Rational Expectations Equilibria of Economies with Local Interactions*

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Abstract

We consider general economies in which rational agents interact locally. The local aspect of the interactions is designed to represent in a simple abstract way social interactions, that is, socioeconomic environments in which markets do not mediate all of agents' choices, which might be in part determined, for instance, by family, peer group, or ethnic group effects. We study static as well as dynamic infinite horizon economies; we allow for economies with incomplete information, and we consider jointly global and local interactions, to integrate e.g., global externalities and markets with peer and group effects. We provide conditions under which such economies have rational expectations equilibria. We illustrate the effects of local interactions when agents are rational by studying in detail the equilibrium properties of a simple economy with quadratic preferences which captures, in turn, local preferences for conformity, habit persistence, and preferences for status or adherence to aggregate norms of behavior.

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1 Introduction

We consider general economies with local interactions. Agents interact locally when each agent interacts with only a finite (small) set of other agents in an otherwise large economy. The local aspect of interactions is designed to capture in a simple abstract way a socioeconomic environment in which markets do not exist to mediate all of agents' choices. In such an environment each agent's ability to interact with others might depend on the position of the agent in a predetermined network of relationships, e.g., a family, a peer group, or more generally any socioeconomic group. Local interactions represent an important aspect of several socioeconomic phenomena. For instance, the decision of a teen to commit a criminal act or to drop out of school is often importantly influenced by the related decisions of peers, as documented by Case and Katz [17], Glaeser, Sacerdote and Scheinkman [33] and Crane [19], respectively. Other phenomena for which relevant peer effects have been identified include out-of-wedlock births (Crane [19]), and smoking habits (Jones [38]). More generally, local interactions occur not only between peers but also between family members, ethnic groups, neighbors in a geographical space. For example, neighborhood effects are important determinants of employment search (Topa [52], Krosnick and Judd [43]), of the pattern of bilateral trade and economic specialization (Kelly [41]), and of local technological complementarities (Ellison and Fudenberg [23], Durlauf [20]) while ethnic group effects play a fundamental role in explaining urban agglomeration, segregation (Benabou [4], Schelling [50]), income inequality and stratification (Durlauf [21]).[1]

The documented empirical evidence of peer and neighborhood effects in socioeconomic phenomena has spurred an interesting theoretical literature. This literature has generated existence and characterization results for important but special classes of static economies: for instance economies with additive quadratic preferences, extreme value distributed shocks, and symmetric interaction effects, introduced by Blume [8] and Brock [12] (see also Brock and Durlauf [13]); or economies with a finite number of agents, studied by Glaeser and Scheinkman [31]. Moreover, when dynamic economies are studied, the analysis is only confined to the case of backward looking myopic dynamics, either as a simple explicit dynamic process with random sequential choice (Brock and Durlauf [13]), or as an equilibrium selection procedure (Glaeser and Scheinkman [31], Blume and Durlauf [9]).[2]

In this paper, we contribute to this literature by extending the class of economies under study in various dimensions. First of all, we study economies in which the distribution of information

[1] Finally, local interactions control the dynamics and spread of ideas and beliefs, and therefore might in particular play an important role in the microstructure of financial markets; see Brock [12] or Horst [36] for theoretical applications to financial markets. See Brock and Durlauf [14], Blume and Durlauf [10], Glaeser and Scheinkman [31] and [32] for good surveys of the theory of social interactions, and its applications.

[2] The only exception is a simple example in Glaeser and Scheinkman [32]. Rational expectations equilibria are instead the focus of the literature on global, or mean-field type, interactions; see Glaeser and Scheinkman [31] for a review.

across the agents, as well as their interactions, are local (economies of *incomplete information*). Economies in which information is distributed locally allow to study environments in which, for instance, only the agents directly interacting with each other observe each other's private characteristics.[3] Most importantly, we study the *rational expectations equilibria* of dynamic economies. While agents may interact locally, they are forward looking, and their choices are optimally based on the past actions of the agents in their neighborhood, as well as on their anticipation of the future actions of their neighbors. We see no valid reason why local interactions should be characterized by myopic behavior more than standard global, e.g., market, interactions. Finally, we also introduce economies characterized by *global interactions* together with local interactions to integrate e.g., global externalities and markets with peer and group effects. This extension allows us, for instance, to consider economies in which agents interact locally and, at the same time, act as price takers in competitive markets, or take as given aggregate norms of behavior, such as specific group cultures.

To pursue these extensions, we restrict in part the analysis to a specific form of local interactions, *one-sided interactions*. This assumption is substantive as it limits the scope of strategic interactions to those which are *directed*, e.g., structured hierarchically inside each social group.[4]

For all the distinct economies we study, static economies with complete and incomplete information, as well as dynamic infinite horizon economies, we provide conditions under which such economies have rational expectations equilibria which depend in a Lipschitz continuous manner on the parameters. For each of these economies, we show that such conditions impose an appropriate bound on the strength of the interactions across agents.[5] They exclude in particular economies in which strategic coordination gives rise to multiple equilibria; for instance economies in which it is always optimal for each agent to match the action of a neighboring agent.

We also illustrate the effects of incomplete information and agents' rationality by studying in detail the equilibrium properties of simple economies with quadratic preferences displaying local preferences for conformity, habit persistence and possibly preferences for *status*. For instance, we compare the magnitudes of the *social multiplier* effects with complete and different degrees of incomplete information. The *social multiplier* summarizes the equilibrium effects of the interactions, and measures the amplification of individual effects in the aggregate due to the correlation

[3]The restriction to economies with complete information is in fact implicitly adopted by the literature only for simplicity: it allows the direct use of mathematical techniques and results from the physics of interacting particle systems and statistical mechanics (Blume [8], Blume and Durlauf [9], Brock [12], Brock and Durlauf [13]; see Liggett [44] for a presentation of such techniques and results).

[4]One-sided interactions are a natural component, for instance, of models of local conformity, when each agent has a preference for behaving as much as possible as some of his peers. They also appear prominently in other forms of social interactions, as e.g., in Glaeser-Sacerdote-Scheinkman's model of crime, [33], in Estigneev-Taksar's model of trade links, [26], or in Ozsoylev's model of information flows in financial markets, [46].

[5]Similar conditions for related economies have appeared in the literature; Glaeser and Scheinkman [32] refers to them as conditions of *Moderate Social Influence*, see also Horst and Scheinkman [37].

across the agents' actions induced by social interactions. We show that incomplete information has the effect of reducing the social multiplier with respect to the complete information case and hence to dampen the aggregate effects of the agents' preferences for conformity. In this context we also attempt at a comparison between the equilibrium actions when agents have rational expectations with the actions of the myopic agents previously studied in the literature. We show that, in the context of our local conformity economy with habit persistence, the effect of rational expectations dynamics is to spread the correlation of equilibrium actions across all agents in the economy. Most importantly though, we show formally that when agents have rational expectations, the effect of the local conformity component of their preferences on their equilibrium actions is reduced with respect to the case in which agents are myopic. We also show then, by means of simulations, that the extent to which rational expectations reduce the agents' reliance at equilibrium on local conformity effects is quite substantial.

The paper proceeds as follows. We first study static economies with local interactions with complete and incomplete information. We then study dynamic economies with local interactions and incomplete information (the results adapt simply to the complete information case). Finally, we introduce global as well as local interactions in the dynamic analysis.

2 Static Economies with Local Interactions

In this section, we introduce our concept of a static economy with local interactions. We consider economies with a countable set $\mathbb{A}$ of *agents*. To each agent $a \in \mathbb{A}$ is associated a *type*, the realization of a random variable θ^a taking values in a set $\Theta \subset \mathbb{R}$. Types are independent and identically distributed across agents with law ν. We assume with no loss of generality that the random variable $\theta := (\theta^a)_{a \in \mathbb{A}}$ is defined on the canonical probability space $(\mathbf{\Theta}, \mathcal{F}, \mathbb{P})$, i.e., $\mathbf{\Theta} := \{(\theta^a)_{a \in \mathbb{A}} : \theta^a \in \Theta\}$. The utility of an agent $a \in \mathbb{A}$ depends on his type θ^a, on an action x^a he chooses from a common compact and convex action set $X \subset \mathbb{R}$, and on the action taken by his neighbor, agent $a+1$. In other words, we assume that each agent $a \in \mathbb{A}$ only interacts with the agent $a+1$. Such a system of local interactions has the property that interactions are one-sided; that is, if the action or the realized type of an agent b affects the choice of agent a, then a's action or type does not affect b's choice.

Remark 2.1 *The one-sidedness of the interaction structure is a substantive assumption as it excludes various forms of strategic interactions* inside neighborhoods. *Akerlof [1], for instance, stresses the importance of models of social interactions encompassing a different range of forms and intensities. In economic applications, one-sided interactions are most appropriate for environments in which agents' interactions are structured hierarchically inside each social group: this is the case for instance when one subset of the agents in each social group looks at the others as role models, as in our study of conformity in Sections 2.2, 4.3, and, in particular, in the model* *with both local and global interactions analyzed in Section 5. One-sided interactions are studied in the literature for simplicity when local rather than global (e.g., mean field) interactions are modelled; see e.g., Glaeser-Scheinkman [32], Glaeser-Sacerdote-Scheinkman [33], Estigneev-Taksar ([24], [25], [26]), and Ozsoylev [46]. We discuss in Section 2.3 how to extend our analysis to general forms of local interactions in the case of static models as, e.g., in Horst and Scheinkman [37]. Our focus, however, is on dynamic economies of local interactions, and we are unaware of any method that would allow us to extend our results derived in Sections 3 and, in particular the existence result for dynamic economies with both local and global interactions established in Section 5, to more complex interaction patterns.*

Any heterogeneity across agents can be incorporated in the probabilistic structure of the types θ^a. Agents can therefore be considered identical ex-ante without loss of generality. Thus, the preferences of each agent $a \in \mathbb{A}$ are described by a utility function u of the form

$$\left(x^a, x^{a+1}, \theta^a\right) \mapsto u\left(x^a, x^{a+1}, \theta^a\right).$$

We assume throughout that $u : X^2 \times \Theta \to \mathbb{R}$ is continuous and strictly concave in its first argument. Prior to his choice, each agent $a \in \mathbb{A}$ observes the realization of his own type θ^a as well as the realizations of the types θ^b of the agents $b \in \{a+1, a+2, ..., a+N\}$. Here $N \in \mathbb{N}^* \cup \{\infty\}$. If $N = \infty$ each agent has *complete information* about the current configuration of types when choosing his action. In particular, each agent $a \in \mathbb{A}$ observes the types of all the agents $b \in \{a+1, a+2, \ldots\}$ with whom he is directly or indirectly linked. When instead $N \in \mathbb{N}$, an agent only has *incomplete information* about the types of the other agents. If, for instance, $N = 1$, then the agents only observe the type of the agent with whom they directly interact.

Definition 2.2 *A static economy with local interactions is a tuple $\mathcal{S} = (X, \Theta, u, \nu, N)$. A static economy with local interactions is an economy with complete information if $N = \infty$, and with incomplete information if $N \in \mathbb{N}$.*

In order to introduce our notion of an equilibrium for a static economy with local interactions $\mathcal{S} = (X, \Theta, u, \nu, N)$, we need some notation. The vector of types whose realization is observed by the agent $a = 0$ is denoted $\theta_N := \{\theta^0, \theta^1, \ldots, \theta^N\}$; by analogy $T^a\theta_N := \left(\theta^a, \ldots, \theta^{a+N}\right)$ denotes the vector of types whose realization is observed by the agent $a \in \mathbb{A}$.[6] In case $N = \infty$, we let $\theta_N = \{\theta^0, \theta^1, \theta^2, \ldots\}$ and $T^a\theta_N = \{\theta^a, \theta^{a+1}, \theta^{a+2}, \ldots\}$. Finally, the set of possible configurations of types of all agents $a \geq 0$ is given by $\mathbf{\Theta^0} := \{(\theta^a)_{a\geq 0} \,:\, \theta^a \in \Theta\}$. We first focus on the simpler case of economies with complete information. Agent $a \in \mathbb{A}$ takes as given his neighbor's policy function g^{a+1} that maps $T^{a+1}\theta_N \in \mathbf{\Theta^0}$ into an element of X. Agent a's choice is then represented

[6] Formally, $T^a : \Theta \mapsto \Theta$ $(a \in \mathbb{A})$ is the a-fold iteration of the canonical right shift operator T; that is, $T^a((\theta^b)_{b\in\mathbb{A}}) = (\theta^{b+a})_{b\in\mathbb{A}}$; furthermore, $T^a\theta_N := \left(\theta^0(T^a\theta), \ldots, \theta^N(T^a\theta)\right) = \left(\theta^a, \ldots, \theta^{a+N}\right)$.

by a function g^a that maps any $T^a\theta_N \in \mathbf{\Theta^0}$ into an element of X.[7] Since the agent observes $T^a\theta_N = \{\theta^a, \theta^{a+1}, \ldots\}$, his optimization problem takes the form

$$\max_{x^a \in X} u\left(x^a, g^{a+1}\left(T^{a+1}\theta_N\right), \theta^a\right).$$

If $\mathcal{S}$ is an economy with incomplete information, by taking as given his neighbor's policy function g^{a+1}, and by observing the realization only of the types $T^a\theta_N := \left(\theta^a, \ldots, \theta^{a+N}\right)$, agent a cannot determine his neighbor's optimal choice. The choice depends on the realization of the random variable θ^{a+N+1} and this information is not available to the agent $a \in \mathbb{A}$. Thus, in a situation with incomplete information, an agent's optimization problem is given by

$$\max_{x^a \in X} \int u\left(x^a, g^{a+1}\left(\theta^{a+1}, \ldots, \theta^{a+N}, \theta\right), \theta^a\right)\nu(d\theta).$$

Since the utility functions are strictly concave with respect to an agent's own action, the maps

$$x^a \mapsto u\left(x^a, g^{a+1}\left(T^{a+1}\theta_N\right), \theta^a\right) \quad \text{and} \quad x^a \mapsto \int u\left(x^a, g^{a+1}\left(\theta^{a+1}, \ldots, \theta^{a+N}, \theta\right), \theta^a\right)\nu(d\theta)$$

are strictly concave, too. Thus, the conditional choice of agent $a \in \mathbb{A}$, given the policy function of agent $a+1$ and given types of all the agents $b \in \{a+1, a+2, \ldots, a+N\}$, is uniquely determined both for economies with complete and with incomplete information. We are now ready to introduce our notion of an equilibrium for static economies with locally interacting agents.

Definition 2.3 *Let $\mathcal{S} = (X, \Theta, u, \nu, N)$ be a static economy with local interactions.*

i. If $\mathcal{S}$ is an economy with complete information, then an equilibrium is a family $(g^{*a})_{a\in\mathbb{A}}$ of measurable mappings $g^{*a} : \mathbf{\Theta}^0 \to X$ such that

$$g^{*a}\left(T^a\theta_N\right) = \arg\max_{x^a \in X} u\left(x^a, g^{*a+1}\left(T^{a+1}\theta_N\right), \theta^a\right) \qquad \mathbb{P}\text{-}a.s. \tag{1}$$

for all $a \in \mathbb{A}$.

ii. If $\mathcal{S}$ is an economy with incomplete information, then an equilibrium is a family $(g^{*a})_{a\in\mathbb{A}}$ of measurable mappings $g^{*a} : \Theta^{N+1} \to X$ such that

$$g^{*a}\left(T^a\theta_N\right) = \arg\max_{x^a \in X} \int_\Theta u\left(x^a, g^{*a+1}\left(\theta^{a+1}, \ldots, \theta^{a+N}, \theta\right), \theta^a\right)\nu(d\theta) \qquad \mathbb{P}\text{-}a.s. \tag{2}$$

for all $a \in \mathbb{A}$.

An equilibrium $(g^{*a})_{a\in\mathbb{A}}$ for an economy $\mathcal{S}$ is symmetric if

$$g^{*a} = g^* \circ T^a \qquad \mathbb{P}\text{-}a.s. \tag{3}$$

for some mapping g^* and each $a \in \mathbb{A}$.

[7] We therefore restrict the arguments of each of the agents' choice to exclude any effects through the realization of the types of the agents to their left. In other words, we exclude any extrinsic effect, that is, we exclude sunspot and correlated equilibria.

An equilibrium for an economy $\mathcal{S}$ can also be viewed as a Nash equilibrium of a game with infinitely many players in which the agents' common strategy set is given by the class $B(\mathbf{\Theta^0}, X)$ of all bounded measurable functions $f : \mathbf{\Theta^0} \to X$. Sufficient conditions for existence of correlated equilibria in games with infinitely many players and compact strategy sets are given in, e.g., Hart and Schmeidler [34]. These results, however, do not apply to our model, because (i) $B(\mathbf{\Theta^0}, X)$ equipped with the usual sup-norm is not compact; and (ii) our focus is on Nash equilibria of the game rather than on correlated equilibria. In what follows, we shall restrict our analysis to symmetric equilibria.

Remark 2.4 *i. As we will see, in our complete information setting, a symmetric equilibrium $a \in \mathbb{A}$, given $T^a\theta_N$, is uniquely determined. Thus, the equilibrium map $g^* : \mathbf{\Theta^0} \to X$ can be identified with the mapping $G^* : \Theta \times X \to X$ defined by*

$$G^*(\theta^a, x^{a+1}) = \arg\max_{x^a \in X} u\left(x^a, x^{a+1}, \theta^a\right).$$

In fact, the previous literature has only studied this case, where each agent reacts to the actions taken by his neighbors; see, e.g. Blume [8], Blume and Durlauf [9], Brock [12], Brock and Durlauf [13], Glaeser and Scheinkman [31], Glaeser, Sacerdote and Scheinkman [33] or Horst and Scheinkman [37].

ii. Suppose that an agent's utility function takes the additive form

$$u(x^a, x^{a+1}, \theta^a) = v(x^a) + x^a x^{a+1} + g(x^a, \theta^a).$$

In such a situation, the agent may as well maximize his utility with respect to the expected action $\mathbb{E}_a x^{a+1}$ of his neighbor. Here, $\mathbb{E}_a$ denotes the conditional expectation operator of the agent $a \in \mathbb{A}$. Thus, we may as well assume that the utility functions are given by

$$u(x^a, x^{a+1}, \theta^a) = v(x^a) + x^a \mathbb{E}_a x^{a+1} + g(x^a, \theta^a).$$

Such preferences are analyzed in, e.g., Brock and Durlauf [13]. In this sense, our framework can also be viewed as an extension of the model by Brock and Durlauf [13].

2.1 Existence, Uniqueness, and Lipschitz Continuity of Equilibrium

In order to guarantee the existence and uniqueness of an equilibrium for static economies with local interactions, we need to impose a form of strong concavity on the agents' utility functions. To this end, we recall the notion of an α-concave function.

Definition 2.5 *Let $\alpha \geq 0$. A real-valued function $f : X \to \mathbb{R}$ is α-concave on X if the map $x \mapsto f(x) + \frac{1}{2}\alpha|x|^2$ from X to $\mathbb{R}$ is concave.*

This definition is first due to Rockafellar [47], and is used for purposes related to ours by Montrucchio [45].

Remark 2.6 *Observe that a twice continuously differentiable map* $f : X \to \mathbb{R}$ *is* α*-concave, if and only if the second derivative is uniformly bounded from above by* $-\alpha$*. For a more detailed discussion of the properties of* α*-concave functions, we refer the reader to Montrucchio [45] and references therein.*

We will also require any agent's marginal utility with respect to his own action to depend in a Lipschitz continuous manner on the action taken by his neighbor. In this sense we impose a qualitative bound on the strength of local interactions between different agents. More precisely, we assume that the following condition is satisfied.

Assumption 2.7 *The utility function* $u : X \times X \times \Theta \to \mathbb{R}$ *satisfies the following conditions:*

i. The map $x \mapsto u(x, y, \theta)$ *is continuous and uniformly* α*-concave for some* $\alpha > 0$.

ii. The map u *is differentiable with respect to its first argument, and there exists a map* $L : \Theta \to \mathbb{R}$ *such that*

$$\left| \frac{\partial}{\partial x} u(x, y, \theta^0) - \frac{\partial}{\partial x} u(x, \hat{y}, \theta^0) \right| \leq L(\theta^0)|\hat{y} - y| \quad \textit{and such that} \quad \mathbb{E}L(\theta^0) < \alpha. \tag{4}$$

A simple example where our Assumption 2.7 can indeed be verified is studied in the next section.

Remark 2.8 *The interpretation of the condition* $\mathbb{E}L(\theta^0) < \alpha$ *is of interest. In the differentiable case, the quantity* $L(\theta^0)$ *defines a bound on* $\frac{\partial^2 u(x,y,\theta)}{\partial x \partial y}$*, whereas* α *may be viewed as a bound on* $\frac{\partial^2 u(x,y,\theta)}{\partial x^2}$*. Thus,* $\mathbb{E}L(\theta^0) < \alpha$ *means that,* on average*, the marginal effect of the neighbor's action on an agent's marginal utility is smaller than the marginal effect of the agent's own choice. It is in this sense that (4) imposes a bound on the strength of the interactions between different agents. A similar condition has also been employed to study uniqueness of equilibria in related environments by Becker [2]; see also Becker and Murphy [3]. The* Moderate Social Influence *conditions in Glaeser and Scheinkman [31] corresponds to the stronger contraction condition* $L(\theta^0) < \alpha$ $\mathbb{P}$*-a.s.*

We are also interested in deriving conditions which guarantee that the economy admits a Lipschitz continuous equilibrium. Lipschitz continuity of the equilibrium map may be viewed as a minimal robustness requirement on equilibrium analysis. In particular, it justifies comparative statics analysis. We therefore introduce the notion of Lipschitz continuity we will use in our analysis. For an arbitrary constant $\eta \geq 0$, we define a metric d_η on the product space $\mathbf{\Theta^0}$ by

$$d_\eta(\theta, \hat{\theta}) := \sum_{a \geq 0} 2^{-\eta|a|} |\theta^a - \hat{\theta}^a| \qquad (\theta = (\theta^a)_{a \in \mathbb{N}},\ \hat{\theta} = (\hat{\theta}^a)_{a \in \mathbb{N}}) \tag{5}$$

and denote by $\mathrm{Lip}_\eta(1)$, the class of all continuous functions $f : \mathbf{\Theta^0} \to X$ which are Lipschitz continuous with respect to the metric d_η with constant 1, i.e.,

$$\mathrm{Lip}_\eta(1) := \{f : \mathbf{\Theta^0} \to X : |f(\theta) - f(\hat{\theta})| \leq d_\eta(\theta, \hat{\theta})\}$$

We are now ready to formulate the main result of this section. Its proof can be found in Appendix B.

Theorem 2.9 *Let $\mathcal{S} = (X, \Theta, u, \nu, N)$ be a static economy with local interactions and complete information, that is with $N = \infty$.*

i. If the utility function $u : X^2 \times \Theta \to \mathbb{R}$ satisfies Assumption 2.7, then $\mathcal{S}$ admits a unique (up to a set of measure zero) symmetric equilibrium g^*.

ii. If, instead of (4), the utility function u satisfies the stronger condition,

$$\left| \frac{\partial}{\partial x} u(x, y, \theta) - \frac{\partial}{\partial x} u(x, \hat{y}, \hat{\theta}) \right| \leq L \max\{|\hat{y} - y|, |\hat{\theta} - \theta|\} \quad \textit{with} \quad L < \alpha, \tag{6}$$

then there exists $\eta^* > 0$ such that the equilibrium g^* is Lipschitz continuous with respect to the metric d_{η^*}:

$$|g^*(\theta) - g^*(\hat{\theta})| \leq \frac{L}{\alpha} d_{\eta^*}(\theta, \hat{\theta}).$$

Establishing the existence of a symmetric equilibrium is equivalent to proving the existence of a measurable function $g^* : \mathbf{\Theta^0} \to X$ which satisfies

$$g^*(\theta) = \arg\max_{x^0 \in X} u(x^0, g^* \circ T(\theta), \theta^0) \qquad \mathbb{P}\text{-a.s.} \tag{7}$$

Observe that each such map is a fixed point of the operator $V : B(\mathbf{\Theta^0}, X) \to B(\mathbf{\Theta^0}, X)$ which acts on the class $B(\mathbf{\Theta^0}, X)$ of bounded measurable functions $f : \mathbf{\Theta^0} \to X$ according to

$$Vg(\theta) = \arg\max_{x^0 \in X} u(x^0, g \circ T(\theta), \theta^0). \tag{8}$$

On the other hand, each fixed point of V is a symmetric equilibrium. It is therefore enough to show that V has a unique fixed point. The existence and uniqueness of equilibrium for economies of incomplete information, where an individual agent only observes a finite number $N < \infty$ of types, requires an additional continuity assumption on the utility function. The proof is analogous to the proof of Theorem 2.9 and is given in Appendix B.

Theorem 2.10 *Let $\mathcal{S} = (X, \Theta, u, \nu, N)$ be a static economy with local interactions and incomplete information, that is with $N \in \mathbb{N}$.*

i. If the utility function $u : X^2 \times \Theta \to \mathbb{R}$ *satisfies Assumption 2.7 and if it is* continuously *differentiable with respect to its first argument, then* $\mathcal{S}$ *admits a unique (up to a set of measure zero) symmetric equilibrium* g^*.

ii. If u *satisfies condition (6), then* g^* *is Lipschitz continuous:*

$$|g^*(\theta_N) - g^*(\hat{\theta}_N)| \leq \frac{L}{\alpha} \max\{|\theta^b - \hat{\theta}^b| : b = 0, 1, \ldots, N\}.$$

2.2 Example: Local Conformity

In this section, we study in detail an economy in which agents have a local preference for conformity: each agent a enjoys utility from behaving as much as possible as his close peers. In many instances of socioeconomic relevance, from smoking to dropping out of school, preferences for conformity act at the level of peers rather than at the level of society as a whole; see Bernheim [6] for a study of preferences for status, where interactions are global rather than local and agents try to match the mean behavior in the society. The economy we analyze is a special case of the general environment studied in the previous sections, in which the agents' equilibrium actions can be given in closed form. We also derive interesting statistical properties of the equilibrium action profile. In particular, we characterize the effects of local conformity on the variance and the correlation structure of individual actions in the population as well as on the variance of the mean action across different economies. When the variance of the mean action across economies is larger than the variance of each action in the population, we say that social interactions generate a *social multiplier* effect; see e.g., Becker [2], Glaeser and Scheinkman [31]. The *social multiplier* summarizes the equilibrium effects of the interactions, and measures the amplification of individual effects in aggregate due to the correlation across the agents' actions induced by social interactions. It is interpreted to explain the large variability across time and space observed in empirical data regarding smoking, criminal activity, dropping out of school, out-of-wedlock pregnancy and other socio-economic decisions in which a social interactions component is relevant; see Glaeser and Scheinkman [32] for a survey.

We compute the *social multiplier* for our local conformity economy under complete and incomplete information, and we study the differential effects of the distribution of information about agents' types across the neighborhood structure. Consider the following utility function:

$$u(x^a, x^{a+1}, \theta^a) := -\alpha_1(x^a - \theta^a)^2 - \alpha_2(x^a - x^{a+1})^2. \tag{9}$$

for α_1 $\alpha_2 \geq 0$. Quadratic utility functions of the form (9) describe preferences in which agents face a trade-off between the utility they receive from matching their own idiosyncratic shocks and the utility they receive from conforming to the action of their peers, which will in general be different. The higher the ratio $\frac{\alpha_2}{\alpha_1}$, the more intense is the agent's desire for conformity. The equilibrium action of a generic agent $a \in \mathbb{A}$ can now be solved for in closed form (we do not report calculations for this section).[8] Let $\beta_1 := \frac{\alpha_1}{\alpha_1+\alpha_2}$ and $\beta_2 := \frac{\alpha_2}{\alpha_1+\alpha_2}$. In the complete and incomplete information cases, respectively, the map g^* is given by

$$g^*\left(T^a\theta_N\right) = \beta_1 \sum_{i=a}^{\infty} \beta_2^{i-a}\theta^i \quad \text{and} \quad g^*\left(T^a\theta_N\right) = \beta_1 \left(\sum_{i=a}^{a+N} \beta_2^{i-a}\theta^i + \frac{\beta_2^{N+1}}{1-\beta_2}\mathbb{E}\theta^a \right).$$

Note that, in either case, the equilibrium action of the agent $a \in \mathbb{A}$ is given by a convex combination of all the observable types and the mean of the random variable θ^a. Table 1 reports the variance of action x^a and the covariance of the actions taken by agents a and b.

	No Interaction ($\alpha_1 > 0, \alpha_2 = 0$)	*Complete Info* ($\alpha_1, \alpha_2 > 0; N = \infty$)	*Incomplete Info* ($\alpha_1, \alpha_2 > 0; N = 1$)
$\mathrm{var}(x^a)$	$\mathrm{var}(\theta^a)$	$\frac{\beta_1^2}{1-\beta_2^2}$ var (θ^a)	$\beta_1^2(1+\beta_2^2)$ var (θ^a)
$\mathrm{cov}(x^a, x^{a+b})$	0	$\frac{\beta_1^2}{1-\beta_2^2}\left(\beta_2\right)^b \mathrm{var}(\theta^a)$	$\begin{cases} \beta_1^2\beta_2 \mathrm{var}(\theta^a) & \text{for } b = 1 \\ 0 & \text{for } b > 1 \end{cases}$
$\mathrm{var}(\overline{x})$	$\mathrm{var}(\theta^a)$	$\mathrm{var}(\theta^a)$	$\beta_1^2\left(1+\beta_2\right)^2 \mathrm{var}(\theta^a)$

Table 1: Statistical Comparison

The variance of action x^a is highest when agents have no preference for conformity. In such a situation, in fact, the whole variability of an agent's idiosyncratic type is reflected on the action he chooses, $x^a = \theta^a$. When agents do interact instead, and therefore attempt to conform to the action of their neighbor, they choose an action which depends on a weighted average of the types of all agents on their right. Because types are i.i.d. across agents this type average has in fact a smaller variance than each single type, and the effect of local interaction is then to reduce the variance of the actions chosen by each agent.[9]

Notice also that when information is incomplete, the variance of a generic action is lower than when information is complete. In Table 1, we only report the case in which each agent observes

[8] It is easy to verify that the map $x^a \mapsto u(x^a, x^{a+1}, \theta^a)$ is α-concave if $\alpha \leq 2(\alpha_1 + \alpha_2)$, and that our Moderate Social Influence assumption holds if $\alpha_1 > 0$.

[9] This is in contrast with the results obtained e.g., by Glaeser and Scheinkman [32] for a related economy with preferences for conformity. In their formulation of conformity, agent a's equilibrium action (with complete information, the only case they study, and in our notation) takes the form: $x^a = \theta^a + \beta_2 x^{a+1}$. In this case, therefore, each agent's attempt to conform to his neighbor adds to his action rather than simply shifting the weight away from his own type, as in our case. Glaeser and Scheinkman's formulation is therefore characterized by more intense preferences for conformity, and local interactions increase the mean and the variance of each agent's action.

the type of the neighbor whose action he wishes to conform to, $N = 1$. However, it is easily shown that the variance increases with the dissemination of information, that is, with the number of neighbors' types each agent observes, N, and converges to the variance of the complete information case as $N \to \infty$. When information is incomplete, in fact, in attempting to predict the action of the neighbor to conform to, an agent necessarily relies on the mean of the types of the agents he does not observe and the variance of his chosen action is reduced.[10] Local interactions in our conformity model have then the effect of reducing the variance of each agent's action, but introduce a correlation across the actions of the agents. In the case of complete information, such correlations extend over all agents, while with incomplete information only the actions of agents at most N spaces apart are correlated. The correlation of actions across agents with preferences for conformity in turn increases the variance of the mean action across economies.[11] In fact, if the mean action is defined as $\overline{x} := \lim_{n\to\infty} \sum_{|a|\leq n} \frac{x^a}{2n+1}$, it can then be shown that $\text{var}\left(\overline{x}\right) = \text{var}(x^0) + 2\lim_{n\to\infty} \sum_{a=1}^{n} \text{cov}(x^0, x^a)$.[12] Therefore preference for conformity decreases $\text{var}(x^a)$, but increases the covariance terms in the variance of the mean action $\overline{x}$. It turns out (see Table 1) that these two effects exactly compensate in our conformity model when information is complete, and hence conformity does not generate a variance multiplier, as e.g., in Glaeser and Scheinkman [32]. In the case of incomplete information, conformity even dampens the variance of the mean action with respect to the case of no interactions. To better understand and summarize the effects of incomplete information on social interactions, we study the social multiplier of our economy with incomplete and complete information. Let the *social multiplier* be defined formally as the ratio of the variance of action x^a and the variance of the mean action $\overline{x}$. From Table 1 it is clear that in the complete information case the multiplier is equal to $\frac{1-\beta_2^2}{\beta_1^2}$ which is larger than the multiplier in the incomplete information case, $\frac{(1+\beta_2)^2}{1+\beta_2^2}$ for $N = 1$ (in fact it can be easily shown that the multiplier with complete information is larger than the multiplier with incomplete information for any $N < \infty$).

2.3 Extensions and Discussion

The existence and continuity results of Theorems 2.9 and 2.10 can be partly extended to economies with more general interaction structures. While for these economies the *Moderate Social Influence* assumption is not enough to guarantee existence, a stronger condition, like condition (6), in fact suffices for existence, uniqueness, and Lipschitz continuity. This is the case for both complete

[10]Extending their analysis to the case of incomplete information, we can show that this effect is also present in Glaeser and Scheinkman [32]'s specification of conformity.

[11]Following Glaeser and Scheinkman [32] and Glaeser, Sacerdote and Scheinkman [33], the different statistical properties of the variance of individual actions and the variance of the mean action across economies could be exploited to empirically identify the intensity of preferences for local conformity, that is, β_2 in our formulation.

[12]Since the process x^a, for $a \in \mathbb{A}$, is stationary and satisfies a strong mixing condition with exponentially declining bounds, central limit behavior results; see Rosenblatt [48].

and incomplete information economies. We illustrate the thrust of the argument in the following subsection. In Section 2.3.2 we characterize equilibria in static economies as stationary solutions of stochastic difference equations.

2.3.1 Equilibria in economies with general interaction structures

In the complete information setting, each symmetric equilibrium can be viewed as a fixed point of the operator V defined by (7). The proofs of the existence results in Theorems 2.9-(i) and 2.10-(i) are based on a mean contraction argument. This argument only applies to economies with one-sided interactions. For instance, in the case of complete information we show that

$$|Vg(\theta) - V\hat{g}(\theta)| \leq \frac{L(\theta^0)}{\alpha}|g \circ T(\theta) - \hat{g} \circ T(\theta)|.$$

In an economy with one-sided interactions the random variables $L(\theta^0)$ and $T(\theta) = (\theta^1, \theta^2, \dots)$ are independent. Under the *Moderate Social Influence* assumption we show that the operator we show that V satisfies the mean-contraction condition

$$\mathbb{E}|Vg(\theta) - V\hat{g}(\theta)| \leq \gamma \mathbb{E}|g(\theta) - \hat{g}(\theta)| \quad \text{where} \quad \gamma := \frac{\mathbb{E}L(\theta^0)}{\alpha} < 1,$$

which turns out to be the key to the existence proof. For general interaction structures, however, best reply functions depend on the entire configuration of taste shocks, and independence of $L(\theta^0)$ and $g \circ T(\theta)$ is typically lost. As a consequence, in this case, there is no reason to expect that a *Moderate Social Influence* condition, as Assumption 2.7, translates into a mean contraction property of the operator V.

To study existence, uniqueness and continuity in the context of economies with more general forms of interactions, we pursue a different strategy. Under the stronger contraction condition (6), in the proof of Theorems 2.9-(ii) and 2.10-(ii) we show that

$$|Vg(\theta) - V\hat{g}(\theta)| \leq \frac{L}{\alpha}|g \circ T(\theta) - \hat{g} \circ T(\theta)| \quad \text{and so} \quad \|Vg - V\hat{g}\|_\infty \leq \frac{L}{\alpha}\|g - \hat{g}\|_\infty,$$

and that Vg is Lipschitz continuous whenever g is. This method therefore delivers existence, uniqueness, and Lipschitz continuity. For a suitable modification of condition (6), it carries over to the case of more general interaction structures. Consider the case in which agents are located on the d-dimensional integer lattice $\mathbb{Z}^d$, and the preferences of the agent $a \in \mathbb{Z}^d$ are described by a utility function of the form

$$\left(x^a, \{x^b\}_{b \in N(a)}, \theta^a\right) \mapsto \hat{u}\left(x^a, \{x^b\}_{b \in N(a)}, \theta^a\right)$$

where $N(a) := \{b \in \mathbb{Z}^d : \|a - b\| = 1\}$ denotes the set of the agent's nearest neighbors. In such a more general model, each symmetric equilibrium is given by a fixed point of the operator

$$Vg(\theta) = \arg\max_{x^0 \in X} \hat{u}\left(x^0, \{g \circ T^a(\theta)\}_{a \in N(0)}, \theta^0\right).$$

If utility function satisfies the contraction condition

$$\left|\frac{\partial}{\partial x^a}\hat{u}\left(x^a, \{x^b\}_{b\in N(a)}, \theta\right) - \frac{\partial}{\partial x^a}\hat{u}\left(x^a, \{\hat{x}^b\}_{b\in N(a)}, \hat{\theta}\right)\right| \leq L \max\{|\hat{x}^b - x^b|, |\theta - \hat{\theta}| : b \in N(a)\},$$

using straightforward modifications of the arguments given in the proof of Theorem 2.9-(ii) and 2.10-(ii), it can be shown that V satisfies the contraction condition

$$|\hat{V}g - \hat{V}\hat{g}| \leq \frac{L}{\alpha} \max\{|g \circ T^b - \hat{g} \circ T^b| : b \in N(a)\}.$$

We then obtain that $\hat{V}$ is a contraction that maps a set of Lipschitz continuous functions continuously into itself. Two-sided interactions are simply a special case of this general model.

2.3.2 Equilibria as stationary solutions to stochastic difference equations

It is possible and instructive, in the class of static economies we have studied, to characterize an equilibrium as a stationary solution to a stochastic difference equation. Consider for illustration an economy with one-sided interactions and complete information. An equilibrium can be defined as a sequence of actions $\{x^a\}_{a\in\mathbb{A}}$ that satisfy the non-linear recursive relation

$$x^a = G^*(\theta^a, x^{a+1}) \qquad (a \in \mathbb{Z}) \tag{10}$$

where $G^*(\theta^a, x^{a+1})$ denotes the conditional best reply of the agent a, given θ^a and his neighbor's action x^{a+1}. Showing existence of a symmetric equilibrium is then equivalent to the existence of a stationary solution to the stochastic difference equation (10), where the index runs over agents $a \in \mathbb{A}$ rather than over time periods as is common in rational expectation models, e.g., Blanchard and Kahn [7], Sargent [49], Burke [15].[13] It follows, e.g. from results given in Chapter 8 of Borovkov [11], that (10) admits a unique stationary solution if the best reply function satisfies the *mean contraction condition*

$$\mathbb{E}|G^*(\theta^a, x^{a+1}) - G^*(\theta^a, \hat{x}^{a+1})| \leq \gamma |x^{a+1} - \hat{x}^{a+1}| \quad \text{for some} \quad \gamma < 1.$$

The proof of Theorem 2.9 shows that this is the case whenever the utility function u satisfies Assumption 2.7 (ii). The mean contraction condition, in turn, guarantees that the difference equation (10) has a unique stationary solution, that is it guarantees a transversality condition of the form

$$\lim_{\tau\to\infty} \frac{\partial}{\partial x^{a+\tau}} G^*\left(\theta^a, G^*\left(\theta^{a+1}, \ldots G^*\left(\theta^{a+\tau-1}, x^{a+\tau}\right)\ldots\right)\right) = 0.^{14}$$

Thus, in an economy with complete information, an equilibrium may be viewed as the unique stationary solution to the difference (10) with state space X. If the map G^* is linear, as is the

[13] We thank an anonymous referee for suggesting us to make this analogy explicit.

[14] Of course differentiability of G^* is not needed; we report the transversality condition only in this case for the sake of notational simplicity.

case in our example with quadratic utility, Example 2.2, for instance, an equilibrium can be represented by a deterministic difference equation of the form

$$x^a = \beta_1 \theta^a + \beta_2 x^{a+1},$$

and the transversality condition holds if its root is explosive, that is, if $\beta_2 < 1$. Similar arguments apply for economies with incomplete information and one-sided interactions. Then, a sequence $\{\hat{g}^a\}_{a \in \mathbb{A}}$ of measurable mappings $\hat{g}^a : \Theta^N \to X$ is an equilibrium if

$$\hat{g}^a(\theta^a, \cdot) = \arg\max_{x^a \in X} \mathbb{E}\left[u(x^a, \hat{g}^{a+1}, \theta^a) | \mathcal{F}_a\right](\cdot) =: \hat{G}(\theta^a, g^{a+1}) \qquad (a \in \mathbb{Z}) \tag{11}$$

where $\mathcal{F}_a$ denotes the information set of agent a, *i.e.*, the σ-field generated by the random variables $\theta^{a+1}, \ldots, \theta^{a+N}$. Again, the best reply function $\hat{G}$ admits a unique stationary solution (on a suitable function space) if it satisfies a mean contraction condition, which in turn we prove is satisfied under the assumptions of Theorem 2.10 (i). If the best reply function $\hat{G}(\theta^0, \cdot)$ is linear the analysis is again straightforward; we developed it above in Example 2.2.

Consider finally the case of economies with general interaction structures, which we discuss previously in this section. In the special case in which information is complete and an arbitrary agent $a \in \mathbb{A}$ interacts directly with agents $a-1$ and $a+1$, an equilibrium satisfies a 2nd order difference equation of the form

$$x^a = G^*\left(\theta^a, x^{a-1}, x^{a+1}\right) \qquad (a \in \mathbb{Z}).$$

Difference equations of this form appear in rational expectations models, see e.g., Benhabib and Farmer [5], and Kehoe and Levine [40]. In this case a symmetric equilibrium can also be represented by a stationary solution to the difference equation, provided appropriate transversality conditions hold (at plus and minus infinity). Results in the literature are generally limited to conditions for local stability; see e.g., Burke [15] and Benhabib and Farmer [5]. Finally, in the case of economies with incomplete information, the stochastic difference equation resulting from the best reply map of an arbitrary agent a is non-standard, as the information sets of agents $a \in \mathbb{A}$ does not form a filtration, and so the results in the rational expectations literature do not apply. Similarly, economies with more general local interactions, discussed previously in this section, or dynamic economies, as we study in the next section, give rise to equilibrium conditions that cannot be reduced to a standard form.

3 Dynamic Economies with Local Interactions

The theoretical literature on dynamic economies with local interactions has so far concentrated on models with *ad hoc* myopic dynamics. One of the main reasons for the widespread use of myopic dynamics is that the resulting equilibrium process for $\{x_t^a\}_{t \in \mathbb{N}}$ has been intensively investigated in the mathematical literature on interacting particle systems. Conditions for asymptotic stability of these processes have been established under suitable weak interaction and average contraction conditions; see e.g., Liggett [44], Kindermann and Snell [42] or Föllmer and Horst [29]. In this paper, we instead study economies with forward looking agents and consider rational expectations equilibrium dynamics. In our economy, therefore, an agent's actual action typically depends on his current type, on his past choices, on the present states of all the other agents and on the expected future behavior of his neighbors. His expectations with respect to the future actions of his neighbors are assumed to be rational, that is, the expectations are assumed to be consistent with the equilibrium dynamics of the neighbors' actions at each time in the future. In the context of a dynamic extension of the local conformity economy of Section 2.2, with quadratic preferences accounting for habit persistence effects, we can identify the characteristics of the behavior of locally interacting agents. This behavior stems from rational expectations about the future and about the behavior of neighbors, which do not exist in the standard analysis of myopic economies; see Section 4.3.

We are now going to introduce our notion of a dynamic economy with locally interacting agents as well as our equilibrium concept. As in the case of static economies, we consider a countable set $\mathbb{A}$ of agents. Each agent is infinitely lived, and is of type $\theta_t^a \in \Theta$ at time $t \in \mathbb{N}$. Types are assumed to be distributed independently and identically across agents and time.[15] The law of the random variables θ_t^a is denoted by ν. The instantaneous utility of agent $a \in \mathbb{A}$ at time $t \in \mathbb{N}$ depends on his current type θ_t^a, on the action x_t^a he chooses from a common compact and convex action set $X \subset \mathbb{R}$, and on his action x_{t-1}^a in the previous period $t-1$. We maintain the simple interaction structure introduced in the previous section, and assume that the agents' momentary utility also depends on the current action x_t^{a+1} of his neighbor, agent $a+1$. His instantaneous preferences at time t are described by a continuous utility function

$$\left(x_t^a, x_{t-1}^a, x_t^{a+1}, \theta_t^a\right) \mapsto u\left(x_t^a, x_{t-1}^a, x_t^{a+1}, \theta_t^a\right) \tag{12}$$

where the map $x_t^a \mapsto u\left(x_t^a, x_{t-1}^a, x_t^{a+1}, \theta_t^a\right)$ is assumed to be strictly concave. An agent's overall utility is the expected sum of future utilities, discounted at a rate $\beta < 1$. Prior to his choice at time t, the agent $a \in \mathbb{A}$ observes the realization only of his own type θ_t^a. In this sense, we focus on economies with incomplete information, in which agents do not observe the type of any other agent except their own. This is just for notational and analytical convenience. We allow agents to observe the entire action profile $\mathbf{x}_\tau = (x_\tau^a)_{a\in\mathbb{A}}$ of previous periods $\tau = t-1, t-2, \ldots$.

Definition 3.1 *A dynamic economy with local interactions is a tuple* $\mathcal{S} = (X, \Theta, u, \nu, \beta)$.

In order to describe an individual agent's optimization problem, we need to introduce some notation. We denote by $\mathbf{X} := \{\mathbf{x} = (x^a)_{a\in\mathbb{A}} : x^a \in X\}$ the space of all configurations of individual

[15] Independence over time simplifies the analysis substantially; see Stokey and Lucas with Prescott [51] for standard dynamic programming techniques to deal with correlated shocks over time.

actions and let $\mathbf{X^0} := \{x = (x^a)_{a\geq 0}\}$. We equip the *configuration spaces* $\mathbf{X}$ and $\mathbf{X^0}$ with the product topologies, and so compactness of the individual action spaces implies compactness of the configuration spaces $\mathbf{X}$ and $\mathbf{X^0}$. At each time t, the agent $a \in \mathbb{A}$ observes the entire action profile $\mathbf{x}_{t-1} \in \mathbf{X}$, in particular, he observes the past actions x^b_{t-1} of all the agents $b > a$ with whom he interacts either directly or indirectly. Even though his instantaneous utility only depends on present and expected future actions of his neighbor, the information contained in all the choices $(x^{a+c}_{t-1})_{c\geq 1}$ is used to predict his neighbor's future actions, and is therefore relevant for the solution to his decision problem. In fact, the choice x^{a+c}_{t-1} of the agent $a+c$ at time $t-1$ affects the action of agent $a+c-1$ in period t; this has an impact on the action of agent $a+c-2$ in period $t+1$, and so on. A rational agent $a \in \mathbb{A}$ anticipates these effects, and so he bases his current decision on all the states $(x^b_{t-1})_{b\geq a}$. In principle, agent a could base his decision at t on the action profiles at time $t-2$, $t-3$ etc., which he observes. In fact though, we study Markov perfect equilibria, in which the policy function of any agent at t will only depend on period $t-1$ actions. Of course actions at $t-2$ also affect future actions, but *at a Markov perfect equilibrium*, they only affect future actions *through the actions at* $t-1$. As a consequence, an agent observing actions at $t-1$ finds the information contained in actions at $t-2$ irrelevant to compute the expectations about future actions that he cares about for optimizing at time t.[16] As in the static case, we shall focus on symmetric equilibria.[17] Thus, we may assume that the optimal action of an economic agent $a \in \mathbb{A}$ is determined by a choice function $g : \mathbf{X^0} \times \Theta \to X$ in the sense that

$$x^a_t = g(T^a x_{t-1}, \theta^a_t) \quad \text{where} \quad T^a x_{t-1} = \{x^b_{t-1}\}_{b\geq a}.$$

In a symmetric situation, it is thus enough to analyze the optimization problem of a single reference agent, say of the agent $0 \in \mathbb{A}$. Given a *continuous* choice function $g : \mathbf{X^0} \times \Theta \to X$, the agent $a \geq 0$ takes as given his neighbor's current choice $g(T^a x_{t-1}, \theta^a_t)$. We denote by $\pi_g(T^a x_{t-1}; \cdot)$ the conditional law of the action x^a_t, given the previous configuration x_{t-1}, and so the choice function $g : \mathbf{X^0} \times \Theta \to X$ induces the Feller kernel

$$\Pi_g(x; \cdot) := \prod_{a=1}^{\infty} \pi_g(T^a x; \cdot). \tag{13}$$

[16] Our economy is formally equivalent to a dynamic game. See Fudenberg and Tirole [30], ch. 13, for the standard justifications for Markov perfect equilibria in dynamic games. Restricting the analysis to the class of Markov perfect equilibria is substantial; trigger strategies and other dynamic punishment strategies, for instance, are excluded. The analysis of these strategies requires different mathematical techniques as in the study of repeated games; see Ellison [22] for a study of repeated games in a local interaction environment where punishment strategies spread by *contagion* across the economy.

[17] We will also, as in the static case, restrict the arguments of each agent's choice to exclude the actions of agents 'on the left', that is, non payoff-relevant state variables and extrinsic effects. For instance, in our formulation of equilibrium, agent a's action at time t cannot affect agent $a+1$'s action at time $t+1$, even though this action affects agent a's utility at time $t+1$. This assumption also precludes the analysis of complex dynamic punishment strategies as, e.g., in Ellison [22], and of other forms of strategic interactions.

If the agent $0 \in \mathbb{A}$ believes that the agents $a > 0$ choose their actions according to g, the kernel Π_g describes the stochastic evolution of the process of individual states $\{(x_t^a)_{a>0}\}_{t\in\mathbb{N}}$. In this case, for any initial configuration of individual states $x \in \mathbf{X^0}$ and for each initial type θ_1^0, the optimization problem of the agent 0 is given by

$$\max_{\{x_t^0\}} \left\{ \int u(x_1^0, x^0, x_1^1, \theta_1^0)\pi_g(Tx; dx^1) + \sum_{t\geq 2} \beta^{t-1} \int u(x_t^0, x_{t-1}^0, x_t^1, \theta_t^0)\Pi_g^t(Tx; dx_t)\nu(d\theta_t^0) \right\}. \quad (14)$$

The value function associated with this dynamic choice problem is defined by the fixed point of the functional equation

$$\begin{aligned} V_g(x_{t-1}, \theta_t^0) = V_g(x_{t-1}^0, Tx_{t-1}, \theta_t^0) &= \max_{x_t^0 \in X} \left\{ \int u\left(x_{t-1}^0, x_t^0, y_t^1, \theta_t^0\right) \pi_g(Tx_{t-1}; dy_t^1) \right. \quad (15) \\ &\left. + \beta \int_{\mathbf{X^0}\times\Theta} V_g(x_t^0, \hat{x}_t, \theta^1)\Pi_g(Tx_{t-1}; d\hat{x}_t)\nu(d\theta^1) \right\}. \end{aligned}$$

The following is a well known result from the theory of dynamic programming, see e.g., Stokey, Lucas, and Prescott [51].

Lemma 3.2 *Assume that the choice map g is continuous. Under our assumptions on the utility function u, the functional fixed point equation (15) has a unique bounded and continuous solution V_g on $\mathbf{X^0} \times \Theta$. Moreover, the map $V_g(\cdot, Tx_{t-1}, \theta_t^0)$ is strictly concave on X and there exists a unique continuous policy function $\hat{g}_g : \mathbf{X^0} \times \Theta \to X$ that satisfies*

$$\begin{aligned} \hat{g}_g\left(x_{t-1}, \theta_t^0\right) &= \arg\max_{x_t^0 \in X} \left\{ \int u\left(x_{t-1}^0, x_t^0, y_t^1, \theta_t^0\right) \pi_g(Tx_{t-1}; dy_t^1) \right. \\ &\left. + \beta \int V_g(x_t^0, \hat{x}_t, \theta^1)\Pi_g(Tx_{t-1}; d\hat{x}_t)\nu(d\theta^1) \right\}. \quad (16) \end{aligned}$$

We can now define a symmetric Markov perfect equilibrium in a dynamic random economy with forward looking interacting agents.

Definition 3.3 *A symmetric Markov perfect equilibrium of a dynamic economy with forward looking and locally interacting agents $\mathcal{S} = (X, \Theta, u, \nu, \beta)$, is a map $g^* : \mathbf{X^0} \times \Theta \to X$ such that*

$$\begin{aligned} g^*\left(x_{t-1}, \theta_t^0\right) &= \arg\max_{x_t^0 \in X} \left\{ \int u\left(x_{t-1}^0, x_t^0, y_t^1, \theta_t^0\right) \pi_{g^*}(Tx_{t-1}; dy_t^1) \right. \quad (17) \\ &\left. + \beta \int V_{g^*}(x_t^0, \hat{x}_t, \theta^1)\Pi_{g^*}(Tx_{t-1}; d\hat{x}_t)\nu(d\theta^1) \right\}. \end{aligned}$$

By analogy to the static case, a symmetric equilibrium might be viewed as a fixed point of a certain operator. Indeed, every fixed point of the operator $\widehat{V}$ that acts on the class of bounded measurable functions $g : \mathbf{X^0} \times \Theta \to X$ by

$$\begin{aligned} \widehat{V}g(x,\theta^0) &= \arg\max_{\hat{x}^0 \in X} \left\{ \int u\left(\hat{x}^0, x^0, y^1, \theta^0\right) \pi_g(Tx; dy^1) \right. \\ &\left. + \beta \int V_g(\hat{x}^0, \hat{x}, \theta^1) \Pi_g(Tx; d\hat{x}) \nu(d\theta^1) \right\}, \end{aligned} \tag{18}$$

defines a symmetric equilibrium; and every symmetric equilibrium g^* satisfies the fixed point relation

$$\widehat{V}g^*(x,\theta^0) = g^*(x,\theta^0).$$

We proceed by establishing a series of general results, on the existence and the convergence of the equilibrium process. Such results require conditions on the policy function $\hat{g}_g$, and hence are not directly formulated as conditions on the fundamentals of the economy. In the next section, we will then introduce an example economy with quadratic preferences which we are able to study in detail. For this economy, we can show that our general conditions are satisfied, and hence they are not vacuous.

3.1 Existence and Lipschitz Continuity of Equilibrium

In order to state a general existence result for equilibria in dynamic random economies with forward looking interacting agents, we need to introduce the notion of a correlation pattern.

Definition 3.4 *For some finite $C > 0$, let*

$$L_+^C := \left\{ \mathbf{c} = (c_a)_{a \geq 0} \ : \ c_a \geq 0, \ \sum_{a \geq 0} c_a \leq C \right\}$$

denote the class of all non-negative sequences whose sum is bounded from above by C. A sequence $\mathbf{c} \in L_+^C$ will be called a correlation pattern with total impact C.

Each strictly positive correlation pattern $\mathbf{c} \in L_+^C$ gives rise to a metric

$$d_{\mathbf{c}}(x,y) := \sum_{a \in \mathbb{N}} c_a |x^a - y^a|$$

that induces the product topology on $\mathbf{X^0}$. Thus, $(d_{\mathbf{c}}, \mathbf{X^0})$ is a compact metric space. In particular, the class

$$\mathrm{Lip}_{\mathbf{c}}^C := \{ f : \mathbf{X^0} \to \mathbb{R} : |f(x) - f(y)| \leq d_{\mathbf{c}}(x,y) \}$$

of all functions $f : \mathbf{X^0} \to \mathbb{R}$ which are Lipschitz continuous with constant 1 with respect to the metric $d_{\mathbf{c}}$ is compact in the topology of uniform convergence.

Remark 3.5 *For a fixed* $\theta^0 \in \Theta$*, let* $g(\cdot, \theta^0) \in \mathrm{Lip}^C_{\mathbf{c}}$ *be the policy function of the agent* $0 \in \mathbb{A}$*. The constant* c_a *may be viewed as a measure for the total impact the current action* x^a *of the agent* $a \geq 0$ *has on the optimal action of agent* $0 \in \mathbb{A}$*. Since* $C < \infty$*, we have* $\lim_{a \to \infty} c_a = 0$*. Thus, the impact of an agent* $a \in \mathbb{A}$ *on the agent* $0 \in \mathbb{A}$ *tends to zero as* $a \to \infty$*. In this sense, we consider economies with weak social interactions. The quantity* C *provides an upper bound for the total impact of the configuration* $x = (x^a)_{a \geq 0}$ *on the current choice of the agent* $0 \in \mathbb{A}$*.*

We are now going to formulate a general existence result for symmetric Markov perfect equilibria in dynamic economies with local interactions.

Theorem 3.6 *Assume that there exists* $C < \infty$ *such that the following holds:*

i. For any $\mathbf{c} \in L^C_+$*, for all* $\theta^0 \in \Theta$ *and for each choice function* $g(\cdot, \theta^0) \in \mathrm{Lip}^C_{\mathbf{c}}$*, there exists* $F(\mathbf{c}) \in L^C_+$ *such that the unique policy function* $\hat{g}_g(\cdot, \theta^0)$ *which solves (16), is Lipschitz continuous with respect to the metric* $d_{F(\mathbf{c})}$ *uniformly in* $\theta^0 \in \Theta$*.*

ii. The map $F : L^C_+ \to L^C_+$ *is continuous.*

iii. We have $\lim_{n \to \infty} \|\hat{g}_{g_n}(\cdot, \theta^0) - \hat{g}_g(\cdot, \theta^0)\|_\infty = 0$ *if* $\lim_{n \to \infty} \|g_n - g\|_\infty = 0$*.*

Then the dynamic economy with local interactions has a symmetric Markov perfect equilibrium g^* *and the function* $g^*(\cdot, \theta^0)$ *is Lipschitz continuous uniformly in* θ^0*.*

Proof: For any $C < \infty$, the convex set L^C_+ may be viewed as a closed, and hence compact (with respect to the product topology) subset of the compact set $[0, C]^{\mathbb{N}}$. Thus, by (ii) the continuous map F has a fixed point $\mathbf{c}^*$. Due to (i) and (iii), the operator $\hat{V}$ defined by (18) maps the compact and convex set $\mathrm{Lip}^{C^*}_+$ continuously into itself where $C^* := \sum_{a \geq 0} c^*_a$. This shows that $\hat{V}$ has a fixed point g^*. □

3.2 Convergence to a Steady State

In the previous section, we have formulated conditions on a dynamic economy with local interactions $\mathcal{S} = (X, u, \beta, \Theta, \nu)$ which guarantee the existence of a symmetric Markov perfect equilibrium g^*. In this section, we study the asymptotic behavior of the process $\{\mathbf{x_t}\}_{t \in \mathbb{N}}$ in equilibrium. To this end, we denote by

$$\Pi_{g^*}(\mathbf{x}; \cdot) = \prod_{a \in \mathbb{A}} \pi_{g^*}(T^a x; \cdot)$$

the stochastic kernel on $\mathbf{X}$ induced by the policy function g^* and by $\Pi^t_{g^*}$, its t-fold iteration. Given an initial configuration $\mathbf{x} \in \mathbf{X}$, the measure $\Pi^t(\mathbf{x}; \cdot)$ describes the distribution of the configuration of individual states at time t. Let us introduce the vector $r^* = (r^*_a)_{a \in \mathbb{A}}$ with components

$$r^*_a := \sup\{\|\pi_{g^*}(x; \cdot) - \pi_{g^*}(y; \cdot)\| : x = y \text{ off } a\}. \tag{19}$$

Here, $\|\pi_{g^*}(x;\cdot) - \pi_{g^*}(y;\cdot)\|$ denotes the total variation of the signed measure $\pi_{g^*}(x;\cdot) - \pi_{g^*}(y;\cdot)$, and $x = y$ off a means that $x^b = y^b$ for all $b \neq a$. The next theorem gives sufficient conditions for convergence of the equilibrium process to a steady state. Its proof follows from a fundamental convergence theorem by Vaserstein [53].

Theorem 3.7 *If $\sum_{a\in\mathbb{A}} r^a_{g^*} < 1$, then there exists a unique probability measure μ^* on the infinite configuration space $\mathbf{X}$ such that, for any initial configuration $\mathbf{x} \in \mathbf{X}$, the sequence $\Pi^t_{g^*}(\mathbf{x};\cdot)$ converges to μ^* in the topology of weak convergence for probability measures.*

4 Example: Local Conformity and Habit Persistence

This section studies a dynamic extension of the local conformity economy introduced in Section 2.2 where the assumptions of Theorem 3.6 and of Theorem 3.7 can indeed be verified. As in the static model analyzed in Section 2.2, agents have a preference for conformity, and each agent a receives utility from conforming his own action to his neighbor's. In the dynamic economy we study in this section, however, agents also face habit persistence: each agent faces a disutility from changing his action over time. Habits and addictions are often associated with social interactions and preferences for conformity: for instance, the consumption of addictive substances, like smoke and several chemical drugs, is often initiated by the desire to conform with peers; the decision to commit criminal acts, partly determined by social interactions and preferences for conformity (see Glaeser and Scheinkman [31]), is difficult to reverse over time. We consider an economy where the preferences of a generic agent a are represented by the utility function

$$u\left(x^a_{t-1}, x^a_t, x^{a+1}_t, \theta^a_t\right) = -\alpha_1\left(x^a_{t-1} - x^a_t\right)^2 - \alpha_2\left(\theta^a_t - x^a_t\right)^2 - \alpha_3\left(x^{a+1}_t - x^a_t\right)^2 \qquad (20)$$

where α_1, α_2 and α_3 are positive constants. The agent's utility can be decomposed into three components. The first term of the utility function $u\left(x^a_{t-1}, x^a_t, x^{a+1}_t, \theta^a_t\right)$ in (20) represents habit persistence, the second the effect of the agent's own type, and the third the local conformity component, that is, social interactions.

4.1 Existence of Equilibria

Our first aim is to prove existence of a Markov perfect equilibrium for this economy.

Theorem 4.1 *Let $X = \Theta = [-1, 1]$. Assume that $\mathbb{E}\theta^0_t = 0$, and that an agent $a \in \mathbb{A}$ only observes his own type θ^a. If the instantaneous utility function takes the quadratic form in (20), the economy has a symmetric Markov perfect equilibrium g^*. The policy function g^* can be chosen to be of the linear form*

$$g^*(x, \theta^0) = c^*_0 x^0 + \gamma\theta^0 + \sum_{b\geq 1} c^*_b x^b$$

for some positive sequence $\mathbf{c}^* = (c^*_a)_{a\geq 0}$ and some constant $\gamma > 0$.

The proof of Theorem 4.1 will be carried out in several steps. In a first step, we prove the existence of an interior solution to an agent's optimization problem in an economy with quadratic utility functions.

Lemma 4.2 *Let $g : \mathbf{X^0} \times \Theta \to X$ be a continuous choice function for the agents $a > 0$. Under the assumptions of Theorem 4.1, the induced policy function, $\hat{g}_g$, of the agent $0 \in \mathbb{A}$ is uniquely determined and*

$$\mathbb{P}\left(\hat{g}_g(x_{t-1}, \theta_t^0) \in \{-1, 1\} \text{ for some } t \in \mathbb{N}\right) = 0. \tag{21}$$

Thus, we have almost surely an interior solution.

PROOF: The existence of a unique policy function follows from continuity of g along with the quadratic form of the utility functions, using standard arguments from the theory of discounted dynamic programming. In order to prove (21), we let

$$\tau := \inf\left\{t > 0 : \; \hat{g}_g(x_{t-1}, \theta_t^0) = 1\right\} \quad \text{and} \quad y_t := \hat{g}_g(x_{t-1}, \theta_t^0).$$

It suffices to show that $\mathbb{P}[\tau < \infty] = 0$. Let us assume to the contrary that $\mathbb{P}[\tau < \infty] > 0$. In such a situation, $y_\tau = 1$ is optimal and this means that

$$\begin{aligned} & -\alpha_1(1 - y_{\tau-1})^2 - \alpha_2(1 - x_\tau^1)^2 - \alpha_3(1 - \theta_\tau^0)^2 - \beta\alpha_1(1 - y_{\tau+1})^2 \\ \geq \; & -\alpha_1(y - y_{\tau-1})^2 - \alpha_2(y - x_\tau^1)^2 - \alpha_3(y - \theta_\tau^0)^2 - \beta\alpha_1(y - y_{\tau+1})^2 \end{aligned}$$

for all $y \in X$. Otherwise $y_\tau < 1$ would lead to a higher payoff. This, however, requires $\theta_\tau^0 = y_{\tau-1} = y_{\tau+1} = 1$. This shows that $y_t = 1 = \theta_t^0$ for all $t \in \mathbb{N}$. This, of course, contradicts $\mathbb{E}\theta_t^0 = 0$. Thus, $\mathbb{P}[\tau < \infty] = 0$. □

Let us now establish a representation of the agents' policy function in terms of the expected behavior of his neighbor. To this end, we denote by $\mathcal{M}(\mathbf{X^0})$ the class of all probability measures on $\mathbf{X^0}$ equipped with the topology of weak convergence. The utility of the agent $0 \in \mathbb{A}$ at time $t \in \mathbb{N}$ depends on the actions x_t^a taken by the agents $a > 0$ only through his neighbor's expected action

$$z_t := \int y^1 \Pi_g(Tx_t; dy) \quad \text{and through} \quad \int (y^1)^2 \Pi_g(Tx_t; dy).$$

We may thus view the agent's dynamic problem as an optimization problem depending only on the *stochastic* sequence $\{\theta_t^0\}_{t \in \mathbb{N}}$, and on the *deterministic* sequence $\left\{\Pi_g^t(Tx; \cdot)\right\}_{t \in \mathbb{N}}$. In fact, in our present setting, we can let $\mu(\cdot) := \Pi_g(Tx; \cdot)$, for any initial configuration $x \in \mathbf{X^0}$, and rewrite his optimization (14) as

$$\max_{\{x_t^0\}_{t \in \mathbb{N}}} \left\{ U(x_1^0, x_0^0, \theta_1^0, \mu) + \sum_{t \geq 2} \beta^{t-1} \int U(x_t^0, x_{t-1}^0, \theta_t^0, \mu\Pi_g^t) \nu(d\theta_t^0) \right\} \tag{22}$$

where

$$U(x_1^0, x_0^0, \theta^0, \mu) := -\alpha_1(x_1^0 - x_0^0)^2 - \alpha_1(x_1^0 - \theta^0)^2 - \alpha_3 \int (x_1^0 - y^1)^2 \mu(dy).$$

This allows us to show that the agent's optimal action is given as a weighted sum of his present type, of his action taken in the previous period and of the expected future actions of his neighbor.

Lemma 4.3 *Let the assumptions of Theorem 4.1 be satisfied. Given an action profile $x \in \mathbf{X^0}$ and a choice function $g : \mathbf{X^0} \times \Theta \to X$ for the agents $a > 0$, the policy function of agent $0 \in \mathbb{A}$ is of the linear form*

$$\hat{g}_g(x, \theta) = \gamma_1 x^0 + \gamma_2 \theta^0 + \sum_{t \geq 1} \delta_{t-1} \int y^1 \Pi_g^t(Tx; dy). \tag{23}$$

With $\lambda := \alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\beta$, the constants $\gamma_1, \gamma_2, \delta_0, \delta_1, \ldots$ are given by

$$\gamma_1 := \frac{\lambda - \sqrt{\lambda^2 - 4\alpha_1^2\beta}}{2\alpha_1\beta}, \quad \textit{and} \quad \gamma_2 := \frac{\alpha_2}{\lambda - \gamma_1\alpha_1\beta}, \tag{24}$$

and by

$$\delta_0 := \frac{\alpha_3}{\lambda - \gamma_1\alpha_1\beta} \quad \textit{and} \quad \delta_{t+1} = \frac{\alpha_1\beta}{\lambda - \gamma_1\alpha_1\beta}\delta_t \quad \textit{for } t \geq 1. \tag{25}$$

The constants in (24) and (25) do not depend on g and satisfy $\gamma_1 + \gamma_2 + \sum_{t \geq 0} \delta_t \leq 1$.

PROOF: Fix an initial configuration $x = (x^a)_{a \geq 0}$ and let $\mu := \Pi_g(Tx; \cdot)$. The value function associated with the optimization problem (22) solves the functional fixed point equation

$$\begin{aligned} V_g(x_0^0, \theta_1^0, \mu) \;&=\; \max_{x_1^0 \in X} \Big\{ -\alpha_1(x_0^0 - x_1^0)^2 - \alpha_2(\theta_1^0 - x_1^0)^2 - \alpha_3 \int (y^1 - x_1^0)^2 \mu(dy) \\ &\quad + \beta \int V_g(x_1^0, \theta_2^0, \mu\Pi_g)\nu(d\theta_2^0) \Big\}. \end{aligned} \tag{26}$$

In view of Lemma 4.2, the fixed point equation (26) has a unique solution $V_g^* : X \times \Theta \times \mathcal{M}(\mathbf{X^0}) \to \mathbb{R}$, the agent's policy function $\hat{g}_g : X \times \Theta \times \mathcal{M}(\mathbf{X^0}) \to X$ is uniquely determined and the optimal solution is almost surely interior. Thus, the first order condition takes the form

$$-2\alpha_1(x_0^0 - x_1^0) - 2\alpha_2(\theta_1^0 - x_1^0)^2 - 2\alpha_3 \int (y^1 - x_1^0)\mu(dy) + \beta \int \frac{\partial}{\partial x_1^0} V_g^*(x_1^0, \theta_2^0, \mu\Pi_g)\nu(d\theta_2^0) = 0,$$

and the envelope theorem gives us

$$\frac{\partial}{\partial x_1^0} V(x_1^0, \theta_2^0, \mu\Pi_g) = -2\alpha_1(x_1^0 - x_2^0) = -2\alpha_1 \left(x_1^0 - \hat{g}_g\left(x_1^0, \theta_2^0, \mu\Pi_g\right)\right). \tag{27}$$

This yields

$$x_1^0 = \frac{1}{\alpha_1 + \alpha_2 + \alpha_3 + \beta\alpha_1} \left(\alpha_1 x_0^0 + \alpha_2 \theta_1^0 + \alpha_3 \int y^1 \mu(dy) + \alpha_1\beta \; \hat{g}_g\left(x_1^0, \theta_2^0, \mu\Pi_g\right) \right). \tag{28}$$

Let us now assume that we have the following alternative representation for the optimal path $\{x_t^0\}_{t\in\mathbb{N}}$:

$$x_t^0 = \gamma_1 x_{t-1}^0 + \gamma_2 \theta_t^0 + \sum_{i=0}^{\infty} \delta_i z_{t+i} \in (0,1) \tag{29}$$

where z_t denotes the expected action of the agent $a = 1$ at time t. Using $\mathbb{E}\theta_t^0 = 0$, it does then follow from the first order condition, from (27) and from (28) that

$$x_1^0 = \frac{1}{\alpha_1 + \alpha_2 + \alpha_3 + \beta_1 \alpha_1} \left(\alpha_1 x_0^0 + \alpha_2 \theta_1^0 + \alpha_3 \int y^1 \Pi_g(x; dy) + \alpha_1 \beta \gamma_1 x_1^0 + \alpha_1 \beta \sum_{i=0}^{\infty} z_{2+i} \right). \tag{30}$$

Now we need to find coefficients $\gamma_1, \gamma_2, \delta_0, \delta_1, \ldots$ such that the representations in (29) and in (30) coincide. This can be accomplished recursively and yields the constants in (24) and (25). In order to prove that the sum of the coefficients is bounded from above by 1, we consider the situation in which the agents maximize the discounted sum of their expected utilities over the periods $t \in \{0, 1, \ldots, \tau\}$ and denote by $g^\tau(x, \theta^0)$ the optimal action of the agent $0 \in \mathbb{A}$. Using a cumbersome, but rather straightforward induction argument along with an argument similar to the one given in the proof of Lemma 4.2, one can easily show that

$$g^\tau(x, \theta^0) = \gamma_1^\tau x^0 + \gamma_2^\tau \theta^0 + \sum_{i=1}^{\tau} \delta_{i-1}^\tau z_i.$$

Here, the coefficients satisfy the recursive relations

$$\gamma_i^\tau = \frac{\alpha_i}{\lambda_\tau} \quad (i = 1, 2), \quad \delta_0^\tau = \frac{\alpha_3}{\lambda_{\tau+1}}, \quad \delta_i^\tau = \frac{\alpha_1 \beta}{\lambda_\tau} \delta_{i-1}^{\tau-1} \quad (i = 1, 2, \ldots) \quad \text{and} \quad \lambda_{\tau+1} = \lambda - \frac{\alpha_1^2 \beta}{\lambda_\tau}$$

with $\lambda^0 = \alpha_1 + \alpha_2 + \alpha_3$. This shows that $\gamma_i^\tau \to \gamma_i$ and $\delta_i^\tau \to \delta_i$ for all $i = 0, 1, 2, \ldots$ as $\tau \to \infty$. Thus,

$$\gamma_1 + \gamma_2 + \sum_{i \geq 0} \delta_i \leq 1 \quad \text{because} \quad \gamma_1^\tau + \gamma_2^\tau + \sum_{i \geq 0} \delta_i^\tau \leq 1 \quad \text{for all } \tau.$$

□

Our representation (23) of the policy function does not yet allow us to apply Theorem 3.6. For this, we need a representation of $\hat{g}_g$ in terms of the sequence $(x^a)_{a\geq 0}$. This, however, can be accomplished as follows: Let us fix a correlation pattern $\mathbf{c} = (c_a)_{a\geq 1} \in L_+^{1-\gamma_2}$ and assume for the moment that the choice function of the agents $a > 0$ takes the linear form

$$\tilde{g}(T^a x, \theta^a) = c_0 x^a + \gamma_2 \theta^a + \sum_{b \geq 1} c_b x^{a+b}. \tag{31}$$

In view of (23), we have $c_0 = \gamma_1$ and the continuous choice function $\tilde{g}$ induces a Feller kernel $\Pi_{\tilde{g}}$ on $\mathbf{X^0}$. Thus, it follows from (31) and from $\mathbb{E}\theta_t^0 = 0$ that

$$\int y^1 \Pi_{\tilde{g}}(x; dy) = \sum_{a \geq 0} c_a x^{a+1}.$$

Hence the expected action of the agent $a = 1$ in the second period is given by

$$\int y^1 \Pi^2_{\tilde{g}}(x; dy) = \sum_{a_1 \geq 0} c_{a_1} \int y^{a+1} \Pi_{\tilde{g}}(x; dy) = \sum_{a_1 \geq 0} c_{a_1} \sum_{a_2 \geq 0} c_{a_2} x^{a_1 + a_2 + 1}.$$

By induction we obtain

$$\int y^1 \Pi^t_{\tilde{g}}(x; dy) = \sum_{a_1 \geq 0} \left(c_{a_1} \sum_{a_2 \geq 0} \left(c_{a_2} \cdots c_{a_{t-1}} \sum_{a_t \geq 0} c_{a_t} x^{a_1 + \cdots + a_t + 1} \right) \cdots \right) \tag{32}$$

for all $t \in \mathbb{N}$. Thus, we have the alternative representation of our policy function:

$$\hat{g}_{\tilde{g}}(x, \theta) = \gamma_1 x^0 + \gamma_2 \theta + \sum_{b \geq 1} l_b x^b$$

where the positive sequence $(l_b)_{b \geq 1}$ is given by

$$l_b = F_b(c_0, c_1, \ldots, c_{b-1}) := \sum_{t \geq 1} \delta_{t-1} \left(\sum_{a_1 = 0}^{b-1} \left(c_{a_1} \sum_{a_2 = 0}^{b-1} c_{a_2} \cdots \right) \sum_{a_t = 0}^{b-1} c_{a_t} \right) \mathbf{1}_{\{\sum_{i=1}^t a_i = b-1\}}. \tag{33}$$

This representation allows us to prove the main result of this section.

Proof of Theorem 4.1: Since $\hat{g}_{\tilde{g}}(x, \theta^0) \in X$, we have $\sum_{b \geq 1} l_b \leq 1 - \gamma_1 - \gamma_2$. Thus, the map F defined by

$$F(\mathbf{c}) := (F_b(\gamma_1, c_1, \ldots c_{b-1}))_{b \geq 1} \tag{34}$$

maps the set $L_+^{1-\gamma_1-\gamma_2}$ into itself. Since F is continuous in the product topology, it has a fixed point $\mathbf{c}^* = (c^*_a)_{a \geq 1}$ and

$$l_b = F_b(\gamma_1, c^*_1, \ldots, c^*_{b-1}) = c^*_b \quad \text{for all } b \geq 1.$$

Finally, let $c^*_0 = \gamma_1$ and $\gamma = \gamma_2$, as defined in (24). Then the assumptions of Theorem 3.6 are satisfied. This proves the assertion. □

4.2 Convergence to a Steady State

We turn now to study the convergence to a unique steady state for the example economy with quadratic preferences. To this end, we consider the representation

$$g^*(x; \theta^0) = c^*_0 x^0 + \gamma_2 \theta^0 + \sum_{a \geq 1} c^*_a x^a.$$

of the policy function g^*. For any two configurations $x, y \in \mathbf{X^0}$ which differ only at site $a \in \mathbb{A}$,

$$|g^*(x, \theta^0) - g^*(y, \theta^0)| \leq c^*_a |x^a - y^a|,$$

Assuming that the taste shocks are uniformly distributed on $[-1, 1]$, we obtain

$$|\pi_{g^*}(x; A) - \pi_{g*}(y; A)| \leq 2c_a^*$$

for all $A \in \mathcal{B}([-1, 1])$, and so $\sum_{a \geq 0} r_{g^*}^a < 1$ if $\sum_{a \geq 0} c_a^* < \frac{1}{2}$. This yields convergence to a steady state if α_1 is big enough and if α_3 is small enough, i.e., if the interaction between different agents is not too strong.

4.3 Rational Expectations, Local Conformity, and Habit Persistence

As already noticed, the literature has only studied the myopic dynamics of economies with local interactions. In particular, the specific form of myopic expectations assumed in the literature contains two components:

i. an agent $a \in \mathbb{A}$, when choosing x_t^a at time t, is assumed not to consider that he or his neighbors will choose again at time $t + 1$;

ii. an agent $a \in \mathbb{A}$, when choosing x_t^a at time t, is assumed to expect his neighbors, agents $b > a$, not to change their previous actions.

In this case, the dynamics describe a backward looking behavior of the agents since the configuration $(x_t^a)_{a \in \mathbb{A}}$ only depends on the current configuration of types and on the previous action profile $(x_{t-1}^a)_{a \in \mathbb{A}}$. In this section, we study a simple 2 period version of the local conformity and habit persistence introduced in the previous section and solve it for both myopic and rational expectations, to illustrate the effects of rationality of expectations on the dynamics of actions. Each agent $a \in \mathbb{A}$ chooses at time t and $t+1$, respectively, actions x_t^a, x_{t+1}^a. The initial condition of the dynamic economy, that is, the configuration of actions at time $t - 1$, is $\{\underline{x}^a\}_{a \in \mathbb{A}}$. Consider first, as a benchmark, the case in which a generic agent $a \in \mathbb{A}$ chooses x_t^a at time t in a fully myopic manner: he does not expect to choose at $t + 1$, that is, he expects $x_{t+1}^a = x_t^a$; and he expects his neighbors' actions to remain $\underline{x}^b$ both at t and at $t + 1$. In this case it is easy to show (we do not report the calculations for this section) that his choice will satisfy:

$$x_t^a = \frac{\alpha_2}{c_0}\theta_t^a + \beta\frac{\alpha_2}{c_0}\mathbb{E}(\theta_t^a) + \frac{\alpha_1}{c_0}\underline{x}^a + (1 + \beta)\frac{\alpha_3}{c_0}\underline{x}^{a+1} \tag{35}$$

where $c_0 = \alpha_1 + \alpha_2 + \alpha_3 + \beta\frac{\alpha_2+\alpha_3}{\alpha_1+\alpha_2+\alpha_3}$. Note that action x_t^a is chosen as a convex combination of the arguments $(\theta_t^a, \mathbb{E}(\theta_t^a), \underline{x}^a, \underline{x}^{a+1})$. That is, the weights on the arguments sum to unity. Consider now the case in which agent a still expects his neighbors' actions to remain $\underline{x}^b$ at both t and at $t + 1$, but he now realizes that he will choose again at time $t + 1$ and that his choice will be optimal (conditionally on $x_t^b = x_{t+1}^b = \underline{x}^b$, for all agents $b > a$). In this case, his choice at time t will satisfy:

$$x_t^a = \frac{\alpha_2}{c_1}\theta_t^a + \beta\frac{\alpha_2}{c_1}\frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3}\mathbb{E}(\theta_t^a) + \frac{\alpha_1}{c_1}\underline{x}^a + \left(1 + \beta\frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3}\right)\frac{\alpha_3}{c_1}\underline{x}^{a+1} \tag{36}$$

where $c_1 := \alpha_1 + \alpha_2 + \alpha_3 + \alpha_1 \beta \frac{\alpha_2+\alpha_3}{\alpha_1+\alpha_2+\alpha_3}$. Thus, an agent's choice is given in terms of a convex combination of his type, his expected type and in terms of his own and his neighbor's action in the previous period. Because $c_1 < c_0$, though, agent a's own type θ_t^a and his past choice $\underline{x}^a$ have now a larger weight in his choice, while the mean action and the action of the neighbor he wishes to conform to have a smaller weight. This effect is due to the fact that the agent now rationally anticipates that he can re-optimize at time $t+1$, and hence at time t, he will attempt matching more directly those arguments which change at time $t+1$ (remember he myopically expects his neighbor not to change actions at t nor at $t+1$). Rational expectations of the agent's own dynamic choice therefore reduce the dependence of the agents' actions on the local conformity effect, but on the other hand strengthen the effect of habits. Consider now the case in which the expectations of the agents in the local conformity and habits economy are fully rational. Agent a in this case, when choosing at time t, not only anticipates rationally his own choices at time $t+1$, but also the choices of his neighbor, agent $a+1$, at time t and $t+1$. In this case, his choice will satisfy:

$$x_t^a = \frac{\alpha_2}{c_1}\theta_t^a + \left(\beta\frac{\alpha_2}{c_1}\frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_3} + A\right)\mathbb{E}(\theta_t^a) + \frac{\alpha_1}{c_1}\underline{x}^a + \sum_{b=a+1}^{\infty} f_b \underline{x}^b \tag{37}$$

where A and f_b ($b \geq a+1$) are positive constants, and x_t^a is given in terms of a convex combination of the arguments $\left(\theta_t^a, \mathbb{E}(\theta_t^a), \underline{x}^a, \{\underline{x}^b\}_{b\geq a+1}\right)$. Taking into account the rational expectations of agent a regarding the behavior of his neighbors has the effect that his choice at t depends on the past actions of all the agents to his right, and not only on the choice of his immediate neighbor, as in the previous case. This introduces long spatial correlation terms in the resulting configuration of actions. But most importantly, comparing (36) and (37), it is apparent that the fully rational choice of agent a is more weighted on the mean shock $\mathbb{E}(\theta_t^a)$ and less on the past action of the neighbors. This property of the equilibrium actions is the consequence of the rational expectations of agent a regarding the persistence of the actions of the agents $b > a$: even though all agents face habits, they still will, in general, change their actions at time t, and hence agent a's action will depend less on the past actions of his neighbors; this further limits the component of local conformity in the choice of agents in this economy. Closed form solutions for the policy function (37), that is, solutions for A and $\{f_b\}_{b\geq a+1}$, are hard to derive. We have therefore run some simulations to better illustrate the properties of the policy function; Figure 1 reports the shape of $\{f_b\}_{b\geq a+1}$ in two representative simulations.[18] First of all, notice that when local conformity is not the predominant component of the agent's preferences (that is, when $\alpha_1 = \alpha_2 = \alpha_3 = 1$ in the simulations), the number of neighbors that significantly affect each agent's action is relatively limited, of the order of 7 or 8. On the contrary, when local

[18]The code for the simulations, available from the authors, uses a recursive algorithm to compute the weights associated to $\underline{x}^b$, for any arbitrary $b > a$, in agent a's policy function. The code does not perform any truncation or approximation, but rather computes the exact weights.

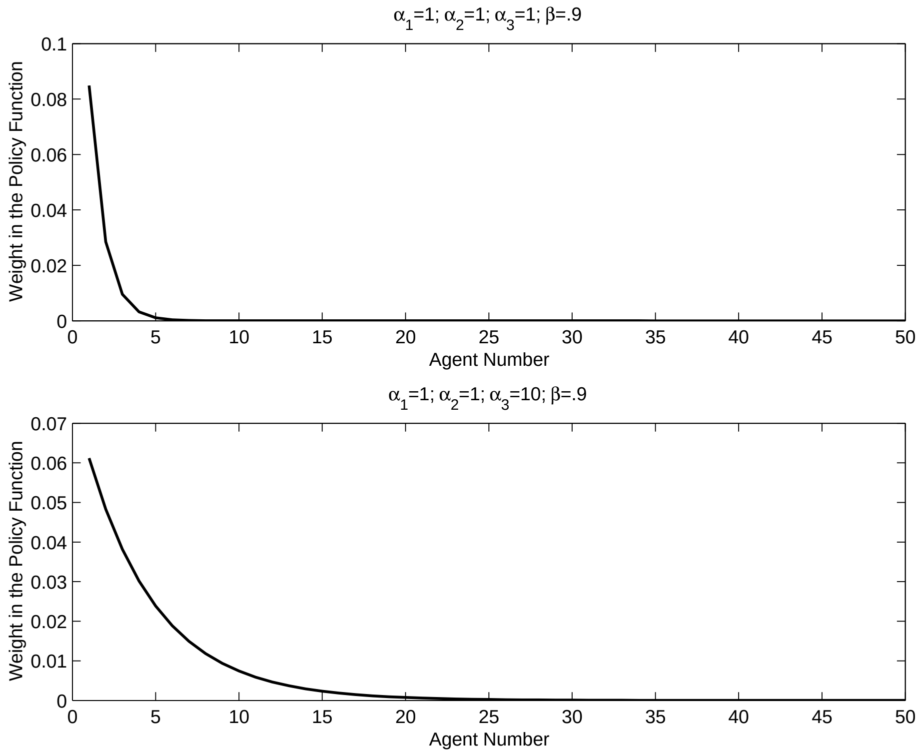

Figure 1: Weights of the Neighbors' Past Actions in the Policy Function

conformity is predominant (that is, when $\alpha_1 = \alpha_2 = 1$, and $\alpha_3 = 10$ in the simulations) the number of neighbors that affect each agent's action increases substantially, about three times in our parametrization of preferences. As we noted, the policy function of myopic agents (35) overestimates, with respect to the policy function of fully rational agents (37), the dependence of equilibrium actions on the agents' neighbors, that is, it overestimates the local conformity effect. In our simulations such overestimation is quite substantial. When local conformity is predominant in preferences the weight on $\underline{x}^{a+1}$ in (35) is .8382, while the sum of the weights on $\underline{x}^b$, $b \geq a+1$, in (37) is only .2928. When local conformity is not predominant, instead, the respective weights are .3611 and .1278. We conclude therefore that our analysis of the local conformity and habits economy shows that the effect of rational expectations dynamics is to spread the correlation of equilibrium actions across several agents in the economy, but to substantially reduce the effects of the local interactions, that is of the agents' preferences for local conformity.

5 Dynamic Economies with Local and Global Interactions

This section extends the analysis of dynamic economies with local interactions to economies in which interactions have an additional *global* component. In particular, we study economies in which each agent's preferences depend on the average action of all agents. Such dependence might occur, for instance, if agents have preferences for *social status*. Similarly, preferences to adhere to aggregate norms of behavior, such as specific group cultures like piercing or rap music, also give rise to the form of global interactions we study in this section. More generally, the analysis of global interactions could capture other externality effects as well as price effects. Formally, we study a dynamic economy with quadratic preferences, as in our analysis of local conformity and habit persistence in Section 4.3, in which the preferences of each agent $a \in \mathbb{A}$ also depend on the average action of the agents in the economy,

$$\varrho(\mathbf{x}) := \lim_{n\to\infty} \frac{1}{2n+1} \sum_{a=-n}^{n} x^a,$$

when the limit exists. We denote by $\mathbf{X_e}$ the set of all configurations such that the associated average action exists:

$$\mathbf{X_e} := \left\{ \mathbf{x} \in \mathbf{X} : \exists\ \varrho(\mathbf{x}) := \lim_{n\to\infty} \frac{1}{2n+1} \sum_{a=-n}^{n} x^a \right\}.$$

The preferences of the agent $a \in \mathbb{A}$ in period t are described by the instantaneous utility function $u : \mathbf{X_e} \times \Theta \to \mathbb{R}$ of the quadratic form

$$\begin{aligned} & u\left(x_{t-1}^a, x_t^a, x_t^{a+1}, \theta_t^a, \varrho(\mathbf{x}_t)\right) \\ = \quad & -\alpha_1 \left(x_{t-1}^a - x_t^a\right)^2 - \alpha_2 \left(\theta_t^a - x_t^a\right)^2 - \alpha_3 \left(x_t^{a+1} - x_t^a\right)^2 - \alpha_4 \left(\varrho(\mathbf{x}_t) - x_t^a\right)^2 \end{aligned} \tag{38}$$

for some positive constants $\alpha_1, \ldots, \alpha_4$. The first term in (38) represents habit persistence, the second the own effect of the agent's type, the third the local conformity component, and the last the global conformity component. As before, we assume that $X = \Theta = [-1, 1]$ and that $\mathbb{E}\theta^0 = 0$. We also assume that information is incomplete so that an agent $a \in \mathbb{A}$ at time t only observes his own type θ_t^a, and all agents' past actions. Following the analysis of Section 4.3, we can define a symmetric Markov perfect equilibrium of this economy.

Definition 5.1 *Let $\mathbf{x} \in \mathbf{X_e}$ be the initial configuration of actions. A symmetric Markov perfect equilibrium of a dynamic economy with local and global interactions is a map $g^* : \mathbf{X}^0 \times \Theta \times X \to X$ and a map $F^* : X \to X$ such that:*

$$\begin{aligned} g^*\left(x_{t-1}, \theta_t^0, \varrho_t\right) = \quad & \arg\max_{x_t^0 \in X} \left\{ \int u\left(x_{t-1}^0, x_t^0, y_t^1, \theta_t^0, \varrho_t\right) \pi_{g^*}(Tx_{t-1}; dy_t^1) \right. \\ & \left. + \beta \int V_{g^*}\left(x_t^0, \hat{x}_t, \theta^1, \varrho_{t+1}\right) \Pi_{g^*}(Tx_{t-1}; d\hat{x}_t) \nu(d\theta^1) \right\}. \end{aligned} \tag{39}$$

and

$$\varrho_{t+1} = F^*(\varrho_t),$$

and

$$\varrho_1 = \varrho(\mathbf{x}) \quad \textit{and} \quad \varrho_t = \varrho(\mathbf{x}_t) \ \textit{almost surely.}$$

At a symmetric Markov perfect equilibrium the policy function g^* determines the optimal action of each agent under the rational expectations condition that all agents choose their own action by the policy function g^*. Moreover, each agent rationally expects the sequence of average actions $\{\varrho(\mathbf{x}_t)\}_{t\in\mathbb{N}}$ to be determined recursively via the map F^*. In studying existence of an equilibrium of a dynamic model with local and global interactions several mathematical difficulties arise. First of all, the endogenous sequence of average actions $\{\varrho(\mathbf{x}_t)\}_{t\in\mathbb{N}}$ might not be well-defined for all t (that is, $\mathbf{x}_t$ might not lie in $\mathbf{X_e}$ for some t). Moreover, even when $\mathbf{x_t} \in \mathbf{X_e}$, an agent's utility function depends on the action profile $\mathbf{x_t}$ in a global manner through the average action $\varrho(\mathbf{x}_t)$, and hence will typically not be continuous in the product topology. Thus, standard results from the theory of discounted dynamic programming cannot be applied to solve the agent's dynamic optimization problem in (39). We are going to show though that, when preferences are quadratic and interactions are one-sided, i) the endogenous sequence of average actions $\{\varrho(\mathbf{x}_t)\}_{t\in\mathbb{N}}$ exists almost surely if the exogenous initial configuration $\mathbf{x}$ belongs to $\mathbf{X_e}$, and that ii) it follows a deterministic recursive relation. This allows us to view dynamic models with local and global interactions as purely local interaction models to which a deterministic time-varying component is added.[19] More specifically, in order to establish the existence of an equilibrium for an economy with locally and globally interacting agents, we proceed in three steps. We first consider an economy where the agents' utility depends on some *exogenous* quantity ϱ, constant over time. We then extend the analysis to the case in which the agents' utility depends on some *exogenous* but time-varying quantity $\{\varrho_t\}_{t\in\mathbb{N}}$ described in terms of a possibly non-linear recursive relation. Finally, we show that the recursive structure of $\{\varrho_t\}_{t\in\mathbb{N}}$ is preserved if we require each element of the sequence to be endogenously determined as the average equilibrium action: $\varrho_t = \varrho(\mathbf{x}_t)$, for any t, at the equilibrium configuration $\mathbf{x}_t$. Consider then, first of all, an economy in which the agents' utility depends on some *exogenous* constant quantity ϱ.

Lemma 5.2 *Assume that the agents' instantaneous utility functions take the quadratic form*

$$u\left(x_{t-1}^a, x_t^a, x_t^{a+1}, \theta_t^a, \varrho\right) = -\alpha_1\left(x_{t-1}^a - x_t^a\right)^2 - \alpha_2\left(\theta_t^a - x_t^a\right)^2 - \alpha_3\left(x_t^{a+1} - x_t^a\right)^2 - \alpha_4\left(\varrho - x_t^a\right)^2 \quad (40)$$

[19] Such separation arguments originally appeared in Föllmer and Horst [29] and Horst [35] where the long run dynamics of locally and globally interacting Markov chains are analyzed. Similar separation arguments have also been successfully applied in the context of a static equilibrium model with locally and globally interacting agents in Horst and Scheinkman [37], and in the context of microstructure models for financial markets in Horst [36].

with $\varrho \in X$ and positive constants $\alpha_1, \ldots, \alpha_4$, then there exists a policy function $g^* : \mathbf{X}^0 \times \Theta \times X \to X$ such that

$$g^*\left(x_{t-1}, \theta_t^0, \varrho\right) = \quad \arg\max_{x_t^0 \in X} \left\{ \int u\left(x_{t-1}^0, x_t^0, y_t^1, \theta_t^0, \varrho\right) \pi_{g^*}(Tx_{t-1}; dy_t^1) \right.$$

$$\left. + \beta \int V_{g^*}\left(x_t^0, \hat{x}_t, \theta^1, \varrho\right) \Pi_{g^*}(Tx_{t-1}; d\hat{x}_t) \nu(d\theta^1) \right\}.$$

The policy function g^* can be chosen of the linear form

$$g^*(x, \theta^0, \varrho) = e_0^* x^0 + \epsilon\theta^0 + \sum_{b\geq 1} e_b^* x^b + A(\varrho) \tag{41}$$

where the correlation pattern $\mathbf{e}^* = (e_a^*)_{a\geq 0}$, and the constant $\epsilon > 0$ are independent of ϱ.

Proof: Fix a continuous policy function g for the agents $a \neq 0$. Continuity of g together with our special interaction structure guarantees that the optimization problem of the agent $0 \in \mathbb{A}$ can be solved by standard methods from the theory of discounted dynamic programming. In fact, using the same arguments as in the proof of Lemma 4.3, we see that the optimal action of the agent 0 is given in terms of a weighted average of his neighbor's future action and of ϱ:

$$\hat{g}_g(x, \theta^0, \varrho) = \epsilon_1 x^0 + \epsilon_2 \theta^0 + \sum_{t\geq 1} \left\{ \delta_{t-1} \int y^1 \Pi_\varrho^t(Tx; dy) + \eta_{t-1}\varrho \right\} \tag{42}$$

where ϵ_1, ϵ_2 and (η_t, δ_t) $(t \in \mathbb{N})$ are strictly positive constants satisfying $\epsilon_1 + \epsilon_2 + \sum_{t\geq 1}(\delta_t + \eta_t) \leq 1$. Suppose now that the agent $a = 0$ assumes that the other players' policy function is given by

$$g(T^a x, \theta^a, \varrho) = \epsilon_1 x^0 + \epsilon_2 \theta^0 + \sum_{b\geq 1} e_b x^{a+b} + B(\varrho) \in [-1, 1] \tag{43}$$

where the non-negative correlation pattern $\{e_a\}_{a\geq 1}$ is independent of the states $\{x^a\}_{a\geq 1}$ and where the constant $B(\varrho)$ depends only on ϱ. It is then straightforward to show agent 0's policy function $\hat{g}_g$ defined by (42), takes the form

$$\hat{g}_g(x, \theta^0, \varrho) = \epsilon_1 x^0 + \epsilon_2 \theta^0 + \sum_{a\geq 1} l_a x^a + B(\varrho) \sum_{t\geq 1} \delta_{t-1} C^t + \varrho \sum_{t\geq 1} \eta_{t-1}$$

where the constants $l_a = F_a(\epsilon_1, e_1, \ldots, e_{a-1})$ are given by (33), and $C := \epsilon_1 + \sum_{a\geq 1} e_a$. In order to prove our assertion it is thus enough to find a correlation pattern $\mathbf{e}^* = (e_a^*)_{a\geq 1}$ and some $B(\varrho)$ such that

$$e_a^* = F_a(\epsilon_1, e_1^*, \ldots, e_{a-1}^*) \text{ for all } a \geq 1 \tag{44}$$

and such that

$$B(\varrho) \sum_{t\geq 1} \delta_{t-1} C^t + \varrho \sum_{t\geq 1} \eta_{t-1} = B(\varrho) \quad \text{where} \quad C := \epsilon_1 + \sum_{a\geq 1} e_a^*. \tag{45}$$

To this end, we first consider the case $B(\varrho) = 1 - C$. Condition (45) translates then into

$$(1-C)\sum_{t\geq 1}\delta_{t-1}C^t + \varrho\sum_{t\geq 1}\eta_{t-1} = 1 - C.$$

By continuity, this equation has a solution $C(\varrho) \in [0,1]$, and $C(1) \leq C(\varrho)$. Choose a sequence $c = (c_a)_{a\geq 0}$ with $c_0 = \epsilon_1$ such that $\sum_{a\geq 0} c_a \leq C(\varrho)$. The correlation pattern $l = (l_a)_{a\geq 0}$ in (43) satisfies then $\sum_{a\geq 0} l_a \leq C(\varrho)$, because an agent's optimal action is almost surely interior, and because the quantities l_a are increasing in $c_0, \ldots, c_{a-1}$. Hence the same arguments as in the proof of Theorem 4.1 show that, for any $\varrho \in [-1,1]$, the map F has a fixed point, i.e., there exists a sequence $(e_a^*)_{a\geq 1}$ such that (44) is satisfied. For a given ϱ, we have a fixed point in the set of sequences such that $\sum_{a\geq 0} c_a \leq C(\varrho)$. In fact, we can look for a fixed point such that $\sum_{a\geq 0} c_a \leq C(1) \leq C(\varrho)$. An equilibrium for an arbitrary external condition $\varrho \in [-1,1]$ is then given by

$$g^*(x, \theta^0, \varrho) = e_0^* x^0 + \epsilon\theta^0 + \sum_{b\geq 1} e_b^* x^b + A^*(\varrho)$$

where $\epsilon = \epsilon_2$, $e_0^* = \epsilon_1$ and $A^*(\varrho)$ is given by

$$A^*(\varrho) := \varrho\frac{\sum_{t\geq 1}\eta_{t-1}}{1 - \sum_{t\geq 1}\delta_{t-1}(C^*)^t}.$$

□

Note that the policy function (41) has the property that a change in ϱ has a direct effect on the chosen action but does not affect the dependency of the action on the realized agent's type nor on the neighbors' actions. In other words, both ϵ and the correlation structure $\{e_b^*\}_{b\geq 0}$ are independent of ϱ. It is this property that allows us to proceed to the second step of our analysis, that is, to study the case in which the agents' preferences at time t are described by a quadratic utility function

$$\begin{aligned} & u\left(x_{t-1}^a, x_t^a, x_t^{a+1}, \theta_t^a, \varrho_t\right) \\ = \; & -\alpha_1\left(x_{t-1}^a - x_t^a\right)^2 - \alpha_2\left(\theta_t^a - x_t^a\right)^2 - \alpha_3\left(x_t^{a+1} - x_t^a\right)^2 - \alpha_4\left(\varrho_t - x_t^a\right)^2, \end{aligned} \tag{46}$$

and the dynamics of the process $\{\varrho_t\}_{t\in\mathbb{N}}$ is described by a possibly non-linear recursive relation of the form

$$\varrho_{t+1} = F(\varrho_t) \quad \text{for some continuous function} \quad F : X \to X. \tag{47}$$

Since F is continuous, an agent's optimization problem can again be solved using standard results from the theory of discounted dynamic programming. In fact, we can apply the same arguments as in the proof of the previous Lemma in order to show that for given a continuous policy function g for the agents $a \neq 0$, the optimal action of the agent $a = 0$ is given in terms of a weighted average of his neighbors future actions and of future external conditions. In order to make this more precise, we fix a continuous policy function $g : \mathbf{X}^0 \times \Theta \times X \to X$ for the agents $a \neq 0$ and denote by $\pi(T^a x, \varrho; \cdot)$ the conditional distribution of the agent a's optimal choice, given the states $(x_b)_{b \geq a}$ and given ϱ. Since preferences now depend on ϱ_t, which is no longer constant but evolves according to a deterministic dynamics, the expected action of the agent $1 \in \mathbb{A}$ in period t, given some continuous policy function $g : \mathbf{X}^0 \times \Theta \times X \to X$ is of the form

$$\int y^1 \Pi_{\varrho_1} \cdots \Pi_{\varrho_t}(Tx; dy) \quad \text{where} \quad \Pi_{\varrho_t}(Tx; \cdot) := \prod_{a \geq 1} \pi(T^a x, \varrho_t; \cdot)$$

and $\pi(x, \varrho_t; \cdot)$ denotes the distribution of the random variable $g(x, \cdot, \varrho_t)$. Let $\delta_\varrho(\cdot)$ be the Dirac measure concentrated on ϱ, and assume that the policy function g is continuous. The kernel

$$\widehat{\Pi}(x, \varrho; \cdot) := \Pi_\varrho(x; \cdot) \otimes \delta_{F(\varrho)}(\cdot).$$

which describes the joint evolution of the sequences $\{x_t^a\}_{t \in \mathbb{N}}$ $(a \geq 1)$ and $\{\varrho_t\}_{t \in \mathbb{N}}$ has the Feller property because F is continuous. Hence, an agent's optimization problem can again be solved using methods from discounted dynamic programming and arguments as in the previous section show that the optimal choice of our reference agent $a = 0$ is of the form

$$\hat{g}_g(x, \theta^0, \varrho_1) = \epsilon_1 x^0 + \epsilon_2 \theta^0 + \sum_{t \geq 1} \left\{ \delta_{t-1} \int y^1 \Pi_{\varrho_1} \cdots \Pi_{\varrho_t}(Tx; dy) + \eta_{t-1} \varrho_t \right\}. \tag{48}$$

If agent 0 expects all the other agents' policy functions to take the linear form

$$g^*(T^a x, \theta^a, \varrho) = e_0^* x^a + \epsilon \theta^a + \sum_{b \geq a+1} e_b^* x^b + \sum_{t \geq 1} h_t \varrho_t$$

with the correlation pattern $e^* = (e_a^*)_{a \geq 0}$ derived in Lemma 5.2 and with a suitable sequence $h = (h_t)_{t \geq 1}$, then (48), using tedious but straightforward calculations, can be written as

$$\hat{g}_g(x, \theta^0, \varrho_1) = e_0^* x^0 + \epsilon \theta^0 + \sum_{b \geq 1} e_b^* x^b + \sum_{t \geq 1} G_t(h) \varrho_t$$

for suitable constants $G_t(h)$. By analogy to the proof of Lemma 5.2 one can now show that there exists a sequence $h^* = (h_t^*)_{t \in \mathbb{N}}$ which does not depend on the specific sequence $\{\varrho_t\}_{t \in \mathbb{N}}$ such that

$$G_t(h^*) = h_t^*.$$

Thus, we have the following result.

Lemma 5.3 *Assume that the agents' instantaneous utility functions take the quadratic form (46) and that the sequence $\{\varrho_t\}_{t \in \mathbb{N}}$ follows the deterministic recursive dynamics (47). Then there exists a policy function $g^* : \mathbf{X}^0 \times \Theta \times X \to X$ such that*

$$\begin{aligned} g^*\left(x_{t-1}, \theta_t^0, \varrho_t\right) &= \arg\max_{x_t^0 \in X} \left\{ \int u\left(x_{t-1}^0, x_t^0, y_t^1, \theta_t^0, \varrho_t\right) \pi_{g^*}(Tx_{t-1}; dy_t^1) \right. \\ &\quad \left. + \beta \int V_{g^*}\left(x_t^0, \hat{x}_t, \theta^1, F(\varrho_t)\right) \Pi_{g^*}(Tx_{t-1}; d\hat{x}_t) \nu(d\theta^1) \right\}. \end{aligned}$$

The policy function can be chosen of the linear form

$$g(x, \theta^0, \varrho_1) = e_0^* x^0 + \epsilon\theta^0 + \sum_{b\geq 1} e_b^* x^b + \sum_{t\geq 1} h_t^* \varrho_t$$

for some correlation pattern $e^* = (e_a^*)_{a\geq 0}$ *and a positive sequence* $h^* = (h_t^*)_{t\geq 1}$. *These sequences can be chosen independently of* F *and satisfy*

$$\sum_{a\geq 0} e_a^* + \sum_{t\geq 1} h_t^* \leq 1.$$

We are now ready to prove the existence of a symmetric Markov perfect equilibrium of our economy. Let a continuous function $F : X \to X$ determine recursively the sequence $\{\varrho_t\}_{t\in\mathbb{N}}$ by (47). Assume that the exogenous initial configuration $\mathbf{x}$ has a well defined average $\varrho := \varrho(\mathbf{x})$, that is, assume that $\mathbf{x} \in \mathbf{X_e}$. Let $F^{(t)}$ denote the t-fold iteration of F so that $\varrho_t = F^{(t)}(\varrho)$. Since the agents' types are independent and identically distributed, it follows from the law of large numbers that the average equilibrium action in the following period is almost surely given by

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{a=-n}^{n} g(T^a x, \theta^a, \varrho) = C^* \varrho + \sum_{t\geq 1} h_t^* F^{(t)}(\varrho) =: G(F)(\varrho).$$

Thus, the average action in period $t = 2$ exists almost surely if the average action in period $t = 1$ exists, and an induction argument shows that the average action exists almost surely for all $t \in \mathbb{N}$. In order to establish the existence of an equilibrium, we first show that there exists a continuous function F^* such that, with $\varrho_1 := \varrho(\mathbf{x})$ we have

$$\varrho_2 := F^*(\varrho_1) = G(F^*)(\varrho_1).$$

Lemma 5.4 *Let* $\mathcal{C}(X)$ *be the class of all continuous functions* F *on* X *equipped with the usual sup-norm* $\|\cdot\|_\infty$. *The map* $F \mapsto G(F)$ *on* $\mathcal{C}(X)$ *has a fixed point* F^*.

PROOF: Let Lip $\subset \mathcal{C}(X)$ be the class of Lipschitz continuous function F on X with constant 1. Due to the theorem by Ascoli and Arzela, the set Lip is compact with respect to the topology induced by the norm $\|\cdot\|_\infty$. Thus, the assertion follows from Brouwer's fixed point theorem if we can show that the map $F \mapsto G(F)$ maps the compact convex set Lip continuously into itself. Continuous dependence of $G(F, \varrho)$ on F is obvious. Since F is Lipschitz with constant 1, the iterates $F^{(t)}$ are Lipschitz with constant 1 for all t. Hence $G(F)$ is Lipschitz with constant 1 because $C^* + \sum_{t\geq 1} h_t^* \leq 1$; see Lemma 5.3. □

We are now in a position to prove the main result of this section.

Theorem 5.5 *Let* $X = \Theta = [-1, 1]$. *Assume that* $\mathbb{E}\theta^0 = 0$ *and, that the initial configuration of actions* $\mathbf{x}$ *belongs to* $\mathbf{X_e}$, *that an agent* $a \in \mathbb{A}$ *only observes his own type* θ^a, *and that the instantaneous utility function takes the quadratic form in (38). Then the following hold:*

i. *The economy has a symmetric Markov perfect equilibrium* (g^*, F^*) *where* $g^* : \mathbf{X}^0 \times \Theta \times X \to X$ *and* $F^* : X \to X$.

ii. *In equilibrium, the sequence of average actions* $\{\varrho(\mathbf{x}_t)\}_{t\in\mathbb{N}}$ *exists almost surely.*

iii. *The policy function* g^* *can be chosen of the linear form*

$$g^*(x, \theta^0) = e_0^* x^0 + \epsilon\theta^0 + \sum_{b\geq 1} e_b^* x^b + B^*(\varrho(\mathbf{x})) \tag{49}$$

for some positive sequence $\mathbf{e}^* = (e_a^*)_{a\geq 0}$*, a constant* $\epsilon > 0$*, some constant* $B^*(\varrho(\mathbf{x}))$ *that depends only on the initial average action.*

PROOF: Let $e^* = (e_a^*)_{a\geq 0}$ and $h^* = (h_t^*)_{t\geq 1}$ be the sequences derived in Lemma 5.2 and 5.3, respectively, and let F^* be a fixed point of the operator G studied in Lemma 5.4. If, for a given initial configuration $\mathbf{x} \in \mathbf{X_e}$, the agent $0 \in \mathbb{A}$ expects the policy functions of all the agents $a \in \mathbb{A}$ to take the form

$$g(T^a x, \theta^a, \varrho(\mathbf{x})) = e_0^* x^a + \epsilon\theta^a + \sum_{b\geq 1} e_b^* x^{a+b} + \sum_{t\geq 1} h_t^* F^{*(t)}(\varrho(\mathbf{x})), \tag{50}$$

then his own policy function is given by

$$g(x, \theta^0, \varrho(\mathbf{x})) = e_0^* x^0 + \epsilon\theta^0 + \sum_{b\geq 1} e_b^* x^b + \sum_{t\geq 1} h_t^* F^{*(t)}(\varrho(\mathbf{x})),$$

and the average action in period t is almost surely given by

$$\varrho(\mathbf{x_t}) = \varrho_t = F^{*(t)}(\varrho(\mathbf{x})).$$

□

It is important to re-iterate that, for our analysis in this section, it is essential that the agents' utility function is quadratic (and hence policy functions are linear). Only in this case, in fact, can the dynamics of average actions $\{\varrho(\mathbf{x_t})\}_{t\in\mathbb{N}}$ can be described in terms of a recursive relation. In models with more general local interactions, such a recursive relation typically fails to hold, as shown e.g., by Föllmer and Horst [29]. In such more general cases, the average action typically is not an appropriate state variable, i.e., a sufficient statistic, for the aggregate behavior of the configuration $\mathbf{x}$; and the analysis must be pursued in terms of empirical fields, which require a probabilistic framework that is beyond the scope of the present paper, along the lines of Föllmer and Horst [29].

6 Conclusions

In this paper, we study existence and continuity of rational expectations equilibria of static and dynamic economies in which an infinite number of agents interact locally. Some of our results should be strengthened. For instance, several other examples for which the assumption of Theorems 3.6 and 3.7 can indeed be verified, should be studied. Also, the class of economies we study is restricted in that we impose several simplifying assumptions on preferences, on the agents' choice space, and on the stochastic structure of preference shocks. While these assumptions can be relaxed using standard methods, this is not the case for the restrictions we have imposed on the structure of local interactions of dynamic economies. In particular, we have restricted the analysis to the case of 'one-sided' interactions, which greatly limits the strategic aspect of the interactions between agents. Allowing for more general forms of local interactions in dynamic economies (when agents are rational) will certainly require a non-trivial and independent analysis. No doubt the first steps introduced in this paper will be of use for such extensions, which we plan for future work. In the analysis of the dynamic model of local interactions in this paper, we have only considered Markov perfect equilibria. While the restriction to Markovian strategies is common to most analyses of dynamic games, for technical as well as substantive reasons, we have noted that it possibly misses an important class of equilibria supported by trigger strategies and other complex dynamic punishment strategies, in analogy to the case of repeated games. This is also content for future work. Finally, we have studied the mathematical properties of existence, continuity, and, in the dynamic models, of ergodicity of the equilibrium of our economies with local interactions. A detailed study of the welfare properties of equilibrium would also be exceptionally interesting. While such a study is outside the scope of this paper, we can show in the special context of our static local conformity example that equilibria will in general (almost surely) be Pareto inefficient.

A Maximizing α-Concave Functions

The following theorem plays a central role in our analysis. Even though its proof follows primarily from straightforward modifications of arguments given in the proof of Theorem 3.1 in Montrucchio [45], we prove it to keep the paper self contained.

Theorem A.1 *Let $X \subset \mathbb{R}$ be a closed and convex set and let $(Y, \|\cdot\|_Y)$ be a normed space. Let $F : X \times Y \to \mathbb{R}$ be a continuous function which satisfies the following conditions:*

i. For each $y \in Y$, the map $x \mapsto F(x, y)$ is α-concave on X.

ii. F is differentiable with respect to x and the derivative $F_1(\cdot) := \frac{\partial}{\partial x} F(\cdot)$ satisfies

$$|F_1(x, y_1) - F_1(x, y_2)| \leq G(y_1, y_2)$$

for some function $G : Y \times Y \to \mathbb{R}$.

Then there exists a unique map $f : Y \to X$ such that $\sup_{x \in X} F(x, y) = f(y)$. *Moreover, f satisfies*

$$|f(y_1) - f(y_2)| \leq \frac{1}{\alpha} G(y_1, y_2).$$

In particular, $|f(y_1) - f(y_2)| \leq \frac{L}{\alpha} \|y_1 - y_2\|_Y$ *if* $G(y_1, y_2) = L\|y_1 - y_2\|_Y$.

Proof: Our proof uses modifications of arguments given in Montrucchio [45]. It follows from Lemmas A1 - A3 in Montrucchio [45] that, for any $y_1 \in Y$, the α-concave function $x \mapsto F(x, y_1)$ has a unique maximizer $f(y_1)$ which satisfies

$$\alpha|x - f(y_1)|^2 \leq F_1(x, y_1)(f(y_1) - x). \tag{51}$$

Thus, choosing $x = f(y_2)$ in (51) for some $y_2 \in Y$, we obtain

$$\alpha|f(y_1) - f(y_2)|^2 \leq F_1(f(y_2), y_1)[f(y_1) - f(y_2)]. \tag{52}$$

Since $f(y_2)$ maximizes the differentiable function $x \mapsto F(x, y_2)$, we obtain $F_1(f(y_2), y_2)[f(y_1) - f(y_2)] \leq 0$. Indeed in case of an interior solution $F_1(f(y_2), y_2) = 0$. If we have a boundary solution, strict concavity of $F(\cdot, y_2)$ implies $F_1(f(y_2), y_2)) > 0$ and $f(y_2) > f(y_1)$. Hence

$$\alpha|f(y_1) - f(y_2)|^2 \leq (F_1(f(y_2), y_1) - F_1(f(y_2), y_2)) [f(y_1) - f(y_2)],$$

and so

$$\alpha|f(y_1) - f(y_2)| \leq |F_1(f(y_2), y_2)| \leq G(y_1, y_2).$$

This yields the assertion. □

B Proof of Theorems 2.9 and 2.10

This section gives the proofs of Theorems 2.9 and 2.10. While related, both proofs are reported for the sake of completeness.

B.1 Proof of Theorem 2.9

In this section we are going to prove Theorem 2.9. For this, it is enough to prove the existence of an almost surely uniquely defined fixed point of the operator $V : B(\mathbf{\Theta^0}, X) \to B(\mathbf{\Theta^0}, X)$ defined by (8). In a first step, we establish the following result.

Lemma B.1 *If the utility function u is uniformly α-concave in its first argument and if the contraction condition (6) holds, then the operator V satisfies the following conditions:*

i. There exists $\eta^* > 0$ such that V maps the set $\mathrm{Lip}_{\eta^*}(1)$ continuously into itself.

ii. The operator V has a unique fixed point g^*, and $g^* \in \mathrm{Lip}_{\eta^*}(1)$.

Proof: Let us denote by

$$g(y,\theta^0) = \arg\max_x u(x,y,\theta^0) \tag{53}$$

the conditional optimal action of the agent $0 \in \mathbb{A}$, given his type θ^0 and given the action $y \in X$ of his neighbor. We equip the product space $\Theta \times X$ with the maximum norm, and so it follows from Theorem A.1, from Assumption 2.7 (i) and from (6) that

$$|g(y,\theta^0) - g(\hat{y},\hat{\theta}^0)| \le \gamma \max\{|y-\hat{y}|, |\theta^0 - \hat{\theta}^0|\} \tag{54}$$

where $\gamma := \frac{L}{\alpha} < 1$.

i. Let us first show that V is a continuous operator on the Banach space $(B(\mathbf{\Theta^0}, X), \|\cdot\|_\infty)$. Indeed, due to (54) we have

$$|Vg(\theta) - V\hat{g}(\theta)| \le \gamma |g \circ T(\theta) - \hat{g} \circ T(\theta)| \le \gamma \|g - \hat{g}\|_\infty \tag{55}$$

for all $g, \hat{g} \in B(\mathbf{\Theta^0}, X)$. Thus,

$$\|Vg - V\hat{g}\|_\infty \le \gamma \|g - \hat{g}\|_\infty.$$

Since $L < \alpha$, the operator V is not only continuous, but also a contraction. In particular, it has a unique fixed point.

ii. Let us now choose $\eta > 0$ such that $2^\eta \gamma < 1$ and fix $g \in \mathrm{Lip}_\eta(1)$. It follows from (54) that

$$\begin{aligned}
|Vg(\theta) - Vg(\hat{\theta})| &\le \gamma \max\left\{|\theta^0 - \hat{\theta}^0|, |g \circ T(\theta) - g \circ T(\hat{\theta})|\right\} \\
&\le \gamma \max\left\{|\theta^0 - \hat{\theta}^0|, \sum_{a \ge 1} 2^{-\eta(a-1)} |\theta^a - \hat{\theta}^a|\right\} \\
&\le 2^\eta \gamma \left\{|\theta^0 - \hat{\theta}^0| + \sum_{a \ge 1} 2^{-\eta a} |\theta^a - \hat{\theta}^a|\right\} \\
&\le d_\eta(\theta, \hat{\theta}).
\end{aligned}$$

This shows that V maps $\mathrm{Lip}_\eta(1)$ into itself. Since $\mathrm{Lip}_\eta(1)$ is a closed subset of $B(\mathbf{\Theta^0}, X)$ with respect to the topology of uniform convergence, $(\mathrm{Lip}_\eta(1), \|\cdot\|_\infty)$ is a Banach space. Thus, V may also be viewed as a contraction on the Banach space $(\mathrm{Lip}_\eta(1), \|\cdot\|_\infty)$, and so the unique fixed point g^* of V belongs to $\mathrm{Lip}_\eta(1)$.

This proves the lemma. □

Proof of Theorem 2.9:

i. Due to Theorem A.1 and Assumption 2.7, we have

$$|g(y,\theta^0) - g(\hat{y},\theta^0)| \le \frac{L(\theta^0)}{\alpha} |y - \hat{y}|$$

where the map g is defined by (53). This shows that the operator V satisfies

$$|Vg(\theta) - V\hat{g}(\theta)| \le \frac{L(\theta^0)}{\alpha} |g \circ T(\theta) - \hat{g} \circ T(\theta)|$$

for all $g, \hat{g} \in B(\mathbf{\Theta^0}, X)$. Since the types are distributed independently across agents, the random variables $L(\theta^0)$ and $|g \circ T(\theta) - \hat{g} \circ T(\theta)|$ are independent, and so

$$\mathbb{E}|Vg(\theta) - V\hat{g}(\theta)| \leq \frac{\mathbb{E}L(\theta^0)}{\alpha}\mathbb{E}|g \circ T(\theta) - \hat{g} \circ T(\theta)| \leq \gamma\mathbb{E}|g \circ T(\theta) - \hat{g} \circ T(\theta)|.$$

Here $\gamma := \frac{\mathbb{E}L(\theta^0)}{\alpha} < 1$. As the types are also identically distributed, we obtain

$$\mathbb{E}|Vg(\theta) - V\hat{g}(\theta)| \leq \gamma\mathbb{E}|g(\theta) - \hat{g}(\theta)|.$$

Thus, denoting by V^n the n-fold iteration of the operator V, we see that

$$\lim_{n\to\infty} \mathbb{E}|V^n g(\theta) - V^n\hat{g}(\theta)| = 0. \tag{56}$$

In particular, the sequence $(V^n g)_{n\in\mathbb{N}}$ satisfies

$$\begin{aligned} \mathbb{E}|V^{n+m}g(\theta) - V^n g(\theta)| &= \mathbb{E}|V^n(V^m g)(\theta) - V^n g(\theta)| \\ &\leq \gamma^n\mathbb{E}|V^m g - g| \\ &\leq C\gamma^n \end{aligned}$$

for some $C < \infty$. This shows that $(V^n g)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\mathbb{P})$. Since $L^1(\mathbb{P})$ is a complete space, there exists an almost surely uniquely determined random variable $G^*[g]$ such that

$$V^n g \xrightarrow{L^1} G^*[g].$$

Due to (56), the L^1-limit $G^*[g]$ does almost surely not depend on g. In other words, there exists an almost surely uniquely defined random variable g^* such that

$$V^n g \xrightarrow{L^1} g^* \quad \text{for all } g \in B(\mathbf{\Theta^0}, X).$$

In view of Chebyschev's inequality this yields

$$\mathbb{P}[|V^n g^* - g^*| > \epsilon] \leq \epsilon^{-1}\mathbb{E}|V^n g^* - g^*| \to 0 \quad \text{as} \quad n \to \infty$$

Let us now fix $\hat{\epsilon} > 0$. There exists a sequence of measurable sets $(A_n)_{n\in\mathbb{N}}$ that satisfies $\mathbb{P}[A_n] \to 1$ as $n \to \infty$ and $N \in \mathbb{N}$ such that

$$|V^n g^* - g^*| < \hat{\epsilon} \quad \text{and} \quad |V^{n+1}g^* - g^*| < \hat{\epsilon} \quad \text{on } A_n \text{ for all } n \geq N.$$

Since the best reply functions are continuous, we can choose $\hat{\epsilon}$ so that $|Vg^* - g^*| < \epsilon$ on A_n, and so

$$Vg^* = g^* \qquad \mathbb{P}\text{-a.s.}$$

Recall now that L^1-convergence implies almost sure convergence along a subsequence. Since any symmetric equilibrium $\hat{g}$ satisfies $\mathbb{P}[V\hat{g} = \hat{g}] = 1$, it follows that

$$\hat{g} = \lim_{k\to\infty} V^{n_k} g = g^* \qquad \mathbb{P}\text{-a.s.}$$

alon some subsequence $(n_k)_{k\in\mathbb{N}}$. This shows uniqueness (up to a set of measure zero) of the symmetric equilibrium.

ii. The assertion follows from Lemma B.1.

□

B.2 Proof of Theorem 2.10

We proceed by analogy with the proof of Theorem 2.9. For analytical convenience, we restrict our attention to the case $N = 1$. The more general case $N \in \mathbb{N}$ can easily be established using similar arguments. In the present setting, it is enough to show that there exists a measurable function $g^* : \Theta^2 \to X$ which satisfies

$$g^*(\theta^0, \theta^1) = \arg\max_{x^a \in X} \int u(x^a, g^*(\theta^1, \theta^2), \theta^0)\nu(d\theta^2). \tag{57}$$

Each such function is a fixed point of the operator $\widetilde{V} : B(\Theta^2, X) \to B(\Theta^2, X)$ which acts on the class $B(\Theta^2, X)$ of bounded measurable functions from θ^2 to X according to

$$\widetilde{V}g(\theta^0, \theta^1) = \arg\max_{x^0 \in X} \int u(x^a, g(\theta^1, \theta^2), \theta^0)\nu(d\theta^2). \tag{58}$$

In order to prove that this operator has a Lipschitz continuous fixed point if the utility function satisfies (6), we introduce the class

$$\mathrm{Lip} := \left\{ g : \Theta^2 \to \mathbb{R} : |g(\theta^0, \theta^1) - g(\tilde{\theta}^0, \tilde{\theta}^1)| \leq \max\{|\theta^0 - \tilde{\theta}^0|, |\theta^1 - \tilde{\theta}^1|\} \right\}$$

of all Lipschitz continuous functions on Θ^2 with Lipschitz constant 1. The following lemma establishes some basic properties of the operator $\widetilde{V}$.

Lemma B.2 *If the utility function u is Lipschitz continuous in the sense of (6), then the operator $\widetilde{V}$ defined by (58) satisfies the following conditions:*

i. $\widetilde{V}$ maps the set Lip *continuously into itself.*

ii. $\widetilde{V}$ has a unique fixed point g^* and $g^* \in$ Lip.

Proof: We introduce the continuous mapping $U : \mathrm{Lip} \times X \times \Theta^2 \to \mathbb{R}$ by

$$U(g, x, \theta^0, \theta^1) := \int u(x, g(\theta^1, \theta^2), \theta^0)\nu(d\theta^2). \tag{59}$$

Under the assumptions of Theorem 2.10, the map $x \mapsto u(x, y, \theta^0)$ is uniformly α-concave, and so the mapping $x \mapsto U(g, x, \theta^0, \theta^1)$ is α-concave.

i. Let us first prove that $\widetilde{V}$ maps the class Lip into itself. To this end, we fix $g \in \mathrm{Lip}$. Since the map $(x^0, \theta^0, \theta^1, \theta^2) \mapsto \frac{\partial}{\partial x} u(x^0, g(\theta^1, \theta^2), \theta^0)$ is uniformly continuous, it follows from Assumption 2.7 (ii) that

$$\begin{aligned}
& \left| \frac{\partial}{\partial x} U(g, x^0, \theta^0, \theta^1) - \frac{\partial}{\partial x} U(g, x^0, \hat{\theta}^0, \hat{\theta}^1) \right| \\
\leq\ & \sup_{\theta^2} \left| \frac{\partial}{\partial x} u(x, g(\theta^1, \theta^2), \theta^0) - \frac{\partial}{\partial x} u(x, g(\hat{\theta}^1, \theta^2), \hat{\theta}^0) \right| \\
\leq\ & \sup_{\theta^2} L \left\{ |g(\theta^1, \theta^2) - g(\hat{\theta}^1, \theta^2)| + |\theta^0 - \hat{\theta}^0| \right\} \\
\leq\ & L \max \left\{ |\theta^1 - \hat{\theta}^1|, |\theta^0 - \hat{\theta}^0| \right\}.
\end{aligned} \tag{60}$$

Here, the last inequality follows from Lipschitz continuity of g. This shows that, for any fixed $g \in \mathrm{Lip}$, the map $(x, \theta^0, \theta^1) \mapsto U(g, x, \theta^0, \theta^1)$ satisfies the assumption of Theorem A.1 with $Y := \Theta^2$

equipped with the maximum norm. Thus, Theorem A.1 concludes that $\widetilde{V}g \in \text{Lip}$ because $L < \alpha$. Let us now show continuity of $\widetilde{V}$ with respect to the sup-norm. To this end, we fix $\theta^0, \theta^1 \in \Theta$. For any two functions $g, \hat{g} \in B(\Theta^2, X)$ we have

$$\left| \frac{\partial}{\partial x} U(g, x, \theta^0, \theta^1) - \frac{\partial}{\partial x} U(\hat{g}, x, \theta^0, \theta^1) \right| \leq L \|g - \hat{g}\|_\infty,$$

due to (60). Thus, for each pair (θ^0, θ^1), the mapping $(g, x) \mapsto U(g, x, \theta^0, \theta^1)$ satisfies the assumptions of Theorem A.1 with $Y := B(\Theta^2, X)$. Hence,

$$\|\widetilde{V}g - \widetilde{V}\hat{g}\|_\infty = \sup_{\theta^0, \theta^1} \left| \arg\max_{x \in X} U(g, x, \theta^0, \theta^1) - \arg\max_{x \in X} U(\hat{g}, x, \theta^0, \theta^1) \right| \leq \frac{L}{\alpha} \|g - \hat{g}\|_\infty.$$

This proves continuity of the operator $\widetilde{V}$ in the topology of uniform convergence.

ii. Since $L < \alpha$, it follows from (i) that $\widetilde{V}$ is a contraction on the Banach space $B(\Theta^2, X)$. Thus, $\widetilde{V}$ has a unique fixed point g^*. Since $\widetilde{V}$ maps the closed set Lip into itself, $g^* \in \text{Lip}$.

□

PROOF OF THEOREM 2.10:

i. Using similar arguments to the ones provided in the proof of Theorem 2.9 and applying Theorem A.1 to the function U defined by (59), we obtain

$$\mathbb{E} \left| \widetilde{V}g(\theta^0, \theta^1) - \widetilde{V}\hat{g}(\theta^0, \theta^1) \right| \leq \frac{\mathbb{E}L(\theta^0)}{\alpha} \int \int \left| g(\theta^1, \theta^2) - \hat{g}(\theta^1, \theta^2) \right| \nu(d\theta^2) \nu(d\theta^1) \leq \gamma \mathbb{E}|g - \hat{g}|.$$

Since $\gamma := \frac{\mathbb{E}L(\theta^0)}{\alpha} < 1$, it follows by an argument sufficiently close to the one spelled out in detail in the proof of Theorem 2.9 that the operator has a fixed point g^* which is uniquely defined up to a set of measure 0.

ii. The assertion follows from Lemma B.2.

□

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