,V_{xx}(t,x) + V_t(t,x) = 0, \qquad (20)$$
$$V(T,x) = [K-x]^+ .$$

Knock-out condition says: Option becomes worthless, i.e. $V = 0$, if X_t crosses the lower boundary L.

- Suppose we restrict option pricing to *alive region*. Then PDE (20) still holds for $x \in (L, \infty)$.
- In addition, for $X_t \le L$, we have $V(t, X_t) = 0$.

Knock-out condition translates into a Dirichlet boundary condition

$$V(t,L) = 0 \quad \text{for} \quad t \in [0,T).$$

American put option pricing can be formulated as free boundary value problem

Again, we have the terminal value problem for the put option:

$$\frac{1}{2}\sigma(t,x)\,V_{xx}(t,x) + V_t(t,x) = 0 \tag{21}$$

$$V(T,x) = [K-x]^+ .$$

We know from American option pricing with CRR model:

- There is a stopping region $\mathcal{R}_s$ and a continuation region $\mathcal{R}_c$ which are divided by an exercise boundary $b : [0,T) \to \mathbb{R}$.
  - For $(t,x) \in \mathcal{R}_c$, i.e. no early exercise, the PDE (21) holds.
  - At (and below) the boundary $(t, b(t))$ option price V equals option payoff $[K - X_t]^+$.

American exercise condition translates into a Dirichlet boundary condition

$$V(t, b(t)) = [K - b(t)]^+ \quad \text{for} \quad t \in [0, T)$$

at the (a priori) unknown exercise boundary b.

Outline

When does the option pricing problem exhibit a solution?

- So far, we *assumed* the existence of $V \in C^{1,2}$.
- Now, we analyse when the terminal value problem in Thm. 150 does have a solution.
- For this analysis, we simplify our model to the Bachelier model with

  $$X_t = X_0 + \mu\, t + \sigma\, B_t.$$

  Here, μ and $\sigma > 0$ are constant.

Pricing PDE for Bachelier model is closely related to heat equation

Terminal value problem in Bachelier model becomes

$$\frac{1}{2}\sigma V_{xx}(t,x) + V_t(t,x) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}, \tag{22}$$
$$V(T,x) = f(x).$$

PDE (22) is closely related to heat equation

$$\frac{1}{2}V_{xx}(t,x) - V_t(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{23}$$

It is easy to check (by differentiation) that heat equation is solved by

$$P(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\}. \tag{24}$$

Function $P(t,x)$ is called the fundamental solution to (23).

Fundamental solution can be used to solve the initial value problem (1/.)

Theorem 152 (Smoothing property of Heat Kernel)

Let $P(t,x)$ be the fundamental solution (24) to the heat equation (23) and $g:\mathbb{R}\to\mathbb{R}$ continuous and bounded. Then the function

$$u(t,x)=\int_{\mathbb{R}} g(y)\,P(t,x-y)\,dy$$

satisfies the properties:

1. *u belongs to the class $C^{\infty}((0,\infty)\times\mathbb{R})$.*
2. *u satisfies the heat equation (23) on $(0,\infty)\times\mathbb{R}$.*
3. *u satisfies the terminal condition such that*

$$\lim_{(t,x)\to(0,y),\ t>0} u(t,x)=g(y)\,.$$

Fundamental solution can be used to solve the initial value problem (2/.)

Proof:
First, we prove $u \in C^\infty((0,\infty)\times\mathbb{R})$. For all $\delta > 0$, $P(t,x)$ is infinitely differentiable with all derivatives bounded on $[\delta,\infty)\times\mathbb{R}$. Since g is bounded, we can interchange differentiation and integration. This gives $u \in C^\infty((0,\infty)\times\mathbb{R})$.

Re-using arguments from above, we get

$$u_t - \frac{1}{2}u_{xx} = \int g \underbrace{\left(P_t - \frac{1}{2}P_{xx}\right)}_{0} dy = 0 \quad \text{on} \quad (0,\infty)\times\mathbb{R}.$$

It remains to show the initial condition

$$u(0,\cdot) = g(\cdot).$$

Fundamental solution can be used to solve the initial value problem (3/.)

We fix an $x^0 \in \mathbb{R}$ ($x^0 = y$). Also we fix an $\varepsilon > 0$.

Since g is continuous, there is a $\delta > 0$ such that

$$\left|g(x) - g\left(x^0\right)\right| < \varepsilon \quad \forall x \text{ with } \left|x - x^0\right| < \delta.$$

We define $B_\delta\left(x^0\right) := \left(x^0 - \delta, x^0 + \delta\right)$. Also note that

$$\int_{\mathbb{R}} P(t, x - y)\, dy = 1.$$

Further, we restrict x such that $\left|x - x^0\right| < \delta/2$. Then

$$\begin{aligned}\left|u(t,x) - g\left(x^0\right)\right| \le &\int_{B_\delta(x^0)} P(t, x-y)\left|g(y) - g\left(x^0\right)\right| dy\\ &+ \int_{\mathbb{R}\setminus B_\delta(x^0)} P(t, x-y)\left|g(y) - g\left(x^0\right)\right| dy.\end{aligned}$$

Fundamental solution can be used to solve the initial value problem (4/.)

We get for the first integral

$$\int_{B_\delta(x^0)} P\left(t, x-y\right)\left|g\left(y\right)-g\left(x^0\right)\right| dy \leq \varepsilon \int_{B_\delta(x^0)} P\left(t, x-y\right) dy \leq \varepsilon.$$

We need to also estimate the second integral. Fur this purpose consider

$$\left|x-x^0\right| < \delta/2 \quad \text{and} \quad \left|y-x^0\right| \geq \delta.$$

This implies

$$\left|y-x^0\right| = \left|y-x+x-x^0\right| \leq \left|y-x\right| + \frac{\delta}{2} \leq \left|y-x\right| + \frac{\left|y-x^0\right|}{2}.$$

And we get

$$\left|y-x\right| \geq \frac{1}{2}\left|y-x^0\right|.$$

Fundamental solution can be used to solve the initial value problem (5/.)

For the second integral we may estimate

$$
\begin{aligned}
&\int_{\mathbb{R}\setminus B_\delta(x^0)} P(t, x-y)\left|g(y) - g\left(x^0\right)\right| dy \\
\leq & 2\left\|g\right\|_\infty \int_{\mathbb{R}\setminus B_\delta(x^0)} P(t, x-y)\, dy \\
\leq & \frac{C}{\sqrt{t}} \int_{\mathbb{R}\setminus B_\delta(x^0)} \exp\left\{-\frac{\left|y-x\right|^2}{2t}\right\} dy \\
\leq & \frac{C}{\sqrt{t}} \int_{\mathbb{R}\setminus B_\delta(x^0)} \exp\left\{-\frac{\left|y-x^0\right|^2}{8t}\right\} dy \\
\leq & \frac{C}{\sqrt{t}} \int_{\delta}^{\infty} \exp\left\{-\frac{\left|y\right|^2}{8t}\right\} dy \to 0 \quad \text{for} \quad t \downarrow 0.
\end{aligned}
$$

Fundamental solution can be used to solve the initial value problem (6/.)

In summary, we find that for all $\varepsilon > 0$ there are $\delta > 0$ and $\Delta > 0$ such that for all (t, x) with

$$\left|x - x^0\right| \leq \frac{\delta}{2} \quad \text{and} \quad 0 < t \leq \Delta,$$

we get

$$\left|u\,(t, x) - g\left(x^0\right)\right| \leq 2\,\varepsilon.$$

This proves the assetion.

We give some further remarks on the solutions of the heat equation

- Assumption that payoff g is continuous and bounded is fairly restrictive.
  - It would not even hold for call options with payoff $[X - K]^+$.
- With some modifications the proof of Thm. 152 can be extended to measurable fuctions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$|f(x)| \leq \alpha \left(1 + e^{C|x|}\right)^2 \tag{25}$$

  for some constants $C, \alpha > 0$.
- General PDE theory shows that the solution to the heat equation is also unique.
  - This will allow for a probabilistic interpretation of the solution.

Now we can apply the results from heat equation to Bachelier model

Corollary 153

Suppose the payoff function $f : \mathbb{R} \to \mathbb{R}$ satisfies the growth condition (25). Then

$$V(t,x) := \int_{\mathbb{R}} f(y)\, P\left(\sigma^2 (T-t), x-y\right) dy$$

belongs to $C^\infty([0,T) \times \mathbb{R})$, solves the Bachelier pricing PDE (22), and satisfies the terminal condition

$$\lim_{t \uparrow T} V(t,x) = f(x).$$

- Proof follows by reversing time and scaling by volatility σ. Details are left as an exercise.

Finally, we link the PDE solution to an expectation

It turns out that price process and replication strategy are represented as functions $V(t,x)$ and $V_x(t,x)$ in terms of one-dimensional integrals.

In particular,

$$\begin{aligned} V(t,x) &= \int_{\mathbb{R}} \frac{f(y)}{\sigma\sqrt{2\pi(T-t)}} \exp\left\{-\frac{(y-x)^2}{2\sigma^2(T-t)}\right\} dy \\ &= \int_{\mathbb{R}} \frac{f\left(x+z\sigma\sqrt{T-t}\right)}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz \\ &= \tilde{\mathbb{E}}\left[f\left(x+\sigma\sqrt{T-t}\,Z\right)\right]. \end{aligned}$$

Here, $Z \sim \mathcal{N}(0,1)$ under some probability measure $\tilde{\mathbb{P}}$.

For specific payoffs f, the price function V can be calculated analytically.

Alternatively, V can be computed efficiently by numerical quadrature.

Outline

We already represented option prices as expectations of discounted payoffs - what else is left?

Convergence of CRR model to Black-Scholes model (Thm. 87) as well as PDE solution for Bachelier model yield option price as

$$V_t = \tilde{\mathbb{E}}\left[\tilde{f}\left(Z_{T-t}\right)\right]. \tag{26}$$

Here,

- expectation $\tilde{\mathbb{E}}[\cdot]$ is calculated with a suitable measure $\tilde{\mathbb{P}}$,
- Z_{T-t} is a random variable with distribution $Z_{T-t} \sim \mathcal{N}(0, T-t)$,
- $\tilde{f}$ is a payoff function suitably reformulated to take Z_T as argument.

What are the limitations of these results?

1. We do not know if/how $\tilde{\mathbb{P}}$ is related to the original measure $\mathbb{P}$.
2. We can only price options that depend on the realisation of X_T at maturity T.

We aim at describing option prices for more general derivatives

To keep discussion simple, we model discounted asset via Bachelier model,

$$X_t = X_0 + \mu\, t + \sigma\, B_t$$

with intial value X_0, constant drift μ, positive volatility $\sigma > 0$, and $\mathbb{P}$-Brownian motion B w.r.t. the filtration $(\mathcal{F}_t)$.

With B^* defined via $B^*_t := B_t + \frac{\mu}{\sigma}\, t$ we get (by substitution)

$$X_t = X_0 + \sigma\, B^*_t.$$

We will show that B^* is also a Brownian motion w.r.t. the same filtration $(\mathcal{F}_t)$, but using a different equivalent measure $\mathbb{P}^* \sim \mathbb{P}$.

We will need Bayes' formula for conditional expectations

Lemma 154 (Bayes' formula)

Let $\mathbb{P}^* \sim \mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with density $Z := \frac{d\mathbb{P}^*}{d\mathbb{P}} > 0$. Then, for $H > 0$ with $H\,Z \in L^1(\mathbb{P})$,

$$\mathbb{E}^*[H \mid \mathcal{F}_t] = \frac{\mathbb{E}[H\,Z \mid \mathcal{F}_t]}{\mathbb{E}[Z \mid \mathcal{F}_t]}.$$

For a proof, see e.g. [10, Lemma 5.2.2.].

Bayes' formula allows us to prove a first version of the change of measure formula.

We state and prove a change of measure formula for Brownian motions (1/.)

Theorem 155 (Cameron-Martin-Girsanov formula)

Let B be a Brownian motion under $\mathbb{P}$ and $\alpha \in \mathbb{R}$. Then, the following statements hold:

1. *There exists an equivalent measure $\mathbb{P}^* \sim \mathbb{P}$ such that B^* with $B^*_t = B_t + \alpha\, t$ is a $\mathbb{P}^*$-Brownian motion.*
2. *The measure $\mathbb{P}^*$ (w.r.t. $\mathbb{P}$) has the density*

$$Z := \frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left\{-\alpha\, B_T - \frac{1}{2}\,\alpha^2\, T\right\}.$$

- In Bachelier model we will use the result with $\alpha = \frac{\mu}{\sigma}$.

We state and prove a change of measure formula for Brownian motions (2/.)

Proof:

The proof is structured along the following steps

1. We verify that Z is a density and that it is bounded. This allows us to construct $\mathbb{P}^*$.
2. We verify that B^* inherits the properties of Brownian motion from B.
3. In particular, we verify that increments of B^* are normally distributed under $\mathbb{P}^*$.

We state and prove a change of measure formula for Brownian motions (3/.)

We define a stochastic process $(Z_t)_{t \geq 0}$ via

$$Z_t := \exp\left\{-\alpha\, B_t - \frac{1}{2}\,\alpha^2\, t\right\}.$$

From $B_t \sim \mathcal{N}(0, t)$ under $\mathbb{P}$ we get $Z > 0$,

$$\mathbb{E}[Z_t] = \exp\left\{\underbrace{\mathbb{E}\left[-\alpha\, B_t - \frac{1}{2}\,\alpha^2\, t\right]}_{-\frac{1}{2}\,\alpha^2\, t} + \frac{1}{2}\underbrace{\mathrm{Var}\left[-\alpha\, B_t - \frac{1}{2}\,\alpha^2\, t\right]}_{\alpha^2\, t}\right\} = 1$$

and similarly

$$\mathbb{E}\left[Z_t^2\right] = e^{\alpha^2 t}.$$

Consequently, Z is uniformly bounded in L^2. This implies uniformly integrable, which also gves that Z is bounded in L^1.

We state and prove a change of measure formula for Brownian motions (4/.)

We also get that Z is a $\mathbb{P}$-martingale, because

$$Z_s = Z_t \exp\left\{-\alpha\,(B_s - B_t) - \frac{1}{2}\,\alpha^2\,(s-t)\right\}$$

and thus

$$\mathbb{E}\left[Z_s \mid \mathcal{F}_t\right] = Z_t\,\mathbb{E}\left[\exp\left\{-\alpha\,(B_s - B_t) - \frac{1}{2}\,\alpha^2\,(s-t)\right\} \mid \mathcal{F}_t\right] = Z_t.$$

We state and prove a change of measure formula for Brownian motions (5/.)

As a result, we can use Z_T to define an equivalent measure $\mathbb{P}^*$ and (by construction)

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = Z_T = Z.$$

From $\mathbb{P}^* \sim \mathbb{P}$ and $B_t^* = B_t + \alpha\, t$ we get that

- $B_0^* = B_0 = 0$ $\mathbb{P}^*$-a.s.,
- $t \mapsto B_t^*$ is continuous $\mathbb{P}^*$-a.s.

It remains to show that increments of B^* are independent and normally distributed under $\mathbb{P}^*$.

We state and prove a change of measure formula for Brownian motions (6/.)

Independence of increments $B_s^* - B_t^*$ w.r.t. $(B_u^*)_{0 \le u \le t}$ follows from $B_s^* - B_t^* = B_s - B_t + \alpha\,(s-t)$ and independence of B-increments.

Normal distribution of B^*-increments is established by verifying that the characteristic function becomes

$$\mathbb{E}^*\left[\exp\left\{\lambda\left(B_s^* - B_t^*\right)\right\} \mid \mathcal{F}_t\right] = \exp\left\{\frac{1}{2}\lambda^2\,(s-t)\right\} \quad \text{for all } \lambda \in \mathbb{R}. \tag{27}$$

Since B is a Brownian motion under $\mathbb{P}$, equation (27) holds analogously for B and $\mathbb{E}[\cdot]$.

We state and prove a change of measure formula for Brownian motions (7/.)

We use Bayes' formula, $B^*_s - B^*_t = B_s - B_t + \alpha\,(s-t)$ and calculate

$$\mathbb{E}^*\left[\exp\left\{\lambda\left(B^*_s - B^*_t\right)\right\} \mid \mathcal{F}_t\right] = \frac{\mathbb{E}\left[\exp\left\{\lambda\left(B_s - B_t + \alpha\left(s-t\right)\right)\right\} Z_T \mid \mathcal{F}_t\right]}{\mathbb{E}\left[Z_T \mid \mathcal{F}_t\right]}.$$

Tower-law of iterated conditional expectation and martingale property of Z gives

$$\begin{aligned}
\ldots =&\mathbb{E}^*\left[\exp\left\{\lambda\left(B_s - B_t + \alpha\,(s-t)\right)\right\} \mid \mathcal{F}_t\right]\\
=&\mathbb{E}\left[\exp\left\{\lambda\left(B_s - B_t + \alpha\,(s-t)\right)\right\} \frac{Z_s}{Z_t} \mid \mathcal{F}_t\right]\\
=&\mathbb{E}\left[\exp\left\{\lambda\left(B_s - B_t + \alpha\left(s-t\right)\right) - \alpha\left(B_s - B_t\right) - \frac{1}{2}\alpha^2\left(s-t\right)\right\} \mid \mathcal{F}_t\right]
\end{aligned}$$

We state and prove a change of measure formula for Brownian motions (8/.)

Re-arranging terms leads to

$$\begin{aligned}
\ldots &= \exp\left\{\lambda\alpha\,(s-t) - \frac{1}{2}\alpha^2\,(s-t)\right\}\mathbb{E}\left[\exp\left\{(\lambda-\alpha)\,(B_s - B_t)\right\}\ \middle|\ \mathcal{F}_t\right] \\
&= \exp\left\{\lambda\alpha\,(s-t) - \frac{1}{2}\alpha^2\,(s-t)\right\}\exp\left\{\frac{1}{2}\,(\lambda-\alpha)^2\,(s-t)\right\} \\
&= \exp\left\{\frac{1}{2}\lambda^2\,(s-t)\right\}.
\end{aligned}$$

This shows that $(B^*_s - B^*_t) \sim \mathcal{N}(0, s-t)$ under $\mathbb{P}^*$ and concludes the proof that B^* is a Brownian motion under $\mathbb{P}^*$.

How can we apply the result to Bachelier model?

We have the price given as

$$V(t,x) = \tilde{\mathbb{E}}\left[f\left(x + \sigma\sqrt{T-t}\,Z\right)\right]$$

for some measure $\tilde{\mathbb{P}}$ and $Z \sim \mathcal{N}(0,1)$.

It turns out, we can choose $\tilde{\mathbb{P}} = \mathbb{P}^*|_{\mathcal{F}_t}$ and get

$$V(t,x) = \mathbb{E}^*\left[f\left(x + \sigma\left(B_T^* - B_t^*\right)\right) \mid \mathcal{F}_t\right].$$

For a particular path ω this gives

$$\begin{aligned} V(t, X_t(\omega)) &= \mathbb{E}^*\left[f\left(X_t + \sigma\left(B_T^* - B_t^*\right)\right) \mid \mathcal{F}_t\right] \\ &= \mathbb{E}^*\left[f\left(X_T\right) \mid \mathcal{F}_t\right]. \end{aligned}$$

We summarise our results for the Bachelier model

For any (Bachelier) model with drift μ and volatility σ we see:

1. Girsanov theorem yields a probability measure $\mathbb{P}^*$ such that discounted option price is

$$V(t, X_t(\omega)) = \mathbb{E}^* \left[f(X_T) \mid \mathcal{F}_t \right].$$

   - We see that discounted option price is a $\mathbb{P}^*$-martingale.
   - And discounted option price is an expectation of the future payoff.

2. The model for the discounted risky asset becomes

$$X_t = X_0 + \sigma B_t^*.$$

   - The discounted asset is also a $\mathbb{P}^*$-martingale.
   - Under $\mathbb{P}^*$, the evolution of the asset is independent of the real-world drift μ.

The measure $\mathbb{P}^*$ is called the risk-neutral measure or equivalent martingale measure w.r.t. $\mathbb{P}$ in the continuous time model.

Outline

References I

- L. Ballabio.
  Implementing QuantLib.
  leanpub.com/implementingquantlib, 2021.

- L. Ballabio and G. Balaraman.
  QuantLib Python Cookbook.
  leanpub.com/quantlibpythoncookbook, 2021.

- J. Cox, S. Ross, and M. Rubinstein.
  Option pricing: a simple aproach.
  J. Financial Economics, 7, 1979.

- D. J. Duffy.
  Finite Difference Methods in Financial Engineering.
  Wiley, 2006.

- E. F. Fama.
  Efficient capital markets: A review of theory and empirical work.
  Journal of Finance, 25:383–417, 1970.

References II

H. Foellmer and A. Schied.
Stochastic Finance: An Introduction in Discrete Time.
de Gruyter, 2016.

Espen Gaarder Haug.
The complete guide to option pricing formulas.
McGraw-Hill, 2007.

D. Lamberton and B. Lapeyre.
Introduction to Stochastic Calculus Applied to Finance.
Chapman and Hall, 2008.

R. Rebonato.
Volatility and Correlation, 2nd Edition.
Wiley, 2004.

S. Shreve.
Stochastic Calculus for Finance II - Continuous-Time Models.
Springer-Verlag, 2004.

Contact

Dr. Sebastian Schlenkrich
Mail: sebastian.schlenkrich@hu-berlin.de

d-fine GmbH
Mobile: +49-162-263-1525
Mail: sebastian.schlenkrich@d-fine.de