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Stochastic cascades, credit contagion, and large portfolio losses

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Abstract

We analyze an interactive model of credit ratings where external shocks spread by a contagious chain reaction to the entire economy. Counterparty relationships along with discrete adjustments of credit ratings generate a transition mechanism that allows the financial distress of one firm to spill over to its business partners. The spread of financial distress constitutes a source of intrinsic risk for large portfolios of credit sensitive securities that cannot be "diversified away". We provide a characterization of the fluctuations of credit ratings in large economies when adjustments follow a threshold rule and analyze the effects of downgrading cascades on credit portfolios.

JEL classification: G1; E1; E3

Keywords: Credit contagion; Local interactions; Mean-field interaction; Portfolio losses; Threshold rules; Branching processes; Power laws

1. Introduction

It has been well documented by, for example, Duffee (1998) and Keenan (2000) that the number of defaults and credit rating downgrades are strongly correlated with the business cycle. The dependence of aggregate default rates on macroeconomic quantities such as short term interest rates, GDP growth rates, or equity returns has motivated reduced form models such as Duffie and Singleton (1999), Lando (1998) or Frey and McNeil (2001), where the default intensity depends

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on an underlying set of state variables. This approach has the convenient feature that defaults of individual companies and the associated losses of a portfolio of credit sensitive securities are conditionally independent given the state variables. However, it has been argued by, for instance, Hull and White (2001) and Schonbucher and Schubert (2001) that the default correlation obtainable in reduced form models is typically quite low. Jarrow and Yu (2001, 1765) write that "a default intensity that depends linearly on a set of smoothly varying macroeconomic variables is unlikely to account for the (degree of) clustering of defaults around an economic recession".

A more direct form of default dependence arises from counterparty relations. Interactive links between different corporations allow for a spillover of financial distress from one company to another. Lang and Stulz (1992) have shown that bankruptcy filing have an impact on stock returns and, most likely, also on default probabilities of non-defaulted companies. An "infectious" propagation of defaults can in fact generate an autonomous dynamics as documented by the recent financial crises in East Asia. The prevalence of financial crises has led many researchers to conclude that the financial sector is unusually susceptible to shocks. One theory is that small shocks, which initially only affect a few institutions or a particular region of the economy, spread by contagious credit quality deterioration to the rest of the financial sector. But credit contagion phenomena are not limited to the financial sector. They are also a concern in manufacturing where trade credits link suppliers and buyers of goods through a chain of borrowing obligations. If firms are highly interdependent, the default of a single customer can trigger an entire cascade of additional bankruptcies; see, for example, Kiyotaki and Moore (1997). Since the likelihood of default is higher during a recession, this cascading effect is much more likely to be observed then.

Credit contagion rests upon a proper transmission mechanism through which shocks, initially affecting only a small number of firms, can infect the whole economy. Counterparty relations in the form of borrowing and lending contracts constitute a typical distress propagation channel as shown by, for example, Allen and Gale (2000). These authors consider an equilibrium model where different sectors in the banking system have overlapping claims on one another in order to buffer external shocks. The arrangement, however, is quite fragile. Depending on the degree of connectedness of the buffer system, a small shock to one institution can spread through the entire banking sector, causing other institutions to default. Freixas et al. (2000) study the stability of the financial system and the coordination role of supervising authorities if an insolvent institution affects the system in various ways depending on the cross-payment pattern in the interbank market. Jarrow and Yu introduced a "Primary-secondary" approach in order to price defaultable bonds in the presence of counterparty risk. Here, firms split into one of two mutually exclusive types: Primary firms' default intensities only depend on some economy-wide macrovariables, whereas secondary firms' default processes depend on both macrovariables and the credit rating of primary firms.

Chain reactions of credit rating downgrades pose a thread to portfolios of credit sensitive securities as the market value of defaultable claims is particularly vulnerable to excessive fluctuations in default rates. This has important implications for the management of credit risk portfolios, where default correlations need to be explicitly modelled. Infectious defaults were first analyzed by Davis and Lo (1999); see also Hammarlid (2003). However, there are only few theoretical approaches that analyze the resulting portfolio risk in a mathematically rigorous manner. Recently, Dembo et al. (2004) provided a large-deviations approximation of the tail distribution of total financial losses on large portfolios of heterogenous credit securities. Under the assumption that individual losses are conditionally independent given some common "correlating factor," these authors showed how to estimate the expected loss on different positions in the event of a large loss, and how to restructure the portfolio so as to "balance the effects of each type of position on the total financial distress costs" (Dembo et al., 2004, 5). Giesecke and Weber (2004) used methods and techniques from the theory of interacting particle systems in order to study a model with homogeneous locally interacting firms that can be in one of two possible states: "high" or "low". The joint dynamics of credit qualities is then modelled by means of an interacting Markov process that allows for multiple limiting distributions. Despite the strong correlations between individual ratings, the per-capita loss associated with a downgrade of individual firms converges almost surely to the expected loss of a single position. Frey and Backhaus (2003) consider an extension of Jarrow and Yu where default intensities depend on the average rating throughout the whole economy.

The aforementioned approaches are analytically very convenient. However, they have the unsatisfactory feature that the additional uncertainty arising from the interaction of firms can be eliminated by means of simple diversification. Similar to the case of independent defaults, the uncertainty about aggregate losses is small whenever a financial institution holds sufficiently many positions. Recently, Egloff et al. (2004) proposed a model of credit contagion with a rich interaction structure that is more tailored towards business needs. These authors do not obtain analytical results, but numerical simulations suggest that microstructural dependencies have significant effects on the tails of the loss distribution.

This paper suggests an equilibrium model of credit ratings where the risk to large portfolios is *intrinsic*: it cannot be "diversified away". The distribution of credit rating downgrades can be given in closed form; we do not need to rely on a numerical analysis. Our focus is on the interplay between counterparty relations and "lumpy" adjustments of credit ratings. Like many microeconomic problems, credit quality adjustments occur in a discrete manner. This raises the question whether the threshold rules followed by rating agencies when evaluating the credit quality of defaultable bonds can generate a channel of contagious downgrade dynamics. The role of thresholds in individual behavior on the dynamics of aggregate quantities has been extensively investigated in the microeconomic literature on (S, s) economies. Caplin (1985) was the first to develop a general framework to study the implications of an (S, s) inventory policy on the fluctuations of aggregate variables. For economies with an uncountable set of agents, subsequent research by Caballero and Engle (1991) and Caplin and Lehay (1997) has shown that the average behavior throughout the entire population does not differ from that of its frictionless counterpart. This neutrality result, however, rests upon the assumption of an infinite number of market participants. Recent work of Nirei (in press) showed that the picture changes if we consider finite economies. The latter paper very much inspired our own research.

We provide a simple probabilistic framework to study credit contagion phenomena arising from lumpy adjustments of credit ratings in economies with heterogeneous interacting firms. Specifically, we consider an equilibrium model of credit ratings

- where dependencies between different firms generate an intrinsic risk for credit sensitive portfolios that cannot be "diversified away",
- where small shocks, initially affecting only a small number of firms, spread by a contagious chain reaction to the rest of the economy,
- where the additional risk arising from counterparty relations can be quantified, and
- where the sources of large portfolio losses can a priori be identified.

Firms are characterized by pairs (x^i, θ^i). The random variable x^i takes values in a finite set. It denotes the credit rating of firm i. The quantity θ^i summarizes idiosyncratic factors that influence the company's financial situation, but credit ratings will be the only observable quantities. This assumption is justified if we think of firms as being small or medium-sized. An investment bank holding financial positions with medium-sized corporations does typically not have full access to all the internal, company specific factors that affect a firm's financial state. Following Allen and Gale, we finally introduce *buffer variables* s^i describing the firm's ability to absorb additional financial distress without getting downgraded by a rating agency. A downgrade only occurs if a shock exceeds a certain threshold.

In a first step, we assume that individual ratings depend on the ratings of all the other firms only through the average rating. Such an interaction of mean-field type makes sense in the context of portfolio credit risk: if a financial institution has incurred unusually many losses in its loan portfolio, then its credit quality is likely to be downgraded by a rating agency. Unusually many defaults may also have a negative impact on the overall business climate, favoring additional defaults. In a subsequent step we introduce an additional local component into the interaction. It turns out that the results are quantitatively very similar to the results of the mean-field model. This suggests that it is the global interaction that generates a transition channel for the infectious spread of financial distress.

The cascade is initiated by downgrades of those firms whose shock exceeds their respective threshold levels. In the limit of an infinite economy, the number of initial downgrades is Poisson distributed. This captures the idea that the shock initially affects only a small number of firms. Nonetheless, the distribution of the total number of defaults has slowly decaying tails. The interactive structure of credit ratings generates a feedback that may trigger a chain reaction of additional downgrades. A firm might well be able to absorb its shock, but it might not be able to absorb both the shock and the resulting deterioration in the average rating. The initial downgrades may thus trigger additional defaults that, in turn, further deteriorate the average rating, and so on. In a large economy, this cascade can be described by a branching process. Under the assumption that the interaction between different firms is weak enough, the distribution of the total number of defaults can be given in closed form. Our proof uses modifications of arguments given in Nirei and a decoupling argument for economies with local and global interactions that has previously been applied in the context of social interaction models by Horst and Scheinkman (in press).

The market value of a financial institution's position hold with a firm can be severely reduced by adverse changes in the counterparty's credit quality. This calls for an analysis of aggregate portfolio losses due to contagious cascades of credit rating downgrades. To this end, we associate to each firm a random variable describing the loss a bank suffers in case of a downgrade. Following Dembo et al., we assume that individual losses are independent and identically distributed according to some exogenous distribution. The aggregate portfolio losses then take the form of a compound sum where the compounding distribution is generated in an *endogenous* manner by means of a cascade process. We characterize the tail structure of the distribution of aggregate losses. It turns out that the distribution has fat tails if the interaction between different firms is too strong. In this case large portfolio losses are typically due to an unusually large number of individual downgrades.

The rest of this paper is organized as follows. In Section 2, we introduce our interactive model of credit ratings. Section 3 analyzes the effects of counterparty relationships on the distribution of aggregate losses. The proofs are given in Section 4. complementary internet supplement summarizes some useful results about branching processes and compound distributions.

2. A stochastic cascade model of credit contagion

This section introduces an interactive equilibrium model of credit ratings. We identify discrete adjustments of credit qualities as a source of intrinsic risk to credit portfolios. Lumpy adjustments of credit ratings along with counterparty relationships generate a propagation channel through which external shocks, initially affecting only a small number of firms, can spread to the rest of the economy, thereby triggering a *cascade* of credit rating downgrades. Using results from the theory of branching processes, we show that the distribution of the total number of downgrades can be given in closed form.[1]

We consider an economy with N companies. Associated with each firm $i \in \{1, 2, \ldots, N\}$ is a *credit rating* $x^{N,i}$ from a finite set Λ. Firms are *heterogeneous*. Ratings depend on firm specific parameters such as the number of outstanding loans or its debt structure and on the average credit rating throughout the whole economy. Specifically, firm $i \in \{1, 2, \ldots, N\}$ is characterized by a pair

$$(x^{N,i}, \theta^i) \tag{1}$$

where the random variable θ^i summarizes idiosyncratic parameters affecting company i's financial situation. Credit ratings downgrades occur in a discrete manner. The set of possible credit ratings is given by a *finite* subset of $\{0, \pm\lambda, \pm 2\lambda, \ldots\}$. The "spread" λ measures the degree of "lumpiness" of credit rating adjustments. The smaller λ, the finer the classification scheme and the more information about a firm's financial health is encoded in its rating.

2.1. A mean-field model of credit ratings

Our goal is to derive a closed form solution for the distribution of the total number of downgrades in an interactive model of credit contagion. Therefore, we consider the simplest possible form of interaction where individual ratings only depend on the average rating,

$$\bar{x}^N := \frac{1}{N} \sum_{i=1}^{N} x^{N,i}, \tag{2}$$

throughout the whole set of firms. A mean field interaction captures the idea that defaults or downgrades of firms negatively influence the overall business climate. An usually large number of defaults typically deepens an economic recession. It may also prompt banks to call in outstanding loans, either having a negative impact on a firm's financial situation.[2] Assuming a simple linear dependence of individual ratings on both average ratings and firm specific quantities and taking into account that the set of possible ratings is discrete, we are led to the following equilibrium condition of credit rating configurations:

$$x^{N,i} = \alpha \bar{x}^N + \theta^i - (\alpha \bar{x}^N + \theta^i) \bmod \lambda \tag{3}$$

where $y \bmod \lambda$ denotes the remainder on division of y by λ and $\alpha \in [0, 1]$ specifies the *strength of interactions* between different firms. The modulo-λ-arithmetic implies that equilibrium ratings are positive multiples of λ. The special case $\alpha = 0$ corresponds to a situation where credit ratings are independent of each other.

[1] The setup in Section 2 closely follows Nirei. He considers a propagation mechanism in an economy where many individuals follow a threshold rule and interact with a positive feedback. However, in our opinion, for his arguments to hold for every configuration of firm specific parameters, all the credit ratings typically need to take the same value; see Example 2.3. In general, however, there is no reason to expect an equilibrium configuration of credit ratings to be symmetric.

[2] An interaction of mean-field type should only be viewed as a first step towards more general networks of interacting companies. However, as our goal is clarify the role of counterparty relations along with discrete adjustments of credit qualities as a transmission channel for the spread of financial distress, we restrict ourselves to the simplest possible form of interaction. An extension to more general interaction structures is left for future research.

We assume that the idiosyncratic quantities θ^i are uniformly bounded. This allows us to choose a *finite* set of possible credit ratings. It further simplifies our analysis if we assume that the random variables θ^i are independent and uniformly distributed.

Assumption 2.1. The random variables $\{\theta^i\}_{i \in \mathbb{N}}$ are independent and uniformly distributed on the interval $[0, \bar{\theta}]:=[0, n\lambda]$ for some $n \in \mathbb{N}$.

We are now ready to prove the existence of equilibrium configurations of credit ratings. The proof as well as the cascade process introduced below closely follows Nirei.

Lemma 2.2. *Suppose that the interaction between different firms is weak enough in the sense that* $\alpha < 1$. *If Assumption 2.1 is satisfied, then there exists a finite set*

$$\Lambda \subset \{0, \pm\lambda, \pm 2\lambda, \ldots\}$$

and a configuration of credit ratings $\{x^{N,i}(\theta)\}_{i \in I} \in \Lambda^N$ *that satisfies* (3).

Proof. Since $\alpha < 1$, there exist $\underline{y}, \bar{y} \in \mathbb{R}$ such that

$$\underline{y} = \alpha \underline{y} - \lambda \quad \text{and} \quad \bar{y} = \alpha \bar{y} + \bar{\theta}.$$

Let us put $S:=\prod_{i=1}^{N}[\underline{y}, \bar{y}]$ and define functions $g_i(\cdot, \theta) : [\underline{y}, \bar{y}] \to \mathbb{R}$ by

$$g_i(y, \theta) = \alpha y + \theta^i - (\alpha y + \theta^i) \bmod \lambda.$$

For $y^i \in [\underline{y}, \bar{y}]$ we obtain

$$g_i(y^i, \theta) \geq \alpha y^i + \theta^i - \lambda \geq \alpha \underline{y} - \lambda = \underline{y} \quad \text{and} \quad g_i(y^i, \theta) \leq \alpha y^i + \theta^i \leq \alpha \bar{y} + \bar{\theta} = \bar{y}.$$

Hence, $g(\cdot, \theta):=(g_1(\cdot, \theta), \ldots g_N(\cdot, \theta))$ maps the complete lattice S into itself. Since $g(\cdot,\theta)$ is increasing, it has a fixed point $x^N(\theta)$, due to Tarski's theorem. □

The set of possible ratings can be chosen independently of the number of firms and independently of the realizations of the random variables θ^i. Throughout,

$$x^N(\theta) = \{x^{N,i}(\theta)\}_{i=1}^{N} \in \Lambda^N \tag{4}$$

denotes an equilibrium configuration of credit ratings, and $y_{\min} := \min\{y\colon\ y \in \Lambda\}$ and $y_{\max} := \max\{y\colon\ y \in \Lambda\}$ are the best and the worst possible rating, respectively. We assume that credit ratings are the only observable quantities. This assumption is justified if we think of firms as being small or medium-sized. A financial institution holding positions with small companies typically does not has complete information about all the firm specific factors encoded in the random variable θ^i. In particular, the buffer variables are unobservable. A risk manager has only incomplete information about a company's ability to absorb additional distress. The ability of firm i to absorb external shocks is measured by its *buffer variable* or *threshold level*

$$s^{N,i}:=(\alpha \bar{x}^N + \theta^i) \bmod \lambda. \tag{5}$$

The impact of buffer systems on the stability of financial systems has also been analyzed by Allen and Gale.

The actual realization of the firm specific parameters θ^i are unobservable, but their distribution its known. Hence we can calculate the conditional distribution of s^i, given an equilibrium configuration of credit ratings $(x^{N,i})_{i=1}^{N}$.

Example 2.3. Suppose that $\lambda = 1$ and that θ^i is uniformly distributed on the unit interval. In this case,

$$\mathbb{P}[x^{n,i} = 0 \ \text{for all}\, i = 1, 2 \ldots, N] = 1,$$

so the equilibrium is almost surely unique. In particular,

$$\mathbb{P}[s^i = \theta^i \ \text{for all}\, i = 1, 2, \ldots, N] = 1,$$

and so the buffer variables are conditionally independent and identically distributed on $(0, \lambda)$.

The case $\lambda = 1$ and $\theta^i \in (0, 1)$ may be viewed as a situation where a financial institution knows only whether or not a firm is bankrupt. In such a "binary choice model" the random variables s^i are independent and identically distributed, but we want to allow for a finer classification scheme in order to study the impact of the "grid size" λ on the distribution of portfolio losses. In the more general case $\lambda < 1$ the buffer variables are no longer conditionally identically distributed. The following example shows that Assumption 1 in Nirei typically does not hold for all equilibria. A result that holds for almost all realizations of the random variables θ^i requires a considerably more complex analysis.

Example 2.4. Let us assume that $\alpha = 3/4$, that $\lambda = 1/2$, that $N = 2$, that $\theta^i \in (0, 1)$, and that a risk manager observes credit ratings $(x^1, x^2) = (1, 3/2)$. Then (θ^1, θ^2) satisfies

$$1 = \frac{15}{16} + \theta^1 - \left(\frac{15}{16} + \theta^1\right) \bmod \frac{1}{2} \quad \text{and} \quad \frac{3}{2} = \frac{15}{16} + \theta^2 - \left(\frac{15}{16} + \theta^2\right) \bmod \frac{1}{2}.$$

A priori $\theta^i \in (0, 1)$, so all pairs (θ^1, θ^2) with

$$\theta^1 \in \left(\frac{1}{16}, \frac{9}{16}\right) \quad \text{and} \quad \theta^2 \in \left(\frac{9}{16}, 1\right)$$

support (x^1, x^2) as an equilibrium configuration. In view of Assumption 2.1 the random variables θ^1 and θ^2 are independent and conditionally uniformly distributed on (1/16, 9/16) and (9/19, 1), given (x^1, x^2). Thus, the buffer variables

$$s^1 = \left(\frac{15}{16} + \theta^1\right) \bmod \frac{1}{2} = \theta^i - \frac{1}{16} \quad \text{and} \quad s^2 = \left(\frac{15}{16} + \theta^2\right) \bmod \frac{1}{2} = \theta^2 - \frac{9}{16}$$

are conditionally independent and uniformly distributed on (0, 1/2) and (0, 7/16), respectively. Suppose now that firm i is subject to a small external shock of size $y^i < 1/2$ that deteriorates its financial standing by y^i. This means that θ^i is to be replaced by $\hat{\theta}^i := \theta^i + y^i$. If $y^2 < 1/16$, then (x^1, x^2) is still an equilibrium provided that firm 1 is not downgraded. The second company is able to absorb small shocks.

2.2. *Thresholds and downgrade cascades in mean-field models*

We are now going to study the impact of external shocks on the equilibrium configuration of credit ratings. The impact of an economy wide shock of size Y on company i is given by

$$y^{N,i} := \frac{\varepsilon^i}{N} \tag{6}$$

for a sequence of bounded and independent and identically distributed non-negative random variables ε^i with law $Q(Y;\cdot)$ and $E\varepsilon^i = Y$. Hence different firms are affected in different ways by shocks. By the law of large numbers

$$Y = E\varepsilon^i = \lim_{N\to\infty} \sum_{i=1}^{N} y^{N,i} \quad \mathbb{P}\text{-a.s.}$$

In a large economy the actual shock is given by the accumulated impact of many small, conditionally independent shocks.[3]

Assumption 2.5. The random variables ε^i are non-negative, bounded, and, conditioned on the macrovariable Y, independent and identically distributed with $E[\varepsilon^i|Y] = Y$

Initially, firm i is downgraded only if the impact of the external shock exceeds its ability to absorb exogenous distress. This ability is measured by the buffer variable $s^{N,i}$ defined in (5). A downgrade occurs if

$$s^{N,i} + y^{N,i} > \lambda.$$

In a large economy the number of initial downgrades will be approximately Poisson distributed. This captures the idea that a shock initially affects only few firms, but counterparty relationships generate a transmission mechanism through which the shock can spread throughout the whole economy. Individual downgrades deteriorate the average situation, and the feedback effect generated by the interactive dependence of credit ratings favors additional defaults.

2.2.1. The cascade process

Let us now be more specific about the dynamics of the cascade of credit rating downgrades. Following Nirei, we define a stochastic process $\{(x_t^N, s_t^N)\}_{t\in\mathbb{N}} = \{(x_t^{N,i}, s_t^{N,i})_{i=1}^N\}_{t\in\mathbb{N}}$ as follows.

The starting point (x_0^N, s_0^N) is given by an equilibrium configuration of credit ratings and by the corresponding vector of sensitivity parameters:

$$x_0^N = (x_0^{N,i})_{i=1}^N \text{ is a random configuration that satisfies (3)} \quad \text{and} \quad s_0^{N,i} = (\bar{x}_0^N + \theta^i) \bmod \lambda.$$

If a shock hits the economy, the first companies that are downgraded are those who are not able to absorb their respective shocks $y^{N,i}$. Even if a company is not downgraded, the shock leads to a deterioration of its financial standing, so its buffer variable has to be adjusted accordingly. Thus, we put

$$x_1^{N,i} = \begin{cases} x_0^{N,i} + \lambda & \text{if } s_0^{N,i} + y^{N,i} > \lambda \\ x_0^{N,i} & \text{otherwise} \end{cases} \quad \text{and} \quad s^{N,i} = s_0^{N,i} + y^{N,i} - x_1^{N,i} + x_0^{N,i}. \tag{7}$$

Taking into account the feedback effect through the average rating, we define for $t \geq 2$:

$$x_t^{N,i} = \begin{cases} x_{t-1}^{N,i} + \lambda & \text{if } s_{t-1}^{N,i} + \alpha\,\Delta_{t-1} \geq \lambda \\ x_{t-1}^{N,i} & \text{otherwise} \end{cases} \quad \text{and} \quad s_t^{N,i} = s_{t-1}^{N,i} + \alpha\,\Delta_{t-1} - x_t^{N,i} + x_{t-1}^{N,i} \tag{8}$$

[3] In the context of our interactive equilibrium model of credit ratings, the quantity Y plays the role the macrovariables play in the Bernoulli mixture models studied in Lando (1998) or Frey and McNeil (2001).

where $\Delta_{t-1} = \bar{x}^N_{t-1} - \bar{x}^N_{t-2}$ denotes the increase in the average rating in period $t-1$. This way, a company is downgraded at most once per period. We shall see that no firm is downgraded twice if N is large enough. This is in accordance with empirical observations; downgrades by more than one class are rather exceptional.

Remark 2.6. Our cascade process should be viewed as an equilibrium selection procedure as in, for example, Cooper (1994). Initially, the firms' financial situation is described by the vector θ, and the corresponding equilibrium configuration is $x(\theta)$. After a shock hits the economy, a firm's credit quality is revalued. The new equilibrium is given by the limiting configuration of a sequential "best reply" dynamics.

2.2.2. The distribution of the number of downgrades

Introducing the stopping time

$$\tau^N := \inf\{t : \bar{x}^N_t = \bar{x}^N_{t+1}\}, \tag{9}$$

we see that $x^{N,i}_t = x^{N,i}_{t+1} = \dots$ for all $i \in \{1, 2, \dots, N\}$, and each $t \geq \tau^N$. The total number $D^N_{\tau^N}$ of downgrades is then given by

$$D^N_{\tau^N} = \frac{1}{\lambda}\sum_{i=1}^{N}(x^{N,i}_{\tau^N} - x^{N,i}_0) \quad \text{and} \quad D^N_t = \frac{1}{\lambda}\sum_{i=1}^{N}(x^{N,i}_t - x^{N,i}_{t-1})$$

is the number of downgrades in period t. We are interested in the distribution of the total number of defaults in a large economy. To this end, we show that the dynamics of the process $\{D^N_t\}_{t\in\mathbb{N}}$ can asymptotically be described by a branching process where the population reproduces from generation to generation in a Poisson manner.[4] The proof of the following result requires some preparation and will be carried out in Section 4.

Theorem 2.7. *Suppose that Assumptions 2.1 and 2.5 are satisfied. Conditioned on the shock size Y, the following holds*:

(i) *As $N \to \infty$, the sequence of downgrade processes $\{D^N_t\}_{t\in\mathbb{N}}(N \in \mathbb{N})$ converges in distribution to a branching process $\{D_t\}_{t\in\mathbb{N}}$ where the population reproduces from generation to generation in a Poisson manner and where D_0 is Poisson distributed*:[5]

$$D_t \overset{D}{=} D_{t,1} + D_{t,2} + \dots + D_{t,D_{t-1}}$$

where the independent random variables $D_{t,i}$ are Poisson distributed.

(ii) *The sequence of the total number of downgrades $\{D^N_{\tau^N}\}_{N\in\mathbb{N}}$ converges in distribution to a random variable D_τ that follows a compound Poisson distribution. More precisely, there exist constants $\pi_1(\lambda)$ and $\pi_2(\lambda)$ and a random variable Π that is Poisson distributed with*

[4] Basic properties of branching processes are summarized in an internet supplement.

[5] For two random variables X and Y defined on a common probability space we write $X \overset{D}{=} Y$ if X and Y have the same distribution.

parameter $Y\pi_1(\lambda)$ *such that*

$$D_\tau \overset{D}{=} \sum_{t=1}^{\Pi} Z_t. \tag{10}$$

Here $\{Z_t\}_{i\in\mathbb{N}}$ *is a sequence of independent and identically distributed random variables following a Borel-Tanner distribution with parameter* $\nu := \alpha\pi_2(\lambda) \leq 1$,

$$\mathbb{P}[Z_t = k] = \frac{1}{k!}(k\nu)^{\kappa-1}\mathrm{e}^{-k\nu}, \tag{11}$$

and independent of Π. *Conditioned on* $\{\Pi = l\}$ *we have that*

$$\mathbb{P}[D_\tau = k|\Pi = l] = \frac{l}{k}\frac{(k\nu)^{\kappa-l}\,\mathrm{e}^{-k\nu}}{(k-l)!} \quad \text{for } k \geq l, \tag{12}$$

and there exists a multiplicative constant $C > 0$ *such that*

$$\mathbb{P}[D_\tau = k|\Pi = l] = C\nu^k\,\mathrm{e}^{\kappa(1-\nu)}k^{-(3/2)} \quad \text{as } k \to \infty. \tag{13}$$

Moreover, $\lim_{\lambda\to 0}\lambda^2\pi_1(\lambda) = \lim_{\lambda\to 0}\pi_2(\lambda) = 0$.

(iii) *The mean and the variance of* D_τ *are given by*, *respectively*,

$$ED_\tau = \frac{Y\pi_1(\lambda)}{1-\pi_2(\lambda)} \quad \text{and} \quad VD_\tau = \frac{Y\pi_1(\lambda)}{(1-\pi_2(\lambda))^3}. \tag{14}$$

(iv) *If* θ^i *is uniformly distributed on* $(0, \lambda)$, *then* $\pi_1(\lambda) = 1$ *and* $\pi_2(\lambda) = \lambda$. *In this case the distribution of* D_τ *only depends on the strength of interaction between different firms*.

Some comments about the previous result are in order. The number of firms initially hit by the shock is approximately Poisson distributed. This captures the idea that a shock initially affects only a small number of firms. The interaction between individual companies generates a propagation mechanism for shocks to spread through the whole economy. Each initial default triggers additional downgrades through a chain reaction. The number of additional downgrades resulting from an "initial default" follows a *Borel-Tanner distribution*. This explains (10). In view of (13), the law of D_τ is given as a mixture of distributions with slowly decaying tails. As such it has considerably fatter tails than the standard normal distribution. The stronger the interaction between individual firms, the fatter the tails. For the limiting case $\nu = 1$, we obtain the Pareto tail

$$\mathbb{P}[D_\tau = k] = Ck^{-(3/2)} \quad \text{as } k \to \infty.$$

In this case D_τ has infinite variance. In Section 3 we study the impact of credit contagion on aggregate portfolio losses under the simplifying assumption that $\Pi = 1$. This can be viewed as a situation where the downgrade cascade is triggered by the default of some major enterprize. In such a situation the total number of defaults follows a Borel-Tanner-distribution, and the distribution of aggregate portfolio losses is heavy-tailed if the interaction between different firms is too strong.

Remark 2.8.

(i) The number of firms initially hit by the shock is increasing both in the shock size Y and the strength of interaction, α. The number of subsequent defaults in periods $t = 1, 2, \ldots$ also increases with the strength of interactions, but depends on Y only through D_0. The quantities

D_1, D_2, ... are increasing in the strength of interaction, but decrease with λ. The smaller λ, the smaller the impact of an individual default on the average credit rating and the fewer downgrades triggered per initial default.

(ii) The proof of Theorem 2.7 is based on the approximation of the binomial distribution with small success probability by a Poisson distribution. Convergence usually takes place at an exponential rate. Since we do not restrict ourselves to the tail of the distribution of D_τ but obtain a representation of the entire distribution, the number of firms does not need to be "too large" for our approximation to be accurate.

The grid size λ may be viewed as a simple measure for the loss a financial institution associates with a position in case of a downgrade. Aggregate portfolio losses are then given by

$$D_\lambda^N := \sum_{i=1}^{N} \left(x_{\tau^N}^{N,i} - x_0^{N,i} \right).$$

In view of our Theorem 2.7 the random variable D_λ^N converges in distribution to λD and the mean and the variance of λD are given by

$$\lambda \frac{Y\pi_1(\lambda)}{1-\pi_2(\lambda)} \quad \text{and} \quad \lambda^2 \frac{Y\pi_1(\lambda)}{(1-\pi_2(\lambda))^3},$$

respectively. By Theorem 2.7 (ii) the random variables λD converge in distribution to some multiple of Y as $\lambda \to 0$. The limiting case $\lambda \to 0$ may be viewed as a frictionless counterpart to our model with lumpy adjustments. In a model with continuous adjustments, there is no uncertainty about the financial distress shocks cause to credit portfolios. The risk is entirely determined by the shock size. If ratings are discrete, the feedback effect generates an intrinsic risk to credit portfolios.

2.3. *Credit contagion in a model of local interactions*

In this section, we extend our interactive model of credit contagion by introducing an additional local component into the interaction. Models of local interactions have been extensively analyzed in the microeconomic literature on non-market interactions by, for example, Bisin et al. (in press), Durlauf (1993), Glaeser et al. (1996) or Horst and Scheinkman (in press). These models are capable of displaying large multipliers that transform small changes in exogenous variables into large changes in endogenous quantities. The existence of "social multipliers" provides a possible explanation for the observation of large fluctuations in aggregate behavior in the absence of corresponding changes in economic fundamentals.

Our goal is to clarify further the role of mean-field interactions as a transition channel for infectious spreads of financial distress. We show that a purely local interaction cannot generate a heavy tailed distribution of defaults. In this sense credit contagion rests upon global interaction. For mathematical reasons we need to restrict ourselves to "binary choice" models where the random variables θ^i are uniformly distributed on the unit interval (0, 1), and where $\lambda = 1$. Associated with each company $i \in \{1, 2, \ldots, N\}$ is the set $N(i) := \{i+1\}$ of its business partner where we apply modulo-N-arithmetic.[6] We fix positive interaction parameters α_1 and α_2, specifying the strength

[6] The assumption of a one-sided interaction is made merely for analytical convenience. Our results carry over to the case $N(i) := \{i-1, i+1\}$. The analysis, however, becomes much more cumbersome, and we do not really gain additional insight from this more general interaction pattern.

of the *global* and the *local* component in the interaction. For binary choice models the symmetric configuration $x^{N,i} \equiv 0$ satisfies the equilibrium condition

$$x^{N,i} = \alpha_1 \bar{x}^N + \alpha_2 x^{N,i+1} + \theta^i - (\alpha_1 \bar{x}^N + \alpha_2 x^{N,i} + \theta^i) \bmod \lambda. \tag{15}$$

In a large economy the impact of the default of a firm's neighbor on its own rating is much stronger than the impact of any other corporation. The special case $\alpha_2 = 0$ corresponds to the mean-field interaction studied in the previous section.

2.3.1. *Default cascades in a model of local and global interactions*

In a model with locally and globally interacting companies a downgrade of an individual firm is now felt both through the deterioration of the average rating throughout the entire economy and through its impact on the credit quality of its immediate business partners. In the sequel it will be convenient to distinguish between local and global defaults.

Definition 2.9. We say that company i defaults locally if it gets downgraded because of the insolvency of its business partner, firm $i+1$. The corporation defaults globally if the default is due to a deterioration of the overall business climate described by the average rating throughout the whole economy.

In the presence of local interactions there are various possibilities to specify the dynamics of the contagious spread of financial distress. We suggest describing the chain reaction of bankruptcies by an alternating sequence of global and local defaults.[7] The global defaults are described by means of a stochastic process $\{(x_t^N, s_t^N)\}_{t \in \mathbb{N}}$ with initial values

$$x_0^N = (x_0^{N,i})_{i=1}^N \quad \text{where } x^{N,i} = 0, \quad \text{and} \quad s_0^N = (s_0^{N,i})_{i=1}^N \quad \text{where } s_0^{N,i} = \theta^i. \tag{16}$$

As in the mean-field case the cascade is again triggered by an external shock, and the first companies to default are those who are not able to absorb additional financial distress:

$$x_1^{N,i} = \begin{cases} 1 & \text{if } s_0^{N,i} + \varepsilon^i > 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad s_1^{N,i} = s_0^{N,i} + \varepsilon^i - x_1^{N,i}. \tag{17}$$

Due to the local dependencies in credit ratings a default of an individual company has a negative impact on the financial health of its business partner. A global default may trigger local insolvencies. The chain reaction of bankruptcies resulting from global insolvencies of business partners is now specified in an inductive manner. Specifically, we suppose that the pair (x_r^N, s_r^N) is already defined and denote by $H_r^N := \{i : x_r^N = 1\}$ the set of all firms that have defaulted by and up to period $r \in \mathbb{N}$. Next, we define a stochastic process $\{(y_t^N, q_t^N)\}_{t \in \mathbb{N}}$ starting in (x_r^N, s_r^N) by $y_t^{N,i} \equiv q_t^{N,i} \equiv 1$ for $i \in H_r^N$. Thus, there is no recovery of defaulted firms. A firm $i \notin H_r^N$ defaults because of an insolvent business partner if it is unable to absorb the additional financial distress. Thus, for $i \notin H_r^N$ we put

$$y_t^{N,i} = \begin{cases} 1 & \text{if } q_0^{N,i} + \alpha_2 y_{t-1}^{N,i+1} \geq 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad q_t^{N,i} = \begin{cases} 1 & \text{if } y_{t-1}^{N,i} = 1 \\ q_0^{N,i} + \alpha_2 y_{t-1}^{N,i+1} & \text{otherwise.} \end{cases}$$

[7] This way of modelling default cascades guarantees that our techniques carry over to interaction patterns that are more general than the simple one-sided interaction used for analytical convenience.

This way, the limits $y_\infty^{N,i}:=\lim_{t\to\infty} y_t^{N,i}$ and $q_\infty^{N,i}:=\lim_{t\to\infty} q_t^{N,i}$ exist, and the set of firms that defaulted due to a default of a business partner in period r is thus given by $\bar{H}_r^N:=\{1 \le i \le N : i \notin H_r^N \text{ and } y_\infty^{N,i} = 1\}$. The local defaults further deteriorate the overall economic conditions, and we define (x_{r+1}^N, s_{r+1}^N) by $x_r^{N,i} = s_t^{N,1} = 1$ if $i \in H_r^N \cup \bar{H}_r^N$ and

$$x_{r+1}^{N,i} = \begin{cases} 1 & \text{if } q_\infty^{N,i} + \alpha_1 \frac{|\bar{H}|_r^N}{N} \ge 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad s_{r+1}^{N,i} = \begin{cases} q_\infty^{N,i} + \alpha_1 \frac{|\bar{H}|_r^N}{N} & \text{if } x_{r+1}^{N,i} = 0 \\ 1 & \text{otherwise} \end{cases}$$

for those companies i that have not defaulted by period r. We denote the number of defaults "in period r" by $\hat{D}_r^N$, and

$$\hat{\tau}^N := \inf\{t : \bar{x}_t^N = \bar{x}_{t+1}^N\}$$

is the time of extinction of the downgrade process $\{\hat{D}_t^N\}_{t\in\mathbb{N}}$. The total number of defaults triggered by an external shock in a model with locally and globally interacting companies is then given by

$$\hat{D}_{\hat{\tau}^N}^N := \sum_{t=0}^{\hat{\tau}^N} \hat{D}_t^N = \sum_{i=1}^{N} x_{\hat{\tau}^N}^{N,i}.$$

2.3.2. The distribution of downgrades in the presence of local interactions

By analogy to the case of a simple mean-field interaction the sequence $\{\hat{D}_t^N\}_{t\in\mathbb{N}}$ can be approximated in law by a branching process. If we limit the strength of interactions in our economy, then the probability of a large number of defaults can be obtained from a seminal paper by Otter (1949). An explicit representation for the distribution of $D_{\tau^N}^N$, however, is not available. The proof of the following result is given in Section 4.4.

Theorem 2.10. *Suppose that $\lambda = 1$, that the random variables θ^i are uniformly distributed on the unit interval, and that the following weak interaction condition holds*:

$$\frac{\alpha_1}{1-\alpha_2} \le 1.$$

(i) *The sequence of downgrades can be approximated in law by a branching process $\{\hat{D}_t\}_{t\in\mathbb{N}}$*:

$$\hat{D}_t \stackrel{D}{=} \hat{D}_{t,1} + \hat{D}_{t,2} + \cdots + \hat{D}_{t,\hat{D}_{t-1}}$$

where the independent random variables $\hat{D}_{t,i}$ follows a compound Poisson distribution.

(ii) *The sequence $\{\hat{D}_{\hat{\tau}^N}^N\}_{N\in\mathbb{N}}$ converges in distribution to an almost surely finite random variable $\hat{D}$. If, in addition, the equation*

$$\alpha^* = \frac{1}{\alpha_1} \frac{(1-\alpha_2\alpha^*)^2}{1-\alpha_2} \quad \text{has a solution } \alpha^* \in (0, \alpha_2^{-1}),$$

then

$$\mathbb{P}[\hat{D} = k | \hat{D}_0 = l] = C r^{-k-\frac{1}{2}} k^{-\frac{3}{2}} \quad \text{as } k \to \infty \quad \text{where } r := \alpha^* e^{-\alpha_1((\alpha^*-1)/(1-\alpha_2\alpha^*))}.$$

In the case of a simple mean field interaction, we obtained an exponentially truncated power law of the distribution of the total number of downgrades. In the presence of an additional local interaction we derived a qualitatively similar result. In both cases such a distribution emerges naturally from the approximation of the cascade process by branching processes. The rate of decay of the tail of the distribution of the total number of defaults is inversely related to the strength of interaction as the following example illustrates.

Example 2.11.

(i) For $\alpha_1 = \alpha_2 = 1/5$ we obtain $\alpha^* \approx 2.1$ and $r \approx 2.2$. In this case the interaction is very weak, so the tail of the distribution of the total number of downgrades decays rather rapidly.
(ii) For $\alpha_1 = 2/3$ and $\alpha_2 = 1/4$ we have $\alpha^* = 1.072$ and $r \approx 1.14$. The interaction is much stronger than in the previous example, and the tails decay much slower.

3. Large portfolio losses

Consider a financial institution holding a portfolio of financial positions whose market value is subject to the credit rating of the issuer or counterparty. Such positions include loans, bonds, and other debt instruments as well as derivatives written by default-prone business partners. Due to adverse changes in the credit qualities of counterparties, for instance credit rating downgrades, the market value of the corresponding portfolio can be severely reduced. At times where the portfolio is revalued, position $i \in \{1, 2, \ldots, N\}$ experiences a random loss of C_i if its credit quality has been downgraded in reaction to an external shock. An investment bank holding a portfolio of positions with firms $i \in \{1, 2, \ldots, N\}$ then suffers from a loss

$$L_N := \sum_{i=1}^{D_{\tau}N} C_i.$$

In this section, we focus on the effects of downgrade cascades on aggregate portfolio losses. Following Dembo et al. we assume that individual exposures are independent and identically distributed according to some exogenously specified distribution. The actual number of downgrades or defaults, however, is specified by an endogenously generated branching process. In view of our Theorem 2.7, it is straightforward to show that the distribution of aggregate losses follows a compound distribution as $N \to \infty$.

Proposition 3.1. *For $N \to \infty$, the random variables L_N converge in distribution to the random variable*

$$L = \sum_{i=1}^{D_{\tau}} C_i. \tag{18}$$

Compound random sums of the form (18) are of major interest in insurance theory; see, for example, Embrechts et al. (1997) for a detailed discussion of random sums in the context of ruin theory. In an insurance context D_{τ} typically denotes the number of claims over a certain period of time, and $C_1, \ldots, C_{D_{\tau}}$ are the claim-sizes. In the context of credit contagion the quantity L describes the loss suffered by a bank from a series of credit rating downgrades. A risk manager aiming at evaluating the possibilities of large portfolio losses induced by credit quality deterioration is then interested in the tail structure of the random variable L. For this it is essential to locate the sources of large losses. Large aggregate losses may result from (i) a rather small number of large individual losses or (ii) a large number of average losses. The latter case corresponds to a situation where D_τ is large whereas the quantities $C_1, \ldots, C_{D_\tau}$ take values close to their expected values. This case is of particular interest for financial institutions because an active risk management can somewhat control the distribution of individual losses, but typically one cannot control the degree of interrelationships between firms. Hence risk management cannot affect the distribution of the random variable D_τ. In this sense, the possibility of cascades of credit rating downgrades poses a considerable *intrinsic* risk to credit portfolios that cannot be "diversified away".

In order to analyze the distribution of aggregate portfolio losses it will be convenient to classify the random variables C_i according to the tail structure of their distribution.

Definition 3.2. Let $(C_i)_{i \in \mathbb{N}}$ be a sequence of non-negative iid random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with distribution function F.

(i) We say that the random variables C_i have an exponential tail if the moment generating function

$$M(s) := E(e^{sC_i})$$

exists for small enough s.

(ii) *Let* $\bar{F}(x) = 1 - F(x)\,(x \geq 0)$ be the tail of the distribution function F and denote by

$$\bar{F}^{*n}(x) := 1 - F^{*n}(x) = \mathbb{P}[C_1 + \cdots + C_n \geq x]$$

the tail of the n-fold convolution of F. Following Embrechts et al. we say that F has a sub-exponential tail if

$$\lim_{x \to \infty} \frac{\bar{F}^{*n}(x)}{\bar{F}(x)} = n \quad \text{for some (all)} \quad n \geq 2. \tag{19}$$

As shown by Embrechts and Goldie (1982), the random variables C_i have sub-exponential tails if and only if

$$\mathbb{P}[C_1 + \ldots + C_n \geq x] = \mathbb{P}[\max\{C_1, \ldots, C_n\}] \quad \text{as } x \to \infty.$$

The sum of independent and identically distributed sub-exponential random variables is likely to be large if and only if the maximum is likely to be large.

Example 3.3.

(i) Suppose that there exist $\alpha \in (0, 2)$ and $M > 0$ such that

$$\mathbb{P}[C > c] = Mc^{-\alpha} \quad \text{as } c \to \infty.$$

Then C_i has a sub-exponential distribution; we refer the interested reader to the textbook by Embrechts et al. for details.

(ii) If individual losses follow an exponential distribution with parameter β, then

$$M(s) = \frac{\beta}{\beta - s} \quad \text{for } s < \beta.$$

In this case, the random variables C_i have an exponential moment because $\mathbb{P}[C_i \geq c] \leq Me^{-rc}$ implies $E[e^{sC}] < \infty$ for $s < r$.

Exponential bounds on the tail probabilities of compound distributions including the classical Lundberg inequality for the ruin probability of an insurance company have been established by various authors; see, for example, Lin (1996), Willmot and Lin (1994), or Willmot (1997). Upper bounds for the tail probabilities of the random variable L are of particular interest for supervising institutions because they may be used to determine the capital requirements imposed on an investment bank holding a large portfolio of defaultable securities.

In order to simplify the analysis we restrict ourselves to a mean field interaction where the shock initially affects only one firm. More precisely, we assume that $\Pi = 1$, so the random variable D_τ has a Borel-Tanner distribution $(p_k)_{k \in \mathbb{N}}$. We denote the associated tail distribution by

$$q_k := \sum_{l \geq k} p_l \quad (k \in \mathbb{N}).$$

3.1. Large individual losses and the tail structure of portfolio losses

Let us first consider the situation where individual loss distributions are fat tailed. Heavy tailed individual loss distributions translate into a heavy tailed distribution of aggregate losses. In such a situation, large portfolio losses result from large individual losses. More precisely, we have the following result about the tail structure of compound sums of heavy tailed random variables. Its proof is an immediate consequence of Theorem A3.20 in Embrechts et al. and the explicit representation of the Borel-Tanner distribution given in, for example, Eq. (6) of the internet supplement.

Theorem 3.4. *Suppose that the random variables C_i have sub-exponential tails. In the sub-critical case $\nu < 1$ where, on average, each downgrade triggers less than one default,*

$$\mathbb{P}[L > x] = \mathbb{P}[\max\{C_1, C_2, \ldots, C_{D_\tau}\} > x] \quad \text{as } x \to \infty$$

and

$$\mathbb{P}[L > x] = E[D_\tau] \cdot \mathbb{P}[C_1 > x] \quad \text{as } x \to \infty.$$

An active risk manager can control the risk arising from large individual losses by eliminating positions from its portfolio that have heavy tailed loss distributions. In this sense a situation with heavy tailed individual losses may serve as a benchmark model, but for practical purposes, it seems more desirable to have bounds for the tails of compound sums where the compound distribution is light tailed.

3.2. Small individual losses and the tail structure of portfolio losses

Although risk management can control the distribution of individual losses, it typically cannot affect the degree of dependencies between different firms. Losses resulting from cascade phenomena pose a considerable *intrinsic* risk to a financial institution holding large portfolios of defaultable securities.

3.2.1. The subcritical case

In the subcritical case $\nu < 1$ a downgrade triggers on average less than one additional default. In this case the sequence $\{q_k\}_{k \in \mathbb{N}}$ converges to zero at an exponential rate. Indeed, by Theorem 2.7 we have

$$\frac{p_{k+1}}{p_k} = \phi := \frac{e^{\nu-1}}{\nu} \quad \text{as } k \to \infty.$$

This yields

$$q_k = \sum_{l \geq k} p_l \leq (1+\varepsilon)\frac{p_k}{1-\phi}, \quad \text{so} \quad \frac{q_{k+1}}{q_k} = \frac{p_{k+1}}{p_k} = \phi \quad \text{as } k \to \infty. \tag{20}$$

The tail structure of the aggregate loss distribution is now specified in terms of the tails of C_i and the average number of downgrades triggered by a single default. For the random variable L to have an exponential tail, the probability of large individual losses has to decrease at a sufficiently fast rate. This rate is inversely related to the strength of interaction between companies.

Theorem 3.5. *Suppose that the random variables C_i have a continuous distribution function F and that there exists a constant $r > 0$ such that*

$$\phi = \int_0^\infty \mathrm{e}^{rx}\,\mathrm{d}F(x) < \infty. \tag{21}$$

Then there exists $\delta < 1$ such that

$$\delta\phi\mathrm{e}^{-rx} \leq \mathbb{P}[L \geq x] \leq \phi\mathrm{e}^{-rx} \quad \text{as } x \to \infty. \tag{22}$$

Thus, for large $x > 0$, the quantity $\mathbb{P}[L \geq x]$ is approximately proportional to e^{-rx} and L has an exponential tail distribution.

Proof. In order to establish the upper bound in (22) we modify a martingale argument given in Gerber (1994); see also Willmot and Lin. For a fixed $\varepsilon > 0$ we can choose $K \in \mathbb{N}$ such that

$$\frac{q_{k+1}}{q_k} \leq \phi(1+\varepsilon) < 1 \quad \text{for all } k \geq K,$$

so continuity of the distribution function along with (22) yields

$$\phi(1+\varepsilon)^{-1} = \int_0^\infty \mathrm{e}^{\hat{r}x}\,\mathrm{d}F(x)$$

for some $\hat{r} \leq r - \varepsilon$. We put

$$\hat{L} := L - \sum_{i=1}^{K \wedge D_\tau} C_i \quad \text{and} \quad \hat{L}_k := C_{K+1} + \cdots + C_{K+k},$$

and introduce the random variables $\hat{Y}_k$ and $\hat{X}_k$ by

$$\hat{Y}_k := \begin{cases} \mathrm{e}^{\hat{r}\hat{L}_k} & \text{if } D_\tau \geq k \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \hat{X}_k := \begin{cases} \mathrm{e}^{\hat{r}C_k} & \text{if } D_\tau \geq k \\ 0 & \text{otherwise} \end{cases}$$

respectively. This yields the recursive relation $\hat{Y}_k = \hat{X}_k \hat{Y}_{k-1}$ and

$$E[\hat{X}_{k+1} | D_\tau \geq k] = E\mathrm{e}^{\hat{r}C_k}\frac{\mathbb{P}[D_\tau \geq k+1]}{\mathbb{P}[D_\tau \geq k]} \leq 1.$$

Thus, the sequence $\{\hat{Y}_k\}_{k\in\mathbb{N}}$ is a supermartingale. Introducing the stopping time

$$\sigma := \inf\{k : \hat{L}_k \geq x \text{ or } D_\tau < k\}$$

we obtain

$$\hat{Y}_1 \geq E[\hat{Y}_\sigma | D_\tau, C_1] \quad \text{or} \quad e^{\hat{r}C_1} \geq E[e^{\hat{r}\hat{L}_\sigma} 1_{\{\hat{L}\geq x\}} | D_\tau, C_1] \geq e^{\hat{r}x}\mathbb{P}[\hat{L} \geq x | D_\tau, C_1].$$

This yields

$$\mathbb{P}[\hat{L} \geq x] = E[\mathbb{P}[\hat{L} \geq x] | D_\tau, C_1] \leq E[e^{\hat{r}C_1} e^{-\hat{r}x}] \leq \frac{\phi}{(1+\varepsilon)} e^{-\hat{r}x}.$$

Hence, the upper bound in (22) follows from $\mathbb{P}[\hat{L} \geq x] = \mathbb{P}[L \geq x]$ as $x \to \infty$ because $\varepsilon > 0$ is arbitrary. The lower bound follows from straightforward modifications of arguments given in the proof of Corollary 2.3 in Lin. □

The previous theorem shows that the tail structure of the distribution of individual losses along with the strength of interactions between different companies completely specifies the tails of the distribution of portfolio losses. Theorem 3.5 also confirms our intuition that the tails of the aggregate loss distribution is the fatter, the slower the decay of the tail of the distribution of individual losses. We further illustrate this by means of the following simple example.

Example 3.6. Suppose that the random variables C_i are distributed exponentially with parameter $\beta > 0$. Thus, C_i is absolutely continuous with density

$$f_\beta(x) := \frac{1}{\beta} e^{-\beta x} \text{ and distribution function } F(x) = e^{-\beta x} \ (x \geq 0).$$

In such a situation (21) translates into

$$\phi = \frac{1}{\beta}\int_0^\infty e^{-(\beta - r)x}\, dx = \frac{\beta - r}{\beta}, \quad \text{that is, into } r = \beta(1-\phi).$$

The slower the decay of the tail of the distribution of individual loss sizes (i.e., the smaller β), the fatter the tails of the law of aggregate portfolio losses.

3.2.2. The critical case

In the critical case $\nu = 1$ each downgrade triggers, on average, another downgrade, and the distribution of the total number of downgrades is fat tailed; see Corollary 1.5 in the internet supplement. Under mild technical assumptions on the distribution of the individual losses one can then show that the distribution of aggregate losses has sub-exponential tails and that large portfolio losses typically result from an unusually large number of individual defaults. More precisely we have the following result. Its proof follows immediately from results given in Schmidli (1999) along with the representation (7) in the internet supplement.

Theorem 3.7. *Suppose that the distribution function F of the random variables C_i satisfies*

$$\liminf_{x\to\infty} \frac{\bar{F}^{*(n+1)}(x)}{\bar{F}^{*n}(x)} \geq a \quad \text{for all } n \in \mathbb{N} \text{ and some } a > 1. \tag{23}$$

If $\nu = 1$, then the random variable L has a sub-exponential distribution. More precisely

$$\mathbb{P}[L \geq x] = \mathbb{P}\left[D_t \geq \frac{x}{EC_i}\right] \quad \text{as } x \to \infty.$$

All light tailed distribution functions of practical interest satisfy (23). For instance, if F has a Gamma tail (i.e., if $\bar{F}(x) = cx^{\gamma-1}\,\mathrm{e}^{-ax}$ for some $\gamma \geq 0$ as $x \to \infty$), then F satisfies (23), due to Lemma 1 in Schmidli (1999). However, even if the distribution of individual losses is light tailed, the portfolio loss distribution can be heavy tailed. If the dependencies between individual credit ratings are too strong, then the distribution of D_τ is fat tailed. This translates into a heavy-tailed distribution of aggregate losses.

4. Approximation of cascade processes in the mean field case

In this section, we prove the approximation result for the process of credit rating downgrades for the case of a mean field interaction. Our proof extends arguments given in Nirei. In a first step we prove that average ratings converge almost surely to some deterministic limit as $N \to \infty$. In a second step we show that the sequence $\{D_t^N\}_{t \in \mathbb{N}}$ can be approximated in law by a simple Galton-Watson process $\{D_t\}_{t \in \mathbb{N}}$. Using general results about the long run behavior of branching processes we finally show that a shock triggers only a finite number of downgrades.

4.1. Ergodicity of equilibria

The following three sections prepare the proof of the approximation result stated in Theorem 2.7. In a first step we show that the average rating converges almost surely to a unique deterministic limit. From this we will deduce that individual ratings will be conditionally independent in the limit of an infinite economy given the average rating.

Lemma 4.1. *For any $N \in \mathbb{N}$, let $x^N(\theta) = \{x^{N,i}(\theta)\}_{i=1}^N \in \Lambda^N$ be a configuration of credit ratings that satisfies (3). The associated sequence of average ratings $\{\bar{x}^N(\theta)\}_{N \in \mathbb{N}}$ converges almost surely to some constant $\bar{x}$. More precisely,*

$$\lim_{N\to\infty} \bar{x}^N(\theta) = \frac{\bar{\theta} - \lambda}{2 - 2\alpha} \qquad \mathbb{P}\text{-a.s.}$$

Proof. Let $\{x^N(\theta)\}_{N \in \mathbb{N}}$ be a sequence of configurations of equilibrium credit ratings. Individual ratings take values in a finite set, so the sequences

$$\{\bar{x}^N(\theta)\}_{N \in \mathbb{N}} \quad \text{and} \quad \left\{ \frac{1}{N}\sum_{i=1}^{N}(\alpha\bar{x}^N + \theta^i) \bmod \lambda \right\}_{N \in \mathbb{N}}$$

are bounded. Hence there is a subsequence $(N_k)_{k \in \mathbb{N}}$ possibly depending on θ such that

$$\bar{x} := \lim_{k\to\infty} \bar{x}^{N_k}(\theta) \quad \text{and} \quad \lim_{k\to\infty} \frac{1}{N_k}\sum_{i=1}^{N_k}(\alpha\bar{x}^{N_k} + \theta^i) \bmod \lambda \tag{24}$$

exist. In view of (3) and because the random variable θ^i are independent and identically distributed,

$$\begin{aligned}
\bar{x} &= \frac{1}{1-\alpha}\left(\lim_{k\to\infty}\frac{1}{N_k}\sum_{i=1}^{N_k}\theta^i - \lim_{k\to\infty}\frac{1}{N_k}\sum_{i=1}^{N_k}(\alpha\bar{x}^{N_k} + \theta^i) \bmod \lambda \right) \\
&= \frac{1}{1-\alpha}\left(\frac{\bar{\theta}}{2} - \lim_{k\to\infty}\frac{1}{N_k}\sum_{i=1}^{N_k}(\alpha\bar{x}^{N_k} + \theta^i) \bmod \lambda \right).
\end{aligned}$$

Thus, it suffices to prove that

$$\lim_{k\to\infty}\frac{1}{N_k}\sum_{i=1}^{N_k}(\alpha\bar{x}^{N_k}+\theta^i)=\frac{\lambda}{2}\qquad \mathbb{P}\text{-a.s.}$$

To this end, observe first that the bounded random variables $(\bar{x}+\theta^i)\text{mod}\,\lambda$ are independent and uniformly distributed on $(0,\lambda)$ due to the modulo-λ-arithmetic and because the random variables θ^i are iid. Thus, the law of large numbers yields

$$\lim_{k\to\infty}\frac{1}{N_k}\sum_{i=1}^{N_k}(\alpha\bar{x}+\theta^i)\text{mod}\,\lambda=\frac{\lambda}{2}\qquad \mathbb{P}\text{-a.s.}$$

It is therefore enough to show that

$$\lim_{k\to\mathbb{R}}\frac{1}{N_k}\sum_{i=1}^{N_k}\{(\alpha\bar{x}+\theta^i)\text{mod}\,\lambda-(\alpha\bar{x}^{N_k}+\theta^i)\text{mod}\,\lambda\}=0\qquad \mathbb{P}\text{-a.s.}$$

Since x^N is an equilibrium configuration,

$$(\alpha\bar{x}^{N_k}+\theta^i)\text{mod}\,\lambda=(\alpha\bar{x}+\theta^i)\text{mod}\,\lambda$$

whenever there exists $n^i\in\mathbb{N}$ such that

$$(n^i-1)\lambda\le\alpha\bar{x}^{N_k}+\theta^i<n^i\lambda\quad\text{and}\quad(n^i-1)\lambda\le\alpha\bar{x}+\theta^i<n^i\lambda.\tag{25}$$

We may with no loss of generality assume $\bar{x}^{N_k}\le\bar{x}$. In view of (24), for each $\varepsilon>0$ there exists K such that condition (25) is violated for $k\ge K$ if and only if

$$\theta^i\in((n^i-1)\lambda-\alpha\bar{x},(n^i-1)\lambda-\alpha\bar{x}^{N_k})\cup(n^i\lambda-\alpha\bar{x},n^i\lambda-\alpha\bar{x}^{N_k})\subset((n^i-1)\lambda-\alpha\bar{x},$$
$$(n^i-1)\lambda-\alpha\bar{x}\varepsilon)\cup(n^i\lambda-\alpha\bar{x},n^i\lambda-\alpha\bar{x}+\varepsilon)=:B_\varepsilon^i.$$

Because the random variables $x^{N,i}$ takes values in a finite set, the law of large numbers for independent and identically distributed random variables yields

$$\lim_{k\to\infty}\frac{1}{N_k}\sum_{i=1}^{N_k}1_{\{(\alpha\bar{x}^{N_k}+\theta^i)\text{mod}\,\lambda\neq(\alpha\bar{x}+\theta^i)\text{mod}\,\lambda\}}(i)\le\lim_{k\to\infty}\frac{1}{N_k}\sum_{i=1}^{N_k}1_{B_\varepsilon^1}(\theta^i)\le\varepsilon\qquad\mathbb{P}\text{-a.s.}\tag{26}$$

This proves our assertion. □

4.2. A benchmark model with infinitely many firms

The previous lemma showed that each sequence of equilibria $\{x^N(\theta)\}_{N\in\mathbb{N}}$ converges almost surely to a configuration $\hat{x}=(\hat{x}^i)_{i\in\mathbb{N}}$ defined by

$$\hat{x}^i:=\alpha\bar{x}+\theta^i-(\alpha\bar{x}+\theta^i)\,\text{mod}\,\lambda.\tag{27}$$

The convergence takes place both locally (i.e., on the level of individual ratings) and globally (that is, on the level of average ratings) because

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \hat{x}^i = \bar{x} \qquad \mathbb{P}\text{-a.s.} \tag{28}$$

In a finite economy the buffer variables $s^{N,i}$ are conditionally independent given the actual configuration of credit ratings, but typically not identically distributed as shown by Example 2.4. The same applies to our benchmark model with infinitely many firms as illustrated by the following example.

Example 4.2. Suppose that θ^i is uniformly distributed on (0, 1), that $\lambda = 1/2$, and that $\alpha = 3/4$. Then $\bar{x} = 1$, and the equilibrium configuration $\hat{x} = (\hat{x}^i)_{i\in\mathbb{N}}$ takes the form

$$\hat{x}^i = \frac{3}{4} + \theta^i - \left(\frac{3}{4} + \theta^i\right) \mod \frac{1}{2} = \begin{cases} \frac{1}{2} & \text{if } \theta^i \in \left(0, \frac{1}{4}\right) \\ 1 & \text{if } \theta^i \in \left(\frac{1}{4}, \frac{3}{4}\right) \\ \frac{3}{2} & \text{if } \theta^i \in \left(\frac{3}{4}, 1\right) \end{cases} \quad \text{and} \quad \hat{s}^i \in \begin{cases} \left(\frac{1}{4}, \frac{1}{2}\right) & \text{if } \hat{x}^i = \frac{1}{2} \\ \left(0, \frac{1}{4}\right) & \text{if } \hat{x}^i = \frac{3}{2} \\ \left(0, \frac{1}{2}\right) & \text{otherwise.} \end{cases}$$

Under Assumption 2.1 the buffer variables $\hat{s}^i$ are conditionally uniformly distributed on (0, 1/4), (1/4, 1/2) or (0, 1/2), depending on the observable value of $\hat{x}^i$. Since the random variables θ^i are independent and uniformly distributed on the unit interval, we see that 1/4th of the buffer variables take values in (0, 1/4) and (1/4, 1/2), respectively, and 1/2th of the random variables s^i are conditionally uniformly distributed on the interval (0, 1/2).

In the context of the previous example we were able to classify firms into different groups such that, within these groups, the respective buffer variables are conditionally independent and identically distributed. In order to formulate such a classification scheme in a general framework we put

$$\hat{x}_{\min} := \alpha\bar{x} - (\alpha\bar{x}) \bmod \lambda \quad \text{and} \quad \hat{x}_{\max} := \alpha\bar{x} + \bar{\theta} - (\alpha\bar{x} + \bar{\theta}) \bmod \lambda$$

so that $\hat{x}^i \in [\hat{x}_{\min}, \hat{x}_{\max}]$. Now we introduce the following classes of firms:

$$M_1 := \{i : \hat{x}^i = \hat{x}_{\min}\}, \quad M_2 := \{i : \hat{x}^i = \hat{x}_{\max}\} \quad \text{and} \quad M_0 := \mathbb{N}/(M_1 \cup M_2)$$

where we use the convention that $M_0 = \mathbb{N}$ if $M_0 = M_1 = M_2$. The set M_1 may be viewed as the class of firms with the best possible rating whereas M_2 represents the companies with the worst credit quality. For a given configuration x^N, the sets M_0^N, M_1^N and M_2^N are defined accordingly.

Example 4.3.

(i) Let us return to the situation studied in Example 2.4. In this case $M_1 = \{i : \hat{x}^i = 1/2\}$ and $M_2 = \{i : \hat{x}^i = 2\}$.
(ii) In the context of the binary choice model analyzed in Example 2.3 we have $\hat{x}^i = 0$ for all $i \in \mathbb{N}$, and by convention $M_0 = \mathbb{N}$.

The following lemma shows that the firms' sensitivity parameters are conditionally identically distributed within the respective groups.

Lemma 4.4. *For almost all equilibrium configurations of credit ratings* $\hat{x}(\theta) = \{\hat{x}^i(\theta)\}_{i \in \mathbb{N}}$ *and* $x^N(\theta) = \{x^{N,i}(\theta)\}_{i=1}^N$ *the following holds*:

(i) *For* $j = 0, 1, 2$, *the random variables* $\{\hat{s}^i\}_{i \in M_j}$ *and* $\{s^{N,i}\}_{i \in M_j^N}$ *are conditionally independent and identically distributed given the respective equilibrium configurations of credit ratings.*
(ii) *For* $j = 0, 1, 2$, *the following limits exist almost surely*:

$$r_j := \lim_{N \to \infty} \frac{|M_j|}{N} = \lim_{N \to \infty} \frac{|M_j^N|}{N}. \tag{29}$$

Moreover, $r_0 = 1$ *if* $\alpha\bar{x}$ *is a multiple of* λ.

Proof.

(i) We prove the assertion for the benchmark case with infinitely many firms; the case of finite economies follows accordingly. For $i \in M_0$, the random variables $\hat{s}^i$ are conditionally uniformly distributed on the interval $[0, \lambda]$ because of Assumption 2.1 and the modulo-λ-arithmetic. If $i \in M_1$, then

$$\hat{x}^i = \alpha\bar{x} - (\alpha\bar{x}) \bmod \lambda = \alpha\bar{x} + \theta^i - (\alpha\bar{x} + \theta^i) \bmod \lambda,$$

so

$$\theta^i \in (0, \hat{\theta}) \quad \text{where } \hat{\theta} := \max\{\theta \in (0, \lambda) : -(\alpha\bar{x}) \bmod \lambda = \theta^i - (\alpha\bar{x} + \theta^i) \bmod \lambda\}.$$

The random variable θ^i is conditionally uniformly distributed on $(0, \hat{\theta})$ given $\hat{x}$. This implies that the random variables $\{\hat{s}^i\}_{i \in M_1}$ are conditionally identically distributed on the interval $((\alpha\bar{x}) \bmod \lambda, \lambda)$. By analogy, the random variables $\{\hat{s}^i\}_{i \in M_2}$ are conditionally uniformly distributed on $(0, (\alpha\bar{x} + \bar{\theta}) \bmod \lambda)$.

(ii) The first limit in (29) exists almost surely because the random variables $\hat{x}^i$ are independent and identically distributed. The ergodicity result for sequences of finite economies follows from Lemma 4.1 and (26) because $\mathbb{P}$-a.s.

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N |x^{N,i} - \hat{x}^i| \le \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \{|\alpha\bar{x}^N - \alpha\bar{x}| + |(\alpha\bar{x}^N + \theta^i) \bmod \lambda - (\alpha\bar{x} + \theta^i) \bmod \lambda|\} = 0.$$

If $\alpha\bar{x}$ is a multiple of λ, then $\hat{x}^i - \alpha\bar{x}$ is a multiple of λ. In this case Assumption 2.1 guarantees that all the random variables $\hat{s}^i$ are uniformly distributed on $(0, \lambda)$. □

4.3. Downgrade cascades and branching processes in the mean-field case

We are now ready to show that the initial number of downgrades, D_0^N, asymptotically follows a Poisson distribution, given the equilibrium x^N.

Lemma 4.5. *Let $\{x^N\}$ be a sequence of equilibrium credit ratings satisfying (3). For $N \to \infty$, the random variable D_0^N is conditionally Poisson distributed given x^N.*

Proof. Let us fix a sequence of equilibrium configurations $\{x^N\}_{N \in \mathbb{N}}$. After a shock hits the economy, firm i will be downgraded if

$$s^{N,i} + \frac{\varepsilon^i}{N} > \lambda. \tag{30}$$

Let $r_j^N := \left|M_j^N\right| / N$. In view of Lemma 4.4, and because the random $(\varepsilon^i)_{i \in \mathbb{N}}$ are independent and identically distributed, downgrades within the groups M_j^N correspond to the outcome of Nr_j^N independent Bernoulli experiments with a "success probabilities" π_j^N. We need to consider the different classes of firms separately.

(i) For $j = 0$ we obtain

$$\pi_0^N = \mathbb{P}\left[s^{N,i} + \frac{\varepsilon^i}{N} > \lambda\right] = \frac{Y}{\lambda N}.$$

Since $\mathbb{P}[\lim_{N \to \infty} r_0^N = r_0] = 1$ this yields

$$\lim_{N \to \infty} \pi_0^N N r_0^N = r_0 \frac{Y}{\lambda} \qquad \mathbb{P}\text{-a.s.}$$

Thus, the number of downgrades of firms belonging to group M_0^N follows a Poisson distribution with parameter $r_0 \frac{Y}{\lambda}$ as $N \to \infty$; for details we refer the reader to the internet supplement.

(ii) For $j = 1$, we have

$$x^{N,i} = \alpha \bar{x}^N + \theta^i - (\alpha \bar{x}^N + \theta^i) \bmod \lambda = \alpha \bar{x}^N - (\alpha \bar{x}^N) \bmod \lambda,$$

so the random variables $\{s^{N,i}\}_{i \in M_1^N}$ are conditionally uniformly distributed on the interval $[(\alpha \bar{x}^N) \bmod \lambda, \lambda]$ given the configuration x^N. This yields

$$\pi_1^N = \mathbb{P}\left[s^{N,i} + \frac{\varepsilon^i}{N} > \lambda\right] = \frac{Y}{(\lambda - (\alpha \bar{x}^N) \bmod \lambda) N} \qquad \mathbb{P}\text{-a.s.}$$

Since $\bar{x}^N \to \bar{x}$ as $N \to \infty$, it follows from Lemma 4.4 that

$$\lim_{N \to \infty} \pi_1^N N r_1^N = r_1 \frac{Y}{\lambda - (\alpha \bar{x}) \bmod \lambda} \qquad \mathbb{P}\text{-a.s.}$$

if $r_1 > 0$. The number of downgrades of firms belonging to group M_1^N then follows a Poisson distribution with parameter $r_1 (Y / (\lambda - (\alpha \bar{x}) \bmod \lambda))$ as $N \to \infty$.

(iii) For $j = 2$, we have

$$x^{N,i} = \alpha \bar{x}^N + \theta^i - (\alpha \bar{x}^N + \theta^i) \bmod \lambda = \alpha \bar{x}^N + \bar{\theta} - (\alpha \bar{x}^N + \bar{\theta}) \bmod \lambda,$$

so the buffer variables $\{s^{N,i}\}_{i \in M_2^N}$ are conditionally uniformly distributed on the interval $[0, (\alpha\bar{x}^N + \bar{\theta})\mathrm{mod}\lambda]$ given the configuration x^N. If $r_2 > 0$, the quantities $\{s^{N,i}\}_{i \in M_2^N}$ are asymptotically uniformly distributed on $[0, (\alpha\bar{x} + \bar{\theta})\mathrm{mod}\lambda]$. Since individual shocks are bounded, we see that firms belonging to group M_2^N will not be further downgraded if N is large enough.[8]

Since the sum of independent Poisson distributed random variables follow a Poisson distribution it follows from (i)–(iii) that the total number of downgrades in period 1 follows a Poisson distribution with parameter $Y\pi_1(\lambda)$ where

$$\pi_1(\lambda) := \left(\frac{r_0}{\lambda} + \frac{r_1}{\lambda - (\alpha\bar{x}) \bmod \lambda}\right). \quad \square \tag{31}$$

If θ^i is uniformly distributed on $(0, \lambda)$ as in the binary choice model studied in Example 2.3, then $\bar{x} = 0$ and $r_0 = 1$. In such a situation all the buffer variables $s^{N,i}$ are conditionally identically distributed. Moreover the random variable D_0^N is asymptotically Poisson distributed with parameter Y. In this case the number of firms initially affected by the shock only depends on the shock size.

Example 4.6. Let us return to the situation analyzed in Example 4.2. In this case $r_1 = r_2 = 1/4$ and $r_0 = 1/2$. Moreover, $\bar{x} = 1$, and $\pi_1(\lambda) = 2$.

Let us now study the distribution of the number of downgrades in period $t+1$ conditioned on the downgrades thus far.

Lemma 4.7. *Given the number $d_1^j, d_2^j, \ldots, d_t^j$ of downgrades of firms belonging to groups M_j^N for $j \in \{0, 1\}$ up to time t, the random variable D_{t+1}^N is asymptotically Poisson distributed with parameter*

$$\pi_{t+1}(\alpha, \lambda, d_t) := \lambda d_t \alpha (r_0 + r_1) \quad \text{where } d_t := d_t^1 + d_t^2.$$

In the limit of an infinite economy, the number of downgrades in the previous period t is thus a sufficient statistic for the distribution of downgrades in period $t+1$.

Proof. Firm $i \in \{1, 2, \ldots, N\}$ is downgraded in period u if

$$s^{N,i} + \lambda\frac{\alpha}{N}\sum_{j=1}^{u-2} d_j + \frac{\varepsilon^i}{N} < \lambda < s^{N,i} + \lambda\frac{\alpha}{N}\sum_{j=1}^{u-1} d_j + \frac{\varepsilon^i}{N}.$$

Because individual shocks are bounded, it is straightforward to show that a company is downgraded at most once within a finite period of time if the economy is large enough. Let H_s^j be the set of firms belonging to group M_j^N that have been downgraded in period s. Given $d_1^j, \ldots, d_t^j$, we can choose a large enough N such that $H_s^j \cap H_r^j = \emptyset$. Downgrades of group M_j^N-firms correspond to independent outcomes of $Nr_j^N - \sum_{k=0}^{t} d_k^j$ Bernoulli experiments with success probability $\pi_{t+1}^{N,j}$.

[8] Loosely speaking, these firms are already bankrupt.

For $i \in M_0^N$ and $i \notin \cup_{s=1}^t H_s^1$, the conditional probability of a downgrade in period $t+1$ given $d_1^j, \ldots, d_t^j$ takes the form

$$\pi_{t+1}^{N,0} = \frac{\int (\mathbb{P}[s^{N,i} + \varepsilon^i/N + \lambda(\alpha/N)\sum_{k=1}^{t} d_k > \lambda] - \mathbb{P}[s^{N,i} + \varepsilon^i/N + \lambda(\alpha/N)\sum_{k=1}^{t-1} d_k < \lambda])Q(Y; d\varepsilon^i)}{\int \mathbb{P}[s^{N,i} + \varepsilon^i/N + \lambda(\alpha/N)\sum_{k=1}^{t-1} d_k < \lambda]Q(Y; d\varepsilon^i)}.$$

Since the sensitivity parameter $s^{N,i}$ is conditionally uniformly distributed on $(0, \lambda)$, the latter expression simplifies to

$$\pi_{t+1}^{N,0} = \frac{\lambda d_t \alpha}{N - ((Y + \lambda\alpha\sum_{k=1}^{t-1} d_k)/\lambda)} \quad \text{and} \quad \lim_{N\to\infty} \left\{ \pi_{t+1}^{N,0} \left(r^{N,0} N - \sum_{k=1}^{t} d_k^1 \right) \right\} = \lambda r_0 \alpha d_t.$$

Thus, the number of defaults of firms belonging to M_0^N in period $t+1$ asymptotically follows a Poisson distribution with parameter $\lambda r_0 \alpha d_t$. Similar arguments yield

$$\lim_{N\to\infty} \left\{ \pi_{t+1}^{N,1} \left(r_1^N N - \sum_{k=0}^{t} d_k^1 \right) \right\}$$
$$= \lim_{N\to\infty} \frac{d_t \alpha}{N - \lambda\left(\left(Y + \alpha\sum_{k=1}^{t-1} d_k\right)/\lambda\right)} \left\{ \pi_{t+1}^{N,1} \left(r^{N,1} N - \sum_{k=0}^{t} d_k^1 \right) \right\} = \lambda r_1 \alpha d_t.$$

Thus, conditioned on the number of downgrades in period t, the random variable D_{t+1}^N asymptotically follows a Poisson distribution with parameter $\lambda\alpha(r_0 + r_1)d_t$. Thus, each downgrade d_t asymptotically generates a number of "descendants" that is Poisson distributed with parameter $\alpha\pi_2(\lambda)$ where

$$\pi_2(\lambda) := \lambda(r_0 + r_1). \quad \square \tag{32}$$

We are now ready to prove the main results of this paper.

Proof of Theorem 2.7.

(i) By Lemma 4.5 the random variable D_0^N is asymptotically Poisson distributed. In order to show that our cascade process can be approximated in law by a Galton-Watson, we denote by $\{D_t\}_{t\in\mathbb{N}}$ a branching process with l sister anchors that reproduces from generation to generation in a Poisson way. More precisely,

$$D_0 = l \quad \text{and that} \quad D_t = D_{t,1} + \cdots + D_{t,D_{t-1}}$$

where the independent random variables $\{D_{t,j}t,\ j \in \mathbb{N}\}$ are Poisson distributed with parameter $\nu := \lambda\alpha(r_0 + r_1) \leq 1$. Thus, given the size d_t of the generation in period t, the random variable D_{t+1} is conditionally Poisson distributed with parameter $\alpha d_t(r_0 + r_1)$. For $d^{(t)} := (d_1, d_2, \ldots, d_t)$, let $\mu_t^N(\cdot|d^{(t)})$ be the conditional distribution of D_{t+1}^{N} given $(D_1^N, \ldots, D_t^N) = d^{(t)}$ and $\mu_t(\cdot|d_t)$ the conditional distribution of D_{t+1} given $D_{\mathrm{t}} = d_t$. By Lemma 4.7 we have

$$\mu_t^N(\cdot|d^{(t)}) \xrightarrow{w} \mu_t(\cdot|d_t) \quad \text{as } N \to \infty \tag{33}$$

where $\xrightarrow{w}$ denotes weak convergence of probability measures.

We need to show that weak convergence of the conditional distributions implies weak convergence of the unconditional distribution μ_t^N of D_{t+1}^N to the law μ_t of D_t, or that (33) yields

$$\lim_{n\to\infty}\int f d\mu_t^N = \int f d\mu_t \quad (t\in\mathbb{N}) \tag{34}$$

for each function $f:\mathbb{N}^t\to\mathbb{R}$. To this end, we proceed by induction. Weak convergence of the sequence $\{\mu_1^N\}$ to μ_1 as $N\to\infty$ follows from (33). The induction hypothesis is

$$\mu_t^N \xrightarrow{w} \mu_t.$$

Since $\int f d\mu_{t+1}^N = \int\int f d\mu_t^N(\cdot|d^{(t)})d\mu_t^N$ it follows from (33) that

$$f_t^N(\cdot):=\int f d\mu_t^N(.|d^{(t)}) \to f_t(\cdot):=\int f d\mu_t(\cdot|d^{(t)}) \quad \text{as } N\to\infty$$

and the convergence is uniform on compact sets because the state space is countable. Hence, convergence of conditional finite dimensional distributions follows from the induction hypothesis by a standard argument. This shows that, conditioned on $\{\Pi = l\}$, the finite dimensional distributions of the process $\{D_t^N\}_{t\in\mathbb{N}}$ converge to the finite dimensional distributions of the branching process $\{D_t\}_{t\in\mathbb{N}}$. In (ii) below we show that the sequence $\{D^N\}_{N\in\mathbb{N}}$ is also tight:

$$\sup_N \ \mathbb{P}[D_t^N \le c] \to 0 \quad \text{as } c\to 0.$$

This proves convergence in law of the whole process.

(ii) Conditioned on $\{D_0 = 1\}$ the sum$\sum_{i=0}^{\infty} D_t$ follows a Borel-Tanner distribution with parameter ν, due to Theorem 1.4 in the supplement. Since D_0 is Poisson distributed with parameter $\alpha\pi_1(\lambda)$, the random variable $D_\tau = \sum_{t=0}^{\infty} D_t$ coincides in distribution with (10).

In order to show that the sequence $\{D_{\tau^N}^N\}_{N\in\mathbb{N}}$ converges in distribution to D_τ we denote by $\tau := \min\{\text{t}: D_t = 0\}$ the time of extinction of the branching process $\{D_t\}_{t\in\mathbb{N}}$. Since $\nu\le 1$, this stopping time is almost surely finite. Thus, for each $\varepsilon > 0$ we can choose $T_\varepsilon\in\mathbb{N}$ such that

$$\mathbb{P}[\tau \ge T_\varepsilon] < \varepsilon \text{ and, in view of (i),} \quad \mathbb{P}[D_{T_\varepsilon}^N > 0] \le 2\varepsilon \quad \text{for all sufficiently large } N.$$

Thus, $\mathbb{P}[D_{\tau^N}^N \ne \sum_{t=1}^{T_\varepsilon} D_t^N] \le 2\varepsilon$. We obtain from (i) that

$$\left|\mathbb{P}[\sum_{t=1}^{T_\varepsilon} D_t = k] - \mathbb{P}[\sum_{t=1}^{T_\varepsilon} D_t^N = k]\right| < \varepsilon \quad \text{for } N \text{ large enough,}$$

and a 3-ε-argument shows that

$$\left|\mathbb{P}[\sum_{t=1}^{\infty} D_t = k] - \mathbb{P}[\sum_{t=1}^{\infty} D_t^N = k]\right| < \varepsilon \quad \text{for } N \text{ large enough,}$$

The representations of the conditional distributions follow from Theorem 4 in the internet supplement.

Let us now prove tightness of the sequence of processes $\{D_t^N\}_{t\in\mathbb{N}}$. To this end, we fix $K\in\mathbb{N}$. In view of (i) and because $\mathbb{P}[\tau<\infty]=1$ we have that

$$\lim_{N\to\infty}\mathbb{P}[D_t^N\le K]=\lim_{N\to\infty}\mathbb{P}[D_t^N\le K]\sum_{k=0}^{K}\mathbb{P}[D_t^N=k]=\mathbb{P}[D_t\le K]\ge 1-\varepsilon$$

for all sufficiently large K.

(iii) The representations (14) follow from arguments given in Section 2 of the internet supplement.

(iv) The quantities $\pi_1(\lambda)$ and $\pi_2(\lambda)$ are given by (31) and (32), respectively. A tedious but straightforward calculation shows that $\lim_{\lambda\to 0}\lambda^2\pi_1(\lambda)=0$. □

4.4. Extension to local interactions

This subsection proves the distribution of the total number of downgrades in a model with both local and global interactions.

Proof of Theorem 2.10. Since we restrict ourselves to the case where $x^{N,i}=0$ for all $i\in\{1,2,\ldots,N\}$, the buffer variables $s^{N,i}$ are conditionally independent and identically distributed given the initial equilibrium x^N. Thus, we can apply the same arguments as in the case of a simple mean field interaction in order to show that the processes $\{\hat{D}_t^N\}_{t\in\mathbb{N}}$ of defaults can be approximated in law by a branching process $\{\hat{D}_t\}_{t\in\mathbb{N}}$ where $\hat{D}_0$ follows a Poisson distribution.

In order to specify the dynamics of the sequence $\{\hat{D}_t\}_{t\in\mathbb{N}}$, we fix the number $\hat{D}_t^N=d_t$ of global defaults in period t. Asymptotically, a global default triggers a local default with probability α_2, because, for large N, the buffer variables are conditionally uniformly distributed on the unit interval. A local default triggers another local default with probability α_2. Thus, the total number of local defaults triggered by the d_t global defaults corresponds to the outcome of d_t independent random variables $Y_1^t,\ldots,Y_{d_t}^t$ following a geometric distribution with parameter $1-\alpha_2$ and moment generating function

$$F(x)=\frac{1-\alpha_2}{1-\alpha_2 x};$$

see Section 2 of the internet supplement for further details. We can then apply the same arguments as in the proof of Lemma 4.7 in order to show that, given $D_t^N=d_t$,

$$\hat{D}_{t+1}^N\sim\mathcal{P}\left(\alpha_1\sum_{i=1}^{d_t}(1+Y_i^t)\right)\quad\text{for } N\to\infty.$$

Thus, for $N\to\infty$, the dynamics of the process $\{\hat{D}_t^N\}_{t\in\mathbb{N}}$ can be approximated in law by a branching process $\{\hat{D}_t\}_{t\in\mathbb{N}}$ of the form

$$\hat{D}_{t+1}=D_{t,1}+\cdots+D_{t,D_{t-1}}\quad\text{where } D_{t,1}\stackrel{D}{=}\sum_{i=1}^{\Pi_t}(1+X_i).$$

The random variable Π_t is Poisson distributed with parameter α_1 and independent of the sequence $\{X_i\}_{i \in \mathbb{N}}$. It is straightforward to show that $D_{t,1}$ has moment generating function

$$G(x) = \exp\left(\alpha_1 \frac{x-1}{1-\alpha_2 x}\right) \quad \text{and} \quad G'(1) = \frac{\alpha_1}{1-\alpha_2} \leq 1. \tag{35}$$

Thus, the branching process $\{\hat{D}_t\}_{t \in \mathbb{N}}$ becomes almost surely extinct. Under the assumptions of the theorem there exists $\alpha^* \in (0, \alpha_2^{-1})$ that satisfies

$$G'(\alpha^*) = \frac{G(\alpha^*)}{\alpha^*}.$$

With $r = -\alpha/G(\alpha)$ the assertion then follows from a seminal theorem by Otter; see also Theorem I.13.1 in Harris (1989). □

5. Conclusion

Based on results from the theory of (S, s) economies we provided a unified probabilistic framework within which to study the effects of discrete adjustments of credit qualities on the losses associated with large portfolios of credit sensitive securities. We analyzed an interactive model of credit ratings where, initially, external shocks trigger a certain number of downgrades. The interactive structure of credit ratings generates a feedback effect that allows a single firm's financial distress to propagate through the whole economy, causing a cascade of downgrades and defaults. In a large economy the dynamics of the cascade can be described by a simple branching process where each generation reproduces in a Poisson manner. Assuming the interaction between different firms is not too strong, the distribution of the total number of defaults is given by a unimodal distribution with slowly decaying tails. We characterized the tail behavior of the distribution of aggregate portfolio losses. If the interaction between firms are too strong, then aggregate losses turned out to be heavy-tailed even if individual losses are thin-tailed. This illustrates that counterparty relationships are in fact an additional source of intrinsic risk that should be accounted for. However, the risk can be considerably reduced if the firms' financial standing is measured on a finer scale. In the limiting case of continuous adjustments, the additional risk is simply specified by the size of the external shock.

Several avenues are open for future research. The explicit representation of the distribution of the total number of defaults rests upon two simplifying assumption: (i) a mean-field interaction and (ii) the independence of the firm specific quantities θ^i. Both assumptions are quite restrictive and should be relaxed. An interaction of mean field type can only be viewed as a first step towards a much richer interaction structure. Mathematical methods and techniques from, for example, random graph or percolation theory or from the theory of branching processes in random environments may allow us to work with more realistic networks of interacting companies such as the one in Egloff et al. or to embed the "primary-secondary" approach by Jarrow and Yu into a general microeconomic model of local interactions. However, for more general models there might be little hope of obtaining analytic solutions for aggregate loss distributions. Secondly, our approach awaits empirical verification. It would clearly be desirable to introduce additional parameter into our model and to try to fit the distribution of the random variable D_τ to empirical data. Embedded in a dynamic framework, our model may also be used to price baskets of credit securities. For instance, one could think about a financial institution that wants to hedge a large portfolio against outside influences that affect the market values of credit sensitive securities. The institution could then issue a bond whose coupon depends on aggregate portfolio losses in case an external shock hits the economy. To price such a catastrophe bond, contagious downgrading effects need to be explicitly modelled. Finally, following the financial literature, we took credit ratings as given. It is clearly desirable and challenging to derive a microeconomic framework where ratings are the "output" rather than the "input".

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at doi:10.1016/j.jebo.2005.02.005.

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