Buy-and-resell Overpricing and Investor Experience *,†

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We study buy-and-resell overpricing—a situation when an equilibrium asset price exceeds a buy-and-hold price that serves as a fundamental value—in a model without short-selling and with experience-dependent beliefs. In our model, investors with same experience have the same posterior beliefs about how long is left until the terminal time, the only time the asset pays a dividend. Investors' posterior beliefs at any one date come from updating the same beliefs from birth but for different durations. We give a necessary and sufficient condition, on the hazard rate of investors' common beliefs at birth, for buy-and-resell overpricing, which occurs either always or never. When viewed as a function of experience, the hazard rate is a measure of instantaneous optimism about the dividend at different experience levels. With this interpretation, our necessary and sufficient condition for overpricing is that the most experienced investors are not instantaneously the most optimistic.

Keywords: short-sales constraint; belief heterogeneity; overpricing; experience; waiting time; hazard rate

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1 Introduction

Investors' expectations vary with their experience according to empirical evidence (*Greenwood and Nagel*, 2009). In models without short-selling following *Harrison and Kreps* (1978), heterogeneous expectations generally lead to large trading volume and prices above buy-and-hold prices, but can experience dependence help us get sharper results? We extend this type of modeling to overlapping generations to make room for experience heterogeneity and make experience predictive of posterior beliefs by assuming that investors start out with the same beliefs about the future at birth. With such a structure, we give a nontrivial necessary and sufficient condition for buy-and-resell overpricing—the equilibrium price exceeding the buy-and-hold price that serves as a fundamental value. Our condition is nontrivial in the sense of being stronger than belief heterogeneity, complementing existing models that for the sake of their valuable insights mostly have too much structure to include overpricing-neutral belief heterogeneity. Exceptions are the models of *Morris* (1996) and *Werner* (2020).

Among contributions of *Morris* (1996) is also a nontrivial necessary and sufficient condition for what we call buy-and-resell overpricing, but in a special case of the model of *Harrison and Kreps* (1978). The condition is on investors' prior beliefs about the probability $\theta \in [0, 1]$ that the asset pays in any one period a dividend of \$1 as opposed to no dividend, with these prior beliefs being represented by densities over the parameter θ. The condition is that there be no investor with a monotone-likelihood-ratio dominant prior and is nontrivial in the sense of being stronger than prior heterogeneity so that some heterogeneous priors violate this condition and are overpricing-neutral. Moreover, *Werner* (2020) shows that this condition remains sufficient for what we call buy-and-resell overpricing in a model of great economic generality. With experience dependence of investors' beliefs, unlike in *Morris* (1996) and *Werner* (2020), we obtain a different nontrivial necessary and sufficient condition for buy-and-resell overpricing.

In our model, the only strong assumption on beliefs is experience dependence in the sense that they say the same about the future at birth for all investors, but for tractability we make strong assumptions on the uncertainty structure. The uncertainty is only about the terminal time, which is also the only time the asset pays a dividend, so that the realized history stays the same and all investors find themselves in exactly the same position at birth. However, investors' expectations are heterogeneous because if we fix a date (1) overlapping generations allow for experience heterogeneity and (2) posterior beliefs come from updating the same beliefs from birth but for different durations.

Our model has five primitives: the terminal time; the remaining time, as perceived at birth by every investor, in the form of a random variable; the terminal age; the trade-frequency parameter; and the interest rate. The rest is further interpretation. We have already mentioned the asset; it pays a one-shot dividend of \$1 at the terminal time. Before that, overlapping generations come and go: one investor is born each time point and stays until the terminal age or the terminal time, whichever is sooner. Time is continuous, backward infinite so that the set of experience levels is time-invariant, and possibly forward infinite depending on the terminal time. Trade, however, is discrete to avoid dealing with equilibria where the asset might change hands continuously, but our overpricing results are asymptotic as the market tends to opening continuously. Investors are risk-neutral and decide whether to buy/sell the asset or not at an endogenous price every time the market opens for trade, but they cannot sell the asset short.

To define equilibrium, we follow the simpler partial-equilibrium approach of *Harrison and Kreps* (1978) and *Morris* (1996) rather than the more sophisticated general-equilibrium approach of *Werner* (2020). An attractive feature of our model is that we can require the equilibrium price to be time-invariant owing to the stationarity of the model and the fact that investors' posterior beliefs depend only on experience. In this context, summarizing their partial-equilibrium criterion in our formulation affords simple wording. An equilibrium price is one that makes the most optimistic investor(s) break even, but who the most optimistic investors are depends on the sought equilibrium price via expected resale proceeds. If, on the contrary, the most optimistic investors did not break even, that would mean either excessive expected discounted return for the most optimistic or expected loss for all. Equilibrium buyers at a particular time the market opens for trade can be any subset of the most optimistic investors when the resale price they consider is the equilibrium price. Less optimistic investors simply do not, for whatever reason, engage in short-selling of the asset and do not create excess supply in this way. Finally, investors do not have budget constraints, but at the equilibrium price we find all of the asset can cost at most \$1, and thus we do not need the infinite-wealth assumption made in the previous literature using this equilibrium criterion.

To define fundamental value, we also build on the previous literature, which, however, is mostly about infinitely-lived investors and defines it as a buy-and-hold price, where to buy and hold means to hold forever. For overlapping generations, we define fundamental value as what an equilibrium price would be if buyers had to meet a no-retrade constraint and only then could resell at this sought price if the dividend were still unpaid. For nesting the standard definition of fundamental value—the highest estimate of the expected discounted return of buying the asset and holding it forever among all investors—we allow the terminal age to be infinite.

Our overpricing results are asymptotic in the sense that we take these discrete-trade equilibrium price and fundamental value, which exist and are unique, to their continuous-trade limits before comparing them. Mild assumptions on investors' common beliefs at birth, which are about how long is left until the terminal time, are in place for the discrete-trade equilibrium price and fundamental value to exist and to converge. To give our results on buy-and-resell overpricing in this continuous-trade limit, we need the investors' common beliefs at birth about the waiting time to have a density.

Our main result is that, roughly speaking, buy-and-resell overpricing occurs if and only if the most experienced investors are not instantaneously the most optimistic, with instantaneous optimism being measured by the hazard rate. In contrast, in the interpretation of the equilibrium price as where the most optimistic investor(s) break even we speak of optimism accounting for short- and long-term expected discounted returns. To give our necessary and sufficient condition in terms of the hazard rate a name, we call it end-of-life hazard switching, since it requires that with experience the hazard rate switches to and stays at lower levels relative to the highest ones. Otherwise, the hazard rate can be, for instance, strictly increasing so that more experienced investors' beliefs are instantaneously more optimistic, but this belief heterogeneity is overpricing-neutral. Here a standard example is simply a gamma distribution with a suitable shape parameter.

Our necessary and sufficient condition for buy-and-resell overpricing helps us understand how exactly investors' beliefs must differ for overpricing to occur, complementing the results of *Morris* (1996) and *Werner* (2020) on this. Understandably, though, in more applied research it is compelling to simply assume away overpricing-neutral belief heterogeneity and to focus on modeling overpricing. In a seminal contribution to this literature, *Scheinkman and Xiong* (2003) point out that it explains the correlation between overpricing and trading volume when belief heterogeneity comes from overconfidence. This literature also provides a useful benchmark, without short-selling, for studying the effects of short-selling costs (*Nutz and Scheinkman*, 2020) or more generally asset-supply fluctuations (*Hong et al.*, 2006). Finally, in this paper we introduced experience dependence for investors' beliefs, which may be helpful for reconciling this literature's findings with (1) models about learning from experience (*Malmendier et al.*, 2020; *Ehling et al.*, 2017; *Collin-Dufresne et al.*, 2016; *Schraeder*, 2016; *Nakov and Nuño*, 2015) and (2) empirical and experimental evidence on overpricing and experience (*Dufwenberg et al.*, 2005; *Greenwood and Nagel*, 2009; *Akiyama et al.*, 2014; *Xie and Zhang*, 2016).

The paper is organized as follows. Section 2 presents the model, including the equilibrium concept defining equilibrium price. Fundamental value is defined in Section 3. Afterwards, Section 4 answers the questions of the existence, uniqueness, and continuous-trade limits for the introduced discrete-trade equilibrium price and fundamental value. Finally, Section 5 reports our necessary and sufficient condition for buy-and-resell overpricing in the limit as the market tends to opening continuously, finishing with examples of buy-and-resell overpricing occurring or not.

2 Model

The model is partial-equilibrium overlapping generations without short-selling with an equilibrium concept adapted to overlapping generations from the literature on buy-and-resell overpricing initiated by *Harrison and Kreps* (1978). Time is continuous, backward infinite, and possibly forward infinite depending on the terminal-time parameter. Investors are risk-neutral and decide whether to buy/sell the asset or not at an endogenous price under uncertainty about a dividend.

In total, our model has five primitives:

(1) the terminal time $\tau \in (-\infty, \infty]$ at which an asset pays a one-time dividend of \$1;

(2) the remaining time, as perceived at birth by every investor, in the form of a random variable[1] W taking values in $(0, \infty]$;

(3) the terminal age $T \in (0, \infty]$;

[1]We consider extended-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. For a formal definition with extended values, see *Chung* (2001). We use the convention $e^{-\infty} = 0$.

(4) the trade-frequency parameter $\Delta \in (0, T)$;

(5) the interest rate $r \in (0, \infty)$.

An asset pays a one-shot dividend of \$1 at the possibly infinite terminal time τ, where we think of the dividend at infinity as no dividend. Investors do not know (and are uncertain about) the terminal time τ until it comes, but this true terminal time τ is not random. Before it comes, one investor is born each time point and stays until the terminal age T or the terminal time τ, whichever is sooner: at each $t \in (-\infty, \tau)$, a new one is born, and unless $T = \infty$ the time is up for the one born at time $t - T$. Investors start out with the same beliefs at birth, namely that the time left is the random variable W, and update them after birth in the standard Bayesian way at every nonterminal time. In other words, at each time $s \in (-\infty, \tau)$, posterior expectations (if well-defined) of the investor born at an arbitrary time $t \in (-\infty, s)$ of functions of the perceived remaining time W are expectations conditional on the event that $W > s - t$. Naturally, we think of the difference $s - t$ between the present time s and the birth time t as the investor's experience level rather than just age. For the conditioning event corresponding to this experience level to have a nonzero probability in relevant cases, we make the following assumption on the perceived remaining time W:

Assumption 1. At all experience levels $x \in [0, T] \backslash \{\infty\}$, the terminal time remains perceived to be in the future in the sense that $\mathrm{P}(W > x) \neq 0$.

Investors can trade every Δ time units at the time points

$$\{0, \Delta, -\Delta, 2\Delta, -2\Delta, 3\Delta, -3\Delta, \dots\} \cap (-\infty, \tau),$$

which are when the market opens for trade. Finally, investors can borrow and lend at the constant rate r, which will serve as a discount rate.

The equilibrium concept's unknown is the asset's price $p \in [0, 1]$ that is constant over time (thanks to the stationarity of our model) and from which we can directly find who can hold the asset and when. The equilibrium criterion boils down to essentially just an equation in p, which we state first and then explain it in detail (existence conditions are later on in Section 4):

Definition 1 (Equilibrium Price). We say that a price $p^*_\Delta \in [0, 1]$ is an equilibrium price if it satisfies the break-even condition (explained next)

$$p^*_\Delta = \max_{\substack{x \in [0, T-\Delta] \backslash \{\infty\} \\ y \in \{\Delta, 2\Delta, 3\Delta, \dots, \infty\} \cap [\Delta, T-x]}} \mathrm{E}\left(e^{-r(W-x)} I_{\{W \leq x+y\}} + p^*_\Delta e^{-ry} I_{\{W > x+y\}} \middle| W > x\right) \quad (1)$$

and the maximum exists.

In interpreting this equilibrium criterion, the same story applies to every time the market opens for trade, because our model is so stationary that we require the sought equilibrium price to be time-invariant. In the break-even condition (1), we think of x and y, respectively, as an experience level at which an investor might buy the asset and a duration for which the investor might hold the asset. The constraint $x \leq T - \Delta$ on the experience levels ensures that the remaining lifetime $T - x$ from reaching this experience of x time units is long enough for the market to reopen for trade at least once: $T - x \geq \Delta$. On the other hand, the constraint $y \leq T - x$ ensures that the holding duration y falls within the remaining lifetime $T - x$, while the requirement $y \in \{\Delta, 2\Delta, 3\Delta, \ldots, \infty\}$ is just a trade-frequency constraint on the holding duration. The indicator functions inside the expectation operator divide the perceived remaining time W into within y time units and beyond y time units from reaching this investor's experience of x time units. In the former case the asset pays the investor the dividend of \$1 in $W - x \leq y$ time units and otherwise the investor stops waiting for the dividend and sells the asset in y time units at the price p^*_Δ. The expectation is the corresponding (posterior) expected discounted return of a (risk-neutral) investor with experience of x time units of buying the asset and either getting the dividend within y time units or else selling in y time units at the price p^*_Δ. Maximizers of this expectation give us equilibrium buyer's experience levels, possibly consistent with multiple and time-dependent allocations of the asset across different experience levels, depending also on whether and how divisible the asset is. The buyers at a particular time the market opens for trade can be any subset of those investors that expect to achieve the highest expected discounted return after maximizing it over the holding durations when the resale price is p^*_Δ. In this sense, the buyers are the most optimistic investors when the resale price investors consider is the equilibrium price. The break-even condition (1) simply requires the equilibrium price to be such an investor's maximum, over the holding durations, expected discounted return when the resale price is this equilibrium price, making the investor break even. If, on the contrary, the equilibrium price differed from the maximum expected discounted return when the resale price is this equilibrium price, that would mean either shortage or excess supply. Finally, we state three remaining implicit assumptions behind this equilibrium concept:

(i) investors can afford to buy all of the asset (which can cost at most \$1, though);

(ii) short-sales constraints (investors simply do not, for whatever reason, engage in short-selling of the asset and do not drive the price down in this way);

(iii) within a particular nonterminal time, first everyone learns that the asset does not pay, then a new investor arrives, and then the market either opens for trade or not.

3 Fundamental Value

By fundamental value we mean the benchmark buy-and-hold price in the literature on what we call buy-and-resell overpricing initiated by *Harrison and Kreps* (1978), but the literature still says relatively little about finitely-lived investors. For infinitely-lived ones, the definition of fundamental value is simple: the highest estimate of the expected discounted return of buying the asset and holding it forever among all investors. Our overlapping-generations environment demands an extension to finitely lived investors more intricate than in the few existing finite-horizon models, without overlapping generations. Such finite-horizon models are due to *Nutz and Scheinkman* (2020), *Berestycki et al.* (2019), and *Allen and Gorton* (1993), who replaced holding forever in the definition of fundamental value simply with holding to the terminal time. For overlapping generations, however, holding to the terminal time is problematic, but holding for less time means that we need to consider resale values even to define fundamental value. We require that they hold the asset for the rest of the investment horizon only (or just one trading round less for certain marginal investors) and then also resell at the sought fundamental value as long as the dividend is unpaid (Definitions 2–3 followed by verbal explanations):

Definition 2 (No-retrade Durations for Definition of Fundamental Value)**.** The correspondence from the experience levels

$$Y_\Delta : [0, T - \Delta]\backslash\{\infty\} \twoheadrightarrow \{\Delta, 2\Delta, 3\Delta, \ldots, \infty\}$$

defined by (longer no-retrade durations for younger investors)

$$Y_\Delta(x) = \begin{cases} \{\Delta\} & \text{if } T - 2\Delta < x \leq T - \Delta, \\ \{\Delta, 2\Delta\} & \text{if } T - 2\Delta = x, \\ \{2\Delta\} & \text{if } T - 3\Delta < x < T - 2\Delta, \\ \{2\Delta, 3\Delta\} & \text{if } T - 3\Delta = x, \\ \{3\Delta\} & \text{if } T - 4\Delta < x < T - 3\Delta, \\ & \vdots \\ \{\infty\} & \text{if } T = \infty \end{cases}$$

is a no-retrade constraint.

In this constraint for the definition of fundamental value, either infinite lifetime $T = \infty$ is the case and the no-retrade duration is forever or else the no-retrade duration depends on remaining lifetime, the opposite of experience, as follows: one trading round at the top experience bracket, one or two at the threshold for the next bracket (for some continuity of the correspondence), two at the next bracket itself, and so on to as many trading rounds as possible within the finite lifetime $T < \infty$ at the lowest experience bracket. Now we define fundamental value as what an equilibrium price would be if buyers of the asset had to meet the no-retrade constraint (Definition 2) and only then could resell at this price if the dividend were still unpaid:

Definition 3 (Fundamental Value)**.** We say that a price $\bar{p}_\Delta \in [0, 1]$ is a fundamental value if it satisfies the break-even condition with the no-retrade constraint (Definition 2)

$$\bar{p}_\Delta = \max_{\substack{x \in [0, T-\Delta]\backslash\{\infty\} \\ y \in Y_\Delta(x)}} \mathrm{E}\left(e^{-r(W-x)} I_{\{W \leq x+y\}} + \bar{p}_\Delta e^{-ry} I_{\{W > x+y\}} \middle| W > x\right) \tag{2}$$

and the maximum exists.

In the benchmark infinite-lifetime case $T = \infty$, this definition of fundamental value (Definition 3) collapses to the standard one: the most optimistic expected discounted return of buying the asset and holding it forever. Indeed, in the case $T = \infty$ condition (2) becomes

$$\bar{p}_\Delta = \max_{x \in [0,\infty)} \mathrm{E}\left(e^{-r(W-x)} \middle| W > x \right) \tag{3}$$

(a fundamental value exists if and only if the maximum exists, as addressed more generally in Section 4 below).

4 Existence and Continuous-trade Limit

The results of this paper on buy-and-resell overpricing concern continuous-trade limits of the discrete-trade equilibrium prices and fundamental values defined so far (Definitions 1 and 3, respectively). To take the limits, we first show the existence and uniqueness in the discrete-trade case but for this we need further assumptions and terminology:

Definition 4 (Investors' Posterior Distributions, Conditional on Experience Levels)**.** The function $F(\cdot|\cdot) : \mathbb{R} \times [0, T) \to \mathbb{R}$ defined by

$$F(w|x) = \mathrm{P}(W - x \leq w | W > x)$$

is in the first variable the posterior distribution function (of an extended-valued random variable) given the second variable (experience, x), and for every $x \in [0, T)$ we automatically have

$$\lim_{w \to \infty} F(w|x) = \mathrm{P}(W - x < \infty | W > x).$$

For $x = 0$, we denote $F(\cdot|0)$ by F and its limit at infinity $\lim_{w\to\infty} F(w)$ by $F(\infty)$.

Definition 5 (Extended First-order Stochastic Dominance)**.** Given two experience levels $x, x' \in [0, T)$, we say that the posterior $F(\cdot|x)$ first-order stochastically dominates $F(\cdot|x')$ if for every (weakly) increasing function $u : (-\infty, \infty] \to \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} u|_{\mathbb{R}}(w)\, dF(w|x) + u(\infty)\left(1 - \lim_{w\to\infty} F(w|x)\right)$$
$$\geq \int_{-\infty}^{\infty} u|_{\mathbb{R}}(w)\, dF(w|x') + u(\infty)\left(1 - \lim_{w\to\infty} F(w|x')\right).$$

Assumption 2. Continuity of F on $[0, T] \setminus \{\infty\}$.

Assumption 3. Either $T < \infty$ or eventual first-order stochastic dominance in the sense that there exists an experience level $\tilde{x} \in [0, T)$ such that $F(\cdot|x)$ first-order stochastically dominates $F(\cdot|\tilde{x})$ for all $x \in [\tilde{x}, T)$.

Here the important scenario is that of finitely-lived investors ($T < \infty$), but the infinite-lifetime case ($T = \infty$) is for better compatibility with the previous literature. Only for this case's sake we involve first-order stochastic dominance (for extended-valued random variables), which we have defined analogously to that for finite-valued random variables (see, e.g., *Mas-Colell et al.*, 1995). Just as first-order stochastic dominance of one money lottery over another means that expected utility of the first is at least as high as that of the second if one values more over less, so do we order beliefs about the remaining time.

Example 1 (Stochastic Dominance under Incomplete Exponential Distribution)**.** Suppose that $0 < \mathrm{P}(W = \infty) < 1$ and conditionally on $W < \infty$ the distribution of W is exponential, i.e., there is a $\lambda \in (0, \infty)$ such that all $w \in [0, \infty)$ satisfy

$$\mathrm{P}(W \le w | W < \infty) = 1 - e^{-\lambda w}$$

(incomplete exponential distribution with parameters λ and $q = \mathrm{P}(W = \infty)$. The posterior given any experience level $x \in [0, T)$ remains incomplete exponential with the same exponential part but different (updated) probabilities of infinite remaining time, i.e.,

$$\mathrm{P}(W - x = \infty | W > x) = \frac{q}{q + (1 - q) e^{-\lambda x}} \tag{4}$$

and all $w \in [0, \infty)$ satisfy

$$F(w|x) = \left(1 - \frac{q}{q + (1 - q) e^{-\lambda x}}\right) \left(1 - e^{-\lambda w}\right).$$

Relative to the zero experience level, at the experience of x time units the posterior $F(\cdot|x)$ scales down everywhere according to this formula, because the posterior probability (4) of infinite remaining time goes up. In other words, the probability shifts to infinite remaining time from finite ones, hence, precisely as first-order stochastic dominance requires, expectations of increasing functions of the remaining time increase (Definition 5). This means that relative to the zero experience level at the experience of x time units the posterior $F(\cdot|x)$ is first-order stochastically dominant, and thus Assumption 3 holds simply with $\tilde{x} = 0$.

We return to this and other examples, summarized in Table 1, in Section 5.4. Now we are ready to prove the existence and uniqueness of the equilibrium price and fundamental value:

Proposition 1. *There exist:*

(i) a unique equilibrium price;

(ii) a unique fundamental value.

Proof. We unify parts (i) and (ii) into one problem by considering a general form of the constraint on holding durations. In both cases, this constraint has the form of a correspondence Z_Δ from the experience levels $[0, T-\Delta]\backslash\{\infty\}$ to the holding durations $\{\Delta, 2\Delta, 3\Delta, \ldots, \infty\}$ defined by either

$$Z_\Delta(x) = Y_\Delta(x), \text{ for part (ii)},$$

or

$$Z_\Delta(x) = \{\Delta, 2\Delta, 3\Delta, \ldots, \infty\} \cap [\Delta, T-x], \text{ for part (i)}.$$

Now the unified problem is to find a unique price $p \in [0,1]$ such that

$$p = \max_{\substack{x \in [0,T-\Delta]\backslash\{\infty\} \\ y \in Z_\Delta(x)}} \mathrm{E}\left(e^{-r(W-x)} I_{\{W \le x+y\}} + p e^{-ry} I_{\{W > x+y\}} \middle| W > x\right) \tag{5}$$

and the maximum exists. Such a price p is either an equilibrium price (part (i)) or a fundamental value (part (ii)) depending on the choice of Z_Δ, and thus it suffices to solve the unified problem (5). It is a combination of a maximization problem (noncompact if $T = \infty$) and a fixed-point problem. For the noncompact maximization to work we had simply included and will use the assumption of eventual first-order stochastic dominance (Assumption 3), and for the fixed point we will use the Contraction-mapping Theorem.

For solving the unified problem (5) as just outlined, it is first convenient to write the maximand expectation in (5) in terms of the distribution and in two different ways (6) and (7):

$$\begin{aligned}
&\mathrm{E}\left(e^{-r(W-x)} I_{\{W \le x+y\}} + p e^{-ry} I_{\{W > x+y\}} \middle| W > x\right) \\
&= \mathrm{E}\left(e^{-r(W-x)} I_{\{W-x \le y\}} + p e^{-ry} I_{\{W-x > y\}} \middle| W > x\right) \\
&= \int_{-\infty}^{y} e^{-rw} dF(w|x) + \int_{y}^{\infty} p e^{-ry} dF(w|x) + p e^{-ry}\left(1 - \lim_{w \to \infty} F(w|x)\right) \qquad (6) \\
&= \frac{1}{1-F(x)}\left(\int_{x}^{x+y} e^{-r(w-x)} dF(w) + p e^{-ry}(1 - F(x+y))\right), \qquad (7)
\end{aligned}$$

for each experience level $x \in [0, T-\Delta]\backslash\{\infty\}$ and for each holding duration $y \in Z_\Delta(x)$. By the former formula (6) and Assumption 3, either $T < \infty$ and the maximization is over the compact subset

$$\left\{(x,y) \in \mathbb{R}^2 : x \in [0, T-\Delta], y \in Z_\Delta(x)\right\}$$

of $\mathbb{R}^2$ or else the maximum coincides (either both or neither exist) with the maximum over the compact subset

$$[0, \tilde{x}] \times Z_\Delta(0) \tag{8}$$

of $\mathbb{R}$ times its one-point compactification $(-\infty, \infty]$ (see, e.g., *Aliprantis and Border*, 2006). From the latter formula (7) and Assumption 2, it is easy to see continuity of the objective function on these compact sets, implying the existence of the maximum, which we denote by $v_\Delta(p)$, for each price $p \in [0,1]$. This reduces the whole problem (5) to finding a unique fixed point $p \in [0,1]$ of this function $v_\Delta : [0,1] \to [0,1]$. For this final step, it suffices to show that this v_Δ is a contraction, and thus it is enough to show that all prices $p, p' \in [0,1]$ with $p \geq p'$ satisfy

$$\left|v_\Delta(p) - v_\Delta(p')\right| \leq e^{-r\Delta}\left|p - p'\right|.$$

To verify this inequality, we use any maximizer (x,y) of the right-hand side of (5) when the price is p and directly calculate that

$$\begin{aligned}
\left|v_\Delta(p) - v_\Delta(p')\right| &= v_\Delta(p) - v_\Delta(p') \\
&= \frac{1}{1-F(x)}\left(\int_x^{x+y} e^{-r(w-x)}dF(w) + pe^{-ry}(1-F(x+y))\right) - v_\Delta(p') \\
&\leq \frac{1}{1-F(x)}\left(pe^{-ry}(1-F(x+y)) - p'e^{-ry}(1-F(x+y))\right) \\
&= \frac{1-F(x+y)}{1-F(x)}e^{-ry}(p-p') \\
&\leq e^{-ry}(p-p') \leq e^{-r\Delta}(p-p') = e^{-r\Delta}\left|p-p'\right|,
\end{aligned}$$

completing the proof. □

The main result of this section builds on Proposition 1 on the existence and uniqueness of the equilibrium price and fundamental value (just proved) and follows below as Proposition 2. It says that the focus of this paper's attention—the continuous-trade limits of these discrete-trade equilibrium prices and fundamental values—exist but for this we need another set of assumptions and terminology first.

Assumption 4. A density $f : \mathbb{R} \to [0,\infty)$ for F exists: f is integrable and all $w, w' \in \mathbb{R}$ with $w < w'$ satisfy

$$F(w') - F(w) = \int_w^{w'} f(z)\,dz.$$

Assumption 5. Continuity of f on $(0,T)$.

Assumption 4 allows us to introduce two important functions for stating our overpricing results, proving them, and first obtaining the continuous-trade limits of the discrete-trade equilibrium prices and fundamental values in Proposition 2 below. These important functions are the hazard rate, as usual when uncertainty is about waiting time, and what we can call the hazard weight:

Definition 6 (Hazard Rate and Hazard Weight)**.** The hazard rate is the function $h : (0,T) \to [0,\infty)$ defined by

$$h(x) = \frac{f(x)}{1-F(x)}$$

(standard definition apart from the restricted domain). The hazard weight is the function $\tilde{h} : (0, T) \to [0, 1]$ defined by

$$\tilde{h}(x) = \frac{h(x)}{h(x) + r}.$$

Indeed, the name hazard weight we chose for this function stands for the relative weight of the hazard rate h and the interest rate r this function $\tilde{h}$ measures. Our overpricing condition that is both necessary and sufficient (Section 5) is on the hazard rate h, and so is our final Assumption 6 below, but they enter our proofs via the hazard weight $\tilde{h}$.

Assumption 6. One-sided limits of h at the endpoints of its domain or else divergence to infinity:

(i) h has a limit or tends to ∞ as $x \to 0^+$;

(ii) h has a limit or tends to ∞ as $x \to T^-$, where the convention is $\infty^- = \infty$.

Assumption 6 just disciplines the distribution F of the perceived remaining time W at the endpoints of the set of experience levels $[0, T] \setminus \{\infty\}$ in the sense of a sufficiently well-behaved hazard rate h. It makes the analysis of the continuous-trade limits of the discrete-trade equilibrium prices and fundamental values less tedious:

Proposition 2 (Continuous-trade Limits)**.** *Consider (i) the equilibrium price p^*_Δ and (ii) fundamental value $\bar{p}_\Delta$ found in Proposition 1 as (real) functions of the trade-frequency parameter Δ on $(0, T)$. They have (finite) right-hand limits at 0.*

Proof. We continue here our unified treatment of the equilibrium price (part (i)) and fundamental value (part (ii)) set out in the proof of Proposition 1, viewing each of them as the solution of the unified problem (5). In other words, we consider the unique solution p_Δ of problem (5) as a (real) function of the trade-frequency parameter Δ on $(0, T)$ and show that this function has a right-hand limit at 0. To structure this proof, we first recall from the proof of Proposition 1 that what we are taking the limit of are the fixed points p_Δ of the maxima $v_\Delta : [0, 1] \to [0, 1]$. The proof of this proposition builds on that of Proposition 1 and goes in four steps: characterizing the fixed points of the maxima as maxima of fixed points, extending these new maximands to continuous trade, extending the constraints on holding durations to continuous trade, and then obtaining the desired limit using the Berge Maximum Theorem.

Step 1 (Characterizations of Discrete-trade Solutions). We simplify the mathematical structure of the definition of the solution p_Δ of the unified discrete-trade problem (5) by two characterizations (9) and (10) below. The second characterization is just a useful improvement of the first.

First to characterize the discrete-trade solution p_Δ by showing that

$$p_\Delta = \max_{\substack{x \in [0, T-\Delta] \setminus \{\infty\} \\ y \in Z_\Delta(x)}} \frac{\int_x^{x+y} e^{-r(w-x)} dF(w)}{1 - F(x) - e^{-ry}(1 - F(x+y))} \tag{9}$$

and the maximum exists, where the maximand is the fixed point of the original price-dependent maximand in the discrete-trade problem (5). This holds for the solution p_Δ because its definition (5) requires that all $x \in [0, T-\Delta]\backslash\{\infty\}$ and all $y \in Z_\Delta(x)$ satisfy

$$p_\Delta \geq \frac{1}{1-F(x)}\left(\int_x^{x+y} e^{-r(w-x)}dF(w) + p_\Delta e^{-ry}(1-F(x+y))\right),$$

hence

$$p_\Delta \geq \frac{\int_x^{x+y} e^{-r(w-x)}dF(w)}{1-F(x)-e^{-ry}(1-F(x+y))},$$

but both inequalities become equalities when (x, y) is any of the original maximizers in (5).

Now a useful improvement of this characterization (9) of the solution p_Δ follows by restricting the maximization there to a compact set in the infinite-lifetime case $T = \infty$ like in the earlier characterization, the one using (8). To write down this improved characterization in a unified way for both the infinite- and finite-lifetime cases, we denote the highest experience level used by x_Δ: x_Δ is $\tilde{x}$ from Assumption 3 or $T-\Delta$ according as $T = \infty$ or $T < \infty$. The useful fact we were after is that

$$p_\Delta = \max_{\substack{x\in[0,x_\Delta]\\ y\in Z_\Delta(x)}} \frac{\int_x^{x+y} e^{-r(w-x)}dF(w)}{1-F(x)-e^{-ry}(1-F(x+y))} \tag{10}$$

and follows in the same way as the first version (9), except that in the discrete-trade problem (5) we take a maximizer belonging to this smaller set using again the maximization over (8).

Step 2 (Continuous Extension of Maximand from Step 1). To take the desired limit using the characterizations (9)–(10), we first show that the maximands there extend to a continuous real function g on

$$\{(x, y) \in \mathbb{R}\times(-\infty,\infty] : 0 \leq x \leq T, 0 \leq y \leq T-x\}, \tag{11}$$

where $(-\infty, \infty]$ is the one-point compactification of $\mathbb{R}$ (the unclear part is $y = 0$). For this, we define g by

$$g(x,y) = \begin{cases} \int_x^{x+y} e^{-r(w-x)}dF(w) \Big/ \left(1-F(x)-e^{-ry}(1-F(x+y))\right) & \text{if } y > 0, \\ \tilde{h}(x) & \text{if } y = 0 < x < T, \\ \lim_{w\to T^-}\tilde{h}(w) & \text{if } y = 0 < x = T, \\ \lim_{w\to 0^+}\tilde{h}(w) & \text{if } y = 0 = x, \end{cases}$$

so that it only remains to prove continuity of g, the crux of which is the L'Hôpital Rule that works in some sense uniformly, but for which we do not have an off-the-shelf statement and do our own proof. Namely, the key to continuity is to first prove that uniformly on compact subsets of $[0,T]\setminus\{\infty\}\subset\mathbb{R}$ we have

$$\lim_{y\to 0^+} g(x,y) = g(x,0), \tag{12}$$

where allowed y's depend on x, i.e., by (12) we mean that for every compact $X\subset[0,T]\setminus\{\infty\}$ and for every $\varepsilon\in(0,\infty)$ there exists a $\delta\in(0,\infty)$ such that all $x\in X$ and all $y\in[0,T-x]\cap(0,\delta)$ satisfy

$$|g(x,y)-g(x,0)|<\varepsilon. \tag{13}$$

Let us prove this key claim (12) and then return to continuity of g. Since in this claim any $\delta\in(0,\infty)$ works if $x=T$, we only need to find a $\delta\in(0,\infty)$ that works for every $x\in X\backslash\{T\}$. We note that $x<T$ and so for all $y,y'\in(0,T-x)$ with $y>y'$ the Cauchy Mean-value Theorem, whose differentiability hypotheses follow from Assumption 5 by the Fundamental Theorem of Calculus, yields a $z\in(y',y)$ such that

$$\begin{aligned}
&\frac{\int_x^{x+y} e^{-r(w-x)}dF(w)-\int_x^{x+y'} e^{-r(w-x)}dF(w)}{1-F(x)-e^{-ry}(1-F(x+y))-(1-F(x)-e^{-ry'}(1-F(x+y')))}\\
&=\frac{e^{-rz}f(x+z)}{e^{-rz}(1-F(x+z))r+e^{-rz}f(x+z)}\\
&=\frac{f(x+z)}{f(x+z)+(1-F(x+z))r}\\
&=\frac{h(x+z)}{h(x+z)+r}\\
&=\tilde{h}(x+z)\\
&=g(x+z,0).
\end{aligned}$$

Now since the function $g(\cdot,0)$ on $[0,T]\setminus\{\infty\}$ is continuous considering continuity of the density f as per Assumption 5, this very $g(\cdot,0)$ is uniformly continuous on compact subsets of $[0,T]\setminus\{\infty\}$. This means that either $T<\infty$ and $g(\cdot,0)$ is uniformly continuous or else $T=\infty$ and $g(\cdot,0)$ is uniformly continuous on $X+[0,1]$. In both cases, uniform continuity yields a $\delta\in(0,\infty)$ independent of x such that

$$\begin{aligned}
&y\in(0,\delta)\implies\\
&|g(x+z,0)-g(x,0)|<\frac{\varepsilon}{2}\implies\\
&\left|\frac{\int_x^{x+y} e^{-r(w-x)}dF(w)-\int_x^{x+y'} e^{-r(w-x)}dF(w)}{1-F(x)-e^{-ry}(1-F(x+y))-(1-F(x)-e^{-ry'}(1-F(x+y')))}-g(x,0)\right|<\frac{\varepsilon}{2}.
\end{aligned}$$

Passing to the limit as $y'\to 0^+$ shows that

$$|g(x,y)-g(x,0)|\le\frac{\varepsilon}{2}.$$

Since here y was an arbitrary element of $(0, T-x) \cap (0, \delta)$, this desired inequality readily extends to all $y \in [0, T-x] \cap (0, \delta)$. Since here x was an arbitrary element of $X \backslash \{T\}$, we have proved the desired uniform convergence stated in (12).

It remains to verify continuity of g now that we have its convergence (12), as $y \to 0^+$, uniformly on compact subsets of $[0, T] \setminus \{\infty\}$ in the sense of (13). For continuity, it is convenient to use the sequential criterion, but we only look at sequences $\{(x_n, y_n)\}$ in the domain (11) of g converging to (x, y) in its domain with $y = 0$, as the rest are rudimental. It is precisely the former cases where knowing the uniform-convergence property (12) helps, because $\{(x_n, y_n)\} \to (x, 0)$ means that the set $\{x, x_1, x_2, \dots\}$ is a compact subset of $[0, T] \setminus \{\infty\}$ and we have that uniform convergence. Now to complete this step, note that by this property for every $\varepsilon \in (0, \infty)$ there is a $\delta_2 \in (0, \infty)$ such that all indices n satisfy

$$y_n \in (0, \delta_2) \implies |g(x_n, y_n) - g(x_n, 0)| < \frac{\varepsilon}{2}$$

and (recall continuity of $g(\cdot, 0)$ from the previous paragraph) there is a $\delta_1 \in (0, \infty)$ such that all n satisfy

$$|x_n - x| < \delta_1 \implies |g(x_n, 0) - g(x, 0)| < \frac{\varepsilon}{2},$$

hence eventually

$$|g(x_n, y_n) - g(x, 0)| < \varepsilon,$$

as desired.

Step 3 (Continuous-trade Holding Constraints). To furnish a candidate for the desired limit of the discrete-trade solution based on its characterizations (9)–(10) from Step 1, we also take their constraint on holding durations and extend it to continuous trade. Our new constraint is the correspondence Z_0 from the experience levels $[0, T] \setminus \{\infty\}$ to the holding durations $[0, \infty]$ defined by either

$$Z_0(x) = \{T - x\}, \text{ for part (ii)}, \tag{14}$$

or

$$Z_0(x) = [0, T - x], \text{ for part (i)}. \tag{15}$$

In its domain, define x_0 to be $\tilde{x}$ from Assumption 3 or T according as $T = \infty$ or $T < \infty$.

Step 4 (Convergence of Discrete-trade Solutions). The final step is to see from Steps 1–3 that the desired limits are (in unified form for both part (i) and (ii) by means of notation (14)–(15) for the constraint on holding durations)

$$\lim_{\Delta \to 0^+} p_\Delta = \max_{\substack{x \in [0, x_0] \\ y \in Z_0(x)}} g(x, y). \tag{16}$$

Here we only note that what we are taking the limit of are themselves maxima (10) and they converge to this maximum by a version of the Berge Maximum Theorem that only assumes continuity of the correspondence at a single point (see *Moore*, 2010). □

The main question of this paper is to compare the equilibrium price in this continuous-trade limit with the fundamental value in this limit. But this is the subject of the following section.

5 Characterization of Buy-and-resell Overpricing

Let us present our main results on buy-and-resell overpricing—the equilibrium price being greater than the fundamental value, defined as what the equilibrium price would be if there were a no-retrade constraint (Definition 3). First we give two sufficient conditions for buy-and-resell overpricing (Sections 5.1 and 5.2) in the continuous-trade limit of our discrete-trade model and then show the necessity of the second condition (Section 5.3).

5.1 1st Sufficient Condition for Overpricing

To report the characterization of buy-and-resell overpricing later in Sections 5.2–5.3, we first need to recover in our model the sufficient condition in terms of personalized fundamental values from the infinite-lifetime models of *Morris* (1996) and *Werner* (2020). These personalized fundamental values are just the expected discounted returns used to define fundamental value for infinitely-lived investors—the expected discounted returns of buying the asset and holding it forever (Section 3). We show that our finite-lifetime extension of personalized fundamental values remains valid for both purposes: representing the fundamental value and giving a sufficient condition for buy-and-resell overpricing. For this, we replace holding forever with holding for the rest of life and then actually reselling as long as the dividend is unpaid, but at the fundamental value's continuous-trade limit as the resale price, which we take from Proposition 2 and denote by

$$\bar{p}_0 = \lim_{\Delta \to 0^+} \bar{p}_\Delta.$$

In our model, the resulting personalized fundamental values are a function of experience, which we call by the shorter name of a fundamental valuation:

Definition 7 (Fundamental Valuation)**.** The fundamental valuation is the function of experience $V : [0, T] \setminus \{\infty\} \to \mathbb{R}$ defined by

$$\begin{aligned} V(x) &= \mathrm{E}\left(e^{-r(W-x)} I_{\{W \le T\}} + \bar{p}_0 e^{-r(T-x)} I_{\{W > T\}} \middle| W > x\right) \\ &= \frac{1}{1 - F(x)} \left(\int_x^T e^{-r(w-x)} dF(w) + \bar{p}_0 e^{-r(T-x)} (1 - F(T)) \right). \end{aligned}$$

In the benchmark infinite-lifetime case $T = \infty$, this fundamental valuation as a function of experience defines fundamental value by taking the maximum. Indeed, in the case $T = \infty$ all the discrete-trade fundamental values (3), by (standard) definition, together with their continuous-trade limit $\bar{p}_0$ are the same thing as the most optimistic fundamental valuation:

$$\bar{p}_0 = \max_{x \in [0, \infty)} V(x). \tag{17}$$

This remains true in the finite-lifetime case $T < \infty$ for the continuous-trade limit $\bar{p}_0$, but the difference from the standard case is that the most optimistic fundamental valuation itself depends on $\bar{p}_0$:

Proposition 3. *The fundamental value's continuous-trade limit $\bar{p}_0$ equals the most optimistic fundamental valuation:*

$$\bar{p}_0 = \max_{x \in [0,T] \setminus \{\infty\}} V(x)$$

and the maximum exists.

Proof. The argument for the finite-lifetime case $T < \infty$ is different. For this scenario, we carry over from the proof of Proposition 2 formula (16) and note that it requires that all experience levels $x \in [0, T)$ satisfy

$$\bar{p}_0 \geq g(x, T - x) = \frac{\int_x^T e^{-r(w-x)} dF(w)}{1 - F(x) - e^{-r(T-x)} (1 - F(T))}, \tag{18}$$

hence

$$\bar{p}_0 \geq \frac{1}{1 - F(x)} \left(\int_x^T e^{-r(w-x)} dF(w) + \bar{p}_0 e^{-r(T-x)} (1 - F(T)) \right) = V(x), \tag{19}$$

but $\bar{p}_0 = V(T)$, completing the proof. □

We are ready to state and prove the first sufficient condition for buy-and-resell overpricing. It extends to overlapping generations the idea of perpetual valuation switching from the infinite-lifetime models of *Morris* (1996) and *Werner* (2020), who also find it to be sufficient for buy-and-resell overpricing:

Definition 8 (Valuation-switching Condition)**.** The fundamental valuation V exhibits switching if some (relatively low) experience level $\hat{x} \in [0, T)$ maximizes V but another (higher) experience level $\breve{x} \in (\hat{x}, T)$ does not.

In other words, here investors switch from belonging to the most optimistic group when their experience is $\hat{x}$ time units to a less optimistic group when they accumulate $\breve{x}$ time units of experience, as long as the dividend is unpaid. As a result, buy-and-resell overpricing occurs in the continuous-trade limit as the discrete-trade equilibrium prices tend to the limit found in Proposition 2 and denoted by

$$p_0^* = \lim_{\Delta \to 0^+} p_\Delta^*,$$

that is:

Proposition 4 (Buy-and-resell Overpricing 1)**.** *If the fundamental valuation V exhibits switching, then $p_0^* > \bar{p}_0$.*

Proof. First we take $\hat{x}, \breve{x}$ from Definition 8 and, without loss of generality, $\hat{x} \in [0, \tilde{x}]$ if $T = \infty$ (maximizers of (7), where p drops out, over (8) for the case of fundamental value (part (ii)), where Δ drops out, give us maximizers of V that belong to $[0, \tilde{x}]$). Now the proof goes by taking the function g from (16) and showing two inequalities

$$p_0^* \geq g(\hat{x}, \breve{x} - \hat{x}) > \bar{p}_0, \tag{20}$$

but the first of these inequalities is immediate from (16):

$$p_0^* = \lim_{\Delta \to 0^+} p_\Delta^* = \max_{\substack{x \in [0, x_0] \\ y \in [0, T-x]}} g(x, y) \geq g(\hat{x}, \breve{x} - \hat{x}).$$

For the remaining inequality (the strict one in (20)), first note that $V(\hat{x}) = \bar{p}_0$ by Proposition 3, but also

$$V(\hat{x}) = \frac{\int_{\hat{x}}^{\breve{x}} e^{-r(w-\hat{x})} dF(w) + \bar{p}_0 e^{-r(\breve{x}-\hat{x})}(1 - F(\breve{x}))}{1 - F(\hat{x})} + \frac{1 - F(\breve{x})}{1 - F(\hat{x})} e^{-r(\breve{x}-\hat{x})} (V(\breve{x}) - \bar{p}_0)$$

by a direct calculation. It follows from these two conditions that

$$\begin{aligned} \frac{\int_{\hat{x}}^{\breve{x}} e^{-r(w-\hat{x})} dF(w) + \bar{p}_0 e^{-r(\breve{x}-\hat{x})}(1 - F(\breve{x}))}{1 - F(\hat{x})} &= V(\hat{x}) + \frac{1 - F(\breve{x})}{1 - F(\hat{x})} e^{-r(\breve{x}-\hat{x})} (\bar{p}_0 - V(\breve{x})) \\ &= \bar{p}_0 + \frac{1 - F(\breve{x})}{1 - F(\hat{x})} e^{-r(\breve{x}-\hat{x})} (V(\hat{x}) - V(\breve{x})). \end{aligned}$$

Now the time is ripe for applying valuation switching, by which here $V(\hat{x}) - V(\breve{x}) > 0$, to conclude that

$$\frac{1}{1 - F(\hat{x})} \left(\int_{\hat{x}}^{\breve{x}} e^{-r(w-\hat{x})} dF(w) + \bar{p}_0 e^{-r(\breve{x}-\hat{x})}(1 - F(\breve{x})) \right) > \bar{p}_0,$$

and thus that

$$\bar{p}_0 < \frac{\int_{\hat{x}}^{\breve{x}} e^{-r(w-\hat{x})} dF(w)}{1 - F(\hat{x}) - e^{-r(\breve{x}-\hat{x})}(1 - F(\breve{x}))} = g(\hat{x}, \breve{x} - \hat{x}),$$

as desired. □

5.2 2nd Sufficient Condition for Overpricing

Whereas the first sufficient condition (Definition 8) for buy-and-resell overpricing is not directly on primitives of the model, our main condition, given here, is, but there is a striking similarity with the first condition. Our main condition requires something like valuation switching in the first condition, extended from *Morris* (1996) and *Werner* (2020), but requires it of the hazard rate instead of the fundamental valuation. Here is the main condition:

Definition 9 (End-of-life hazard switching)**.** The hazard rate h exhibits end-of-life switching if

$$\lim_{x\to T^-} h(x) < \sup_{x\in(0,T)} h(x).$$

In other words, eventually, with experience, the hazard rate h has to switch to and stay at lower levels relative to the highest ones, whereas we did not require the fundamental valuation V to stay at lower levels in the first condition (Definition 8). Interpreting h as instantaneous optimism about the dividend as a function of experience, the condition says that the most experienced investors are not instantaneously the most optimistic. It is also sufficient for buy-and-resell overpricing:

Proposition 5 (Buy-and-resell Overpricing 2)**.** *If the hazard rate h exhibits end-of-life switching, then $p_0^* > \bar{p}_0$.*

Proof. Taking advantage of Proposition 4, we assume for this proof that the fundamental valuation V does not exhibit switching. With this, the proof goes by showing two things

$$p_0^* > \lim_{x\to T^-} \tilde{h}(x) = \bar{p}_0, \tag{21}$$

but the inequality is almost immediate from end-of-life hazard switching, by which there exists an $\hat{x} \in (0,T)$ such that

$$\lim_{x\to T^-} h(x) < h(\hat{x}),$$

and the fact that p_0^* is the limit in (16) of discrete-trade equilibrium prices with representations (9):

$$\begin{aligned} p_0^* = \lim_{\Delta\to 0^+} p_\Delta^* &\geq \lim_{\Delta\to 0^+} g(\hat{x},\Delta) \qquad\qquad (\Delta\in(0,T-\hat{x}))\\ &= g(\hat{x},0)\\ &= \tilde{h}(\hat{x})\\ &= \frac{h(\hat{x})}{h(\hat{x})+r}\\ &> \frac{\lim_{x\to T^-} h(x)}{r+\lim_{x\to T^-} h(x)}\\ &= \lim_{x\to T^-} \tilde{h}(x). \end{aligned}$$

For the remaining part of the proof (the equality in (21)), we use the fact that we could assume that V does not exhibit switching and consider two cases:

Case 1 ($T<\infty$). By way of contradiction, suppose that $\lim_{x\to T^-} \tilde{h}(x) \neq \bar{p}_0$. It follows that

$$\lim_{x\to T^-} \tilde{h}(x) = g(T,0) < \bar{p}_0 = \max_{x\in[0,T]} g(x,T-x)$$

from (16), and thus that here some $\hat{x} \in [0, T)$ is a maximizer but some $\breve{x} \in (\hat{x}, T)$ is not. The next step is to conclude that $V(\hat{x}) = \bar{p}_0 > V(\breve{x})$ by noting that in (18)–(19) both inequalities becomes equalities or strict inequalities according as we plug $\hat{x}$ or $\breve{x}$. Now $\hat{x}$ maximizes V by Proposition 3 but $\breve{x}$ does not, and thus V exhibits switching, contradicting our assumption for this proof.

Case 2 ($T = \infty$). This goes by showing that V and $\tilde{h}$ are eventually constant at $\bar{p}_0$. First for V, take any of its maximizers $\hat{x}$, which exists with $V(\hat{x}) = \bar{p}_0$ by Proposition 3, and note that now all $x \in [\hat{x}, \infty)$ satisfy $V(x) = V(\hat{x}) = \bar{p}_0$, because we assumed for this proof that V does not exhibit switching. Now for $\tilde{h}$, differentiate V on $(\hat{x}, \infty)$ to see that every $x \in (\hat{x}, \infty)$ satisfies

$$0 = V'(x) = (h(x) + r) V(x) - h(x) = (h(x) + r) \bar{p}_0 - h(x),$$

hence

$$\tilde{h}(x) = \frac{h(x)}{h(x) + r} = \bar{p}_0,$$

completing the proof. □

5.3 Converse: Necessary Condition for Overpricing

The remaining part of our main result is the necessity of our last sufficient condition for buy-and-resell overpricing. The equilibrium price equals the fundamental value in the continuous-trade limit of our discrete-trade model if the sufficient condition—end-of-life hazard switching—fails:

Proposition 6 (Pricing at Fundamental Value)**.** *If the hazard rate h does not exhibit end-of-life switching, then $p_0^* = \bar{p}_0$.*

Proof. First note that it suffices to show that $p_0^* \leq \bar{p}_0$, because the reverse inequality holds by (16):

$$p_0^* = \max_{\substack{x \in [0, x_0] \\ y \in [0, T-x]}} g(x, y) \geq \max_{\substack{x \in [0, x_0] \\ y \in \{T-x\}}} g(x, y) = \bar{p}_0. \tag{22}$$

Now the proof goes by taking any maximizer (x^*, y^*) on the left-hand side of this inequality and showing three more inequalities

$$p_0^* = g(x^*, y^*) \leq g(x^*, 0) \leq \lim_{x \to T^-} \tilde{h}(x) \leq \bar{p}_0. \tag{23}$$

Among these, the middle inequality follows from the absence of end-of-life hazard switching by noting that

$$\lim_{x \to T^-} \tilde{h}(x) = \lim_{x \to T^-} \frac{h(x)}{h(x) + r} \geq \sup_{x \in (0,T)} \frac{h(x)}{h(x) + r} = \sup_{x \in (0,T)} g(x, 0) \geq g(x^*, 0),$$

the last inequality, in (23), holds because either $T < \infty$ and

$$\lim_{x \to T^-} \tilde{h}(x) = g(T, 0) \leq \max_{x \in [0,T]} g(x, T - x) = \bar{p}_0$$

again by (16) or else $T = \infty$ and

$$\begin{aligned}
\lim_{x \to \infty} \tilde{h}(x) &= \lim_{x \to \infty} \frac{h(x)}{h(x) + r} \\
&= \lim_{x \to \infty} \frac{\int_x^\infty e^{-rw} dF(w)}{e^{-rx}(1 - F(x))} && \text{(by the L'Hôpital Rule)} \\
&= \lim_{x \to \infty} \frac{1}{1 - F(x)} \int_x^\infty e^{-r(w-x)} dF(w) \\
&= \lim_{x \to \infty} V(x) \\
&\leq \max_{x \in [0,\infty)} V(x) \\
&= \bar{p}_0 && \text{(by (17))},
\end{aligned}$$

and to prove the first inequality, in (23), consider three cases:

Case 1 ($x^* = 0$). By way of contradiction, suppose that $g(x^*, y^*) > g(x^*, 0)$. It follows by definitions that

$$p_0^* > \lim_{x \to 0^+} \tilde{h}(x). \tag{24}$$

Now it must be that $\lim_{x \to 0^+} \tilde{h}(x) < 1$, because $p_0^* = \lim_{\Delta \to 0^+} p_\Delta^* \leq 1$. It then follows that $\lim_{x \to 0^+} h(x) < \infty$. Furthermore, f has the same limit

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} h(x) < \infty, \tag{25}$$

which is to say $\lim_{x \to 0^+} f(x) < \infty$. Next as a density for F also define $f_0 : \mathbb{R} \to [0, \infty)$ by

$$f_0(w) = \begin{cases} \lim_{x \to 0^+} f(x) & \text{if } w = 0, \\ f(w) & \text{if } w \neq 0. \end{cases}$$

This way, f_0 is continuous on $[0, T)$ by Assumption 5. Now the Fundamental Theorem of Calculus allows us to differentiate $v : [0, y^*) \to \mathbb{R}$ defined by

$$v(x) = g(x, y^* - x) = \frac{\int_x^{y^*} e^{-r(w-x)} dF(w)}{1 - F(x) - e^{-r(y^*-x)}(1 - F(y^*))} \tag{26}$$

to see that its derivative at every $x \in [0, y^*)$ is

$$v'(x) = \frac{(f_0(x) + r(1 - F(x)))\, v(x) - f_0(x)}{1 - F(x) - e^{-r(y^*-x)}(1 - F(y^*))}.$$

Furthermore, we have

$$v'(0) = \frac{\left(\lim_{x\to 0^+} h(x) + r\right) p_0^* - \lim_{x\to 0^+} h(x)}{1 - e^{-ry^*}\left(1 - F(y^*)\right)} > 0,$$

because (24) and (25) show that

$$p_0^* > \lim_{x\to 0^+} \tilde{h}(x) = \frac{\lim_{x\to 0^+} h(x)}{r + \lim_{x\to 0^+} h(x)}.$$

It follows that some $\hat{x} \in (0, y^*)$ satisfies $v(\hat{x}) > v(0) = p_0^*$. Finally, for the contradiction, let Δ be $y^* - \hat{x}$ or 1 according as $y^* < \infty$ or $y^* = \infty$, let p_Δ^* be the corresponding (discrete-trade) equilibrium price found in Proposition 1, and observe, using (9) and (10), that

$$\begin{aligned} p_0^* &< v(\hat{x}) = g(\hat{x}, y^* - \hat{x}) \\ &\le \max_{\substack{x\in[0,T-\Delta]\setminus\{\infty\} \\ y\in\{\Delta,2\Delta,3\Delta,\ldots,\infty\}\cap[\Delta,T-x]}} g(x,y) = p_\Delta^* = \max_{\substack{x\in[0,x_\Delta] \\ y\in\{\Delta,2\Delta,3\Delta,\ldots,\infty\}\cap[\Delta,T-x]}} g(x,y) \qquad (27) \\ &\le \max_{\substack{x\in[0,x_0] \\ y\in[0,T-x]}} g(x,y), \end{aligned}$$

contradicting the formula (used in (22)) for p_0^*.

Case 2 $(0 < x^* < T)$. It goes either trivially if $y^* = 0$ or by taking a first-order necessary condition for an interior maximum in the following way if $y^* > 0$. First of all, as an objective function it is convenient to consider $v : (0, x^* + y^*) \to \mathbb{R}$ defined by

$$v(x) = g(x, x^* + y^* - x) = \frac{\int_x^{x^*+y^*} e^{-r(w-x)} dF(w)}{1 - F(x) - e^{-r(x^*+y^*-x)}\left(1 - F(x^* + y^*)\right)}$$

similarly to (26). Now we show that x^* maximizes v by adjusting (27): let $x' \in (0, x^* + y^*)$, let Δ be $x^* + y^* - x'$ or 1 according as $y^* < \infty$ or $y^* = \infty$, let p_Δ^* be the corresponding (discrete-trade) equilibrium price found in Proposition 1, and observe, using (9) and (10), that

$$\begin{aligned} v(x') &= g(x', x^* + y^* - x') \\ &\le \max_{\substack{x\in[0,T-\Delta]\setminus\{\infty\} \\ y\in\{\Delta,2\Delta,3\Delta,\ldots,\infty\}\cap[\Delta,T-x]}} g(x,y) = p_\Delta^* = \max_{\substack{x\in[0,x_\Delta] \\ y\in\{\Delta,2\Delta,3\Delta,\ldots,\infty\}\cap[\Delta,T-x]}} g(x,y) \\ &\le \max_{\substack{x\in[0,x_0] \\ y\in[0,T-x]}} g(x,y) = g(x^*, y^*) = g(x^*, x^* + y^* - x^*) \\ &= v(x^*). \end{aligned}$$

Next differentiate v at x^* to see that

$$0 = v'(x^*) = \frac{\left(f(x^*) + r\left(1 - F(x^*)\right)\right) g(x^*, y^*) - f(x^*)}{1 - F(x^*) - e^{-ry^*}\left(1 - F(x^* + y^*)\right)},$$

hence

$$g(x^*, y^*) = \frac{f(x^*)}{f(x^*) + r(1 - F(x^*))} = \frac{h(x^*)}{h(x^*) + r} = \tilde{h}(x^*) = g(x^*, 0),$$

proving the first inequality in (23), as desired, for this case.

Case 3 ($x^* = T$). It is immediate from (22) because $y^* = 0$ by the constraints, completing the proof. □

5.4 Examples

We give three examples where our necessary and sufficient condition either fails so that Proposition 6 applies or holds so that Proposition 5 does. For convenience, we summarize them in Table 1, where, in particular, we indicate whether buy-and-resell overpricing and belief heterogeneity co-occur or only the latter occurs and does not cause overpricing (the last column). The possibility of such overpricing-neutral belief heterogeneity, when our necessary and sufficient condition for overpricing of course fails, shows that the condition—end-of-life hazard switching—is nontrivial.

First of all, an example where our necessary and sufficient condition fails so that Proposition 6 applies is to let the perceived remaining time W be finite-valued with an exponential distribution and everything else be arbitrary (see Table 1). Indeed, in this case the hazard rate h is constant and consequently does not exhibit end-of-life switching, as desired.

To give an example where our necessary and sufficient condition holds so that Proposition 5 applies, let the perceived remaining time W have an incomplete exponential distribution, as in Example 1, and everything else be arbitrary (see Table 1). Indeed, in this case the hazard rate h is strictly decreasing and thus exhibits end-of-life switching, as required.

Finally, to see that our necessary and sufficient condition is indeed stronger than belief heterogeneity, let W be finite-valued and gamma-distributed with the shape parameter greater than one, $T < \infty$, and everything else be arbitrary (see Table 1). Indeed, in this case the hazard rate h is strictly increasing, i.e., more experienced investors' beliefs are instantaneously more optimistic, but overpricing does not occur (by Proposition 6).

Table 1: Main Scenarios Ilustrated by Standard Distributions

	Exponential	Incomplete exponential	Gamma(α, λ), $\alpha > 1$
Parameters	$\lambda \in (0, \infty)$	$\lambda \in (0, \infty)$, $q \in (0, 1)$	$\lambda \in (0, \infty)$, $\alpha \in (1, \infty)$
$F(w)$, $w \geq 0$	$1 - e^{-\lambda w}$	$(1 - q)\left(1 - e^{-\lambda w}\right)$	$\frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^w z^{\alpha-1} e^{-\lambda z} dz$
$f(w)$, $w \geq 0$	$\lambda e^{-\lambda w}$	$(1 - q)\lambda e^{-\lambda w}$	$\frac{\lambda^\alpha}{\Gamma(\alpha)} w^{\alpha-1} e^{-\lambda w}$
Assumptions throughout paper	All hold	All hold, Example 1 verifies Assumption 3	All hold if $T < \infty$, Assumption 3 fails if $T = \infty$
$h(x)$, $0 < x < T$	λ	$\left(1 - \frac{q}{q+(1-q)e^{-\lambda x}}\right) \lambda$	$\frac{1}{\int_0^\infty \left(1+\frac{w}{x}\right)^{\alpha-1} e^{-\lambda w} dw}$
Hazard-rate shape	Constant	Strictly decreasing	Strictly increasing
End-of-life hazard switching	No	Yes	No
Buy-and-resell overpricing	*No*	*Yes*	*No*
$F(w\|x)$, $w \geq 0$, $0 \leq x < T$	$F(w)$	$\left(1 - \frac{q}{q+(1-q)e^{-\lambda x}}\right)\left(1 - e^{-\lambda w}\right)$	$1 - \frac{1-F(w+x)}{1-F(x)}$
Belief heterogeneity (across x)	No	Yes	Yes (gamma has memory)

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