Introduction to Mathematical Finance

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Contents

1 Discrete Time Finance **7**

1.1 Introduction . . . 7

1.1.1 The put-call parity . . . 8

1.1.2 Naive approaches to option pricing . . . 8

1.2 Mathematical finance in one period . . . 9

1.2.1 Assets portfolios and arbitrage opportunities . . . 9

1.2.2 The Fundamental Theorem of Asset Pricing . . . 11

1.2.3 Derivative securities . . . 14

1.2.4 Complete market models and perfect replication . . . 18

1.2.5 Good deal bounds and super-deals . . . 19

1.2.6 Indifference valuation and risk sharing . . . 22

1.3 Dynamic hedging in discrete time . . . 23

1.3.1 The multi-period market model . . . 24

1.3.2 Arbitrage opportunities and martingale measures . . . 26

1.3.3 European options and attainable claims . . . 28

1.3.4 Complete markets . . . 31

1.4 Binomial trees and the Cox-Ross-Rubinstein Model . . . 32

1.4.1 Delta-Hedgeing in discrete time . . . 33

1.4.2 Exotic derivatives . . . 34

1.4.3 Convergence to Black-Scholes Prices . . . 36

1.5 Introduction to optimal stopping and American options . . . 40

1.5.1 Motivation and introduction . . . 40

1.5.2 Stopping times . . . 41

1.5.3 The Snell envelope . . . 42

1.5.4 Decomposition of supermartingales and pricing of American options . . . 43

1.5.5 American options in the CRR model . . . 44

1.5.6 Arbitrage-free prices . . . 46

1.5.7 The lower Snell envelope . . . 47

1.6 Introduction to risk measures . . . 50

1.6.1 Risk measures and their acceptance sets . . . 50

1.6.2 Robust representations of risk measures . . . 53

1.6.3 V@R, AV@R and shortfall risk . . . 55
1.6.4 Law invariance . . . 58

2 Continuous Time Finance **59**
2.1 Introduction to continuous time models and Pathwise Itô calculus . . . 59
2.1.1 Basic notions and classes of stochastic processes . . . 59
2.1.2 Brownian motion . . . 60
2.1.3 Quadratic Variation . . . 61
2.1.4 The basic Itô formula . . . 64
2.1.5 The quadratic variation of Itô integrals . . . 65
2.1.6 Application: The Bachelier Model . . . 66
2.2 Stochastic Calculus for Brownian Motion . . . 69
2.2.1 The Itô Integral . . . 70
2.2.2 Itô integral for a larger class of integrands . . . 73
2.2.3 The Generalized Itô Formula . . . 74
2.2.4 The martingale representation theorem . . . 78
2.2.5 Application: The Black Scholes Model . . . 79
2.3 Topics in diffusion theory . . . 83
2.3.1 Stochastic differential equations . . . 83
2.3.2 Solutions to linear SDEs . . . 85
2.3.3 Change of variables and Girsanov's theorem . . . 87
2.4 Risk neutral pricing . . . 90
2.4.1 The market model . . . 91
2.4.2 Equivalent martingale measures and absence of arbitrage . . . 92
2.4.3 Option Pricing and PDE . . . 96
2.5 The fundamental theorem of asset pricing in continuous time . . . 97
2.5.1 Definitions . . . 98
2.5.2 The Kreps-Yan theorem . . . 100
2.5.3 The proof of the fundamental theorem of asset pricing . . . 100
2.5.4 On the weak-*-closedness of C . . . 101

3 Stochastic Optimal Control **105**
3.1 Introduction to basic concepts . . . 105
3.1.1 Examples of stochastic optimization problems . . . 105
3.1.2 Controlled diffusion processes . . . 107
3.1.3 Dynamic programming principle . . . 109
3.1.4 Hamilton-Jacobi-Bellman equation . . . 110
3.1.5 Application: Portfolio optimization for power utilities . . . 111
3.2 Viscosity solutions . . . 113
3.2.1 Framework, Motivation and Definition . . . 114
3.2.2 From the value function to a viscosity solution of HJB via DPP . . . 116
3.2.3 From the viscosity solution to the value function via a comparison principle . . . 118

3.2.4 Comparison principles 119
3.3 Backward Stochastic Equations and Optimal Control 122
3.3.1 Existence and uniqueness results 122
3.3.2 Control and BSDE 125
3.3.3 Application: Portfolio optimization 127

Foreword

The objective of this one year course is to give an introduction to the probabilistic techniques required to understand the most widely used models of mathematical finance. The course is intended for undergraduate and graduate students in mathematics, but it might also be useful for students in economics and operations research.

The focus of the first semester is on stochastic models in discrete time. This has at least two immediate benefits. First, the underlying probabilistic machinery is much simpler; second the standard paradigm of complete financial markets akin to many continuous time models where all derivative securities admit a perfect hedge does typically not hold and we are confronted with incomplete models at a very early stage. Our discussion of (dynamic) arbitrage theory and risk measures follows the textbook by Föllmer & Schied (2004). The chapter on optimal stopping and American options follows Section 2 of Lamberton & Lapeyre (1996).

The second semester treats stochastic finance in continuous time. We discuss pathwise Itô calculus as well as selected topics in stochastic analysis such as Itô integrals, the change of variable formula (Girsanov's theorem), the martingale representation theorem and stochastic differential equations. They constitute the building blocks of continuous time financial mathematics including the famous Black-Scholes option pricing formula. Here we follow the textbooks by Lamberton & Lapeyre (1996) and Øksendal (2003). The latter provides a detailed introduction to Itô calculus and stochastic analysis. The textbook of Lamberton & Lapeyre (1996), on the other hand, yields a more elementary introduction to stochastic calculus but with a clear application to mathematical finance. For a more elaborate discussion of financial mathematics in both discrete and continuous time we also refer to books by Shreve (2005a, 2005b); students with an interest in economics are encouraged to also consult Duffie (1996) and Hull (2000). The last part of the course provides an introduction into stochastic optimal control with applications to utility maximization and portfolio optimization. We introduce both the traditional Hamailton-Jacobi-Bellmann approach that derives value functions and optimal strategies through PDE methods as well as more recent approaches using backward stochastic differential equations (BSDEs). Our discussion is based on the books of Pham (2009) and Touzi (2013). While we mostly follow Pham (2009), Touzi's book contains a fine proof of the dynamic programming principle that does not require any measurable selection arguments.

Chapter 1

Discrete Time Finance

1.1 Introduction

Our presentation concentrates on options and other derivative securities. Options are among the most relevant and widely spread financial instruments. The need to price and hedge options has been the key factor driving the development of mathematical finance. An option gives its holder the right, but not the obligation, to buy or sell

- a financial asset (*underlying*)
- at or before a certain date (*maturity*)
- at a predetermined price (*strike*).

Underlyings include, but are not limited to stocks, currencies, commodities (gold, copper, oil, ...) and options. One usually distinguishes

- *European Options* that can only be exercised at maturity, and
- *American Options* that can be exercised any time before maturity.

An option to buy the underlying is referred to as a *Call Option* while an option to sell is called at *Put Option*. Typically T denotes the time to maturity, K the strike and S_t the price of the underlying at time $t \in [0, T]$. The writer (seller) of European Call option needs to pay the buyer an amount of

$$\max\{S_T - K, 0\} := (S_T - K)^+ \tag{1.1}$$

at time T: if the underlying trades at some price $S_T < K$ at maturity the buyer will not exercise the option while she will exercise her right to buy the option at the predetermined price K when the market price of the underlying at time T exceeds K. As a result the writer of the option bears an unlimited risk. The question, then, is how much the writer should charge the buyer in return for taking that risk (*Pricing the Option*) and how that money should be invested in the bond and stock market to meet his payment obligations at maturity (*Hedging the Option*). To answer these questions we will make some basic assumptions. A commonly accepted assumption in every financial market model is the absence of arbitrage opportunities ("No free Lunch"). It states that there is no riskless profit available in the market.

1.1.1 The put-call parity

Based solely on the assumption of no arbitrage we can derive a formula linking the price of a European call and put option with identical maturities T and strikes K written on the same underlying. To this end, let C_t and P_t be the price of the call and put option at time $t \in [0, T]$, respectively. The no free lunch condition implies that

$$C_t - P_t = S_t - Ke^{-r(T-t)}$$

where $r \geq 0$ denotes the risk free interest rate paid by a government bond. In fact, if we had

$$C_t - P_t > S_t - Ke^{-r(T-t)}$$

we would buy one stock and the put and sell the call. The net value of this transaction is

$$C_t - P_t - S_t.$$

If this amount is positive, we would deposit it in a bank account where it earns interest at the rate r; if it were negative we would borrow the amount paying interest at rate r. If

$$S_T > K$$

the option will be exercised and we deliver the stock. In return we receive K and close the bank account. The net value of the transaction is positive:

$$K + e^{r(T-t)}(C_t - P_t - S_t) > 0.$$

If, on the other hand,

$$S_T \leq K$$

then we exercise our right to sell our stock at K and close the bank account. Again the net value of the transaction is positive:

$$K + e^{r(T-t)}(C_t - P_t - S_t) > 0.$$

In both cases we locked in a positive amount without making any initial investment which were an arbitrage opportunity. Similar considerations apply when

$$C_t - P_t < S_t - Ke^{-r(T-t)}.$$

1.1.2 Naive approaches to option pricing

The "no free lunch condition" implies that discounted conditional expected payoffs are typically no appropriate pricing schemes as illustrated by the following example.

Example 1.1.1 *Consider a European option with strike K and maturity $T = 1$ written on a single stock. The asset price π_0 at time $t = 0$ is known while the price at maturity is given by the realization of a random variable S defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. More specifically, consider a simple "coin flip" where $\Omega = \{w_-, \omega_+\}$ and $\mathbb{P} = (0.5, 0.5)$ and assume that*

$$\pi = 100, \quad K = 100, \quad \text{and} \quad S(\omega) = \begin{cases} 120 & \text{if } \omega = \omega_+ \\ 90 & \text{if } \omega = \omega_- \end{cases}.$$

If the price $\pi(C)$ of the call option with payoff $C = (S - 100)^+$ were $\frac{10}{1+r}$, then the following is an arbitrage opportunity as long as the risk-less interest rate is less than 5%:

- *sell the option at $\frac{10}{1+r}$;*
- *borrow the dollar amount $\frac{60}{1+r}$ at the risk free rate;*
- *buy $2/3$ shares of the stock.*

A direct calculation shows that the value of the portfolio at time $t = 1$ is zero in any state of the world while it yields the positive cash-flow $\frac{70}{1+r} - \frac{2}{3}100$ at time $t = 0$.

A second approach that dates back to Bernoulli is based on the idea of indifference valuation. The idea is to consider an investor with an initial wealth W whose preferences over income streams are described by a strictly concave, increasing utility function $u : \mathbb{R} \to \mathbb{R}^+$ and to price the option by its *certainty equivalent*. The certainty equivalent $\pi(C)$ is defined as the unique price that renders an expected-utility maximizing investor indifferent between holding the deterministic cash amount $W + \pi(C)$ and the random payoff $W + C$; that is:

$$u(W + \pi(C)) = \mathbb{E}\left[u(W + C)\right].$$

Among the many drawbacks of this approach is the fact that $\pi(C)$ is not a market price. Investors with heterogeneous preferences are charged different prices. It will turn out that options and other derivative securities can in fact be priced without any preference to preferences of market participants.

1.2 Mathematical finance in one period

Following Chapter 1 of Föllmer & Schied (2004), this section studies the mathematical structure of a simple one-period model of a financial market. We consider a finite number of assets whose initial prices at time 0 are known while their future prices at time 1 are described by random variables on some probability space. Trading takes place at time $t = 0$.

1.2.1 Assets portfolios and arbitrage opportunities

Consider a financial market model with $d + 1$ assets. In a one-period model the assets are priced at the initial time $t = 0$ and the final time $t = 1$. We assume that i-th asset is available at time 0 for a price $\pi^i \geq 0$ and introduce the *price system*

$$\bar{\pi} = (\pi^0, \dots, \pi^d) \in \mathbb{R}^{d+1}_+.$$

In order to model possible *uncertainty* about prices in the following period $t = 1$ we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and describe the asset prices at time 1 as non-negative measurable functions

$$S^0, S^1, \ldots, S^d$$

on $(\Omega, \mathcal{F})$. Notice that not all asset prices are necessarily uncertain. In fact, there is usually a *riskless bond* that pays a *sure* amount at time 1. We include such a bond assuming that

$$\pi^0 = 1 \quad \text{and} \quad S^0 \equiv 1 + r$$

for a constant deterministic rate of return $r \geq 0$. To distinguish the bond from the risky assets we conveniently write

$$\bar{S} = (S^0, S^1, \ldots, S^d) = (S^0, S) \quad \text{and} \quad \bar{\pi} = (1, \pi).$$

Every $\omega \in \Omega$ corresponds to a particular scenario of market evolution, and $S^i(\omega)$ is the price of the i-th asset at time 1 if the scenario ω occurs.
A *portfolio* is a vector $\bar{\xi} = (\xi^0, \xi) \in \mathbb{R}^{d+1}$ where ξ^i represents the number of shares of the i-th asset. Notice that ξ^i may be negative indicating that an investor sells the asset short. The price for buying the portfolio $\bar{\xi}$ equals

$$\bar{\pi} \cdot \bar{\xi} = \sum_{i=0}^{d} \pi^i \xi^i.$$

At time $t = 1$ the portfolio has the value

$$\bar{\xi} \cdot \bar{S}(\omega) = \xi^0(1 + r) + \xi \cdot S(\omega)$$

depending on the scenario ω. With this we are now ready to formally introduce the notion of an arbitrage opportunity, an investment strategy that yields a positive profit in some states of the world without being exposed to any downside risk.

Definition 1.2.1 *A portfolio $\bar{\xi}$ is called an arbitrage opportunity if $\bar{\pi} \cdot \bar{\xi} \leq 0$ but*

$$\mathbb{P}[\bar{\xi} \cdot \bar{S} \geq 0] = 1 \quad \textit{and} \quad \mathbb{P}[\bar{\xi} \cdot \bar{S} > 0] > 0.$$

We notice that the "real-world-probability-measure" $\mathbb{P}$ enters the definition of arbitrage only through the null sets of $\mathbb{P}$. In particular, the assumption can be stated without reference to probabilities if Ω is countable (or finite). In this case we can with no loss of generality assume that

$$\mathbb{P}[\{\omega\}] > 0 \quad \text{for all} \quad \omega \in \Omega$$

and an arbitrage strategy is simply a portfolio $\bar{\xi}$ such that $\bar{\pi} \cdot \bar{\xi} \leq 0$ but

$$\bar{\xi} \cdot \bar{S}(\omega) \geq 0 \quad \text{for all } \omega \in \Omega \quad \text{and} \quad \bar{\xi} \cdot \bar{S}(\omega_0) > 0 \quad \text{for at least one } \omega_0 \in \Omega.$$

Definition 1.2.2 *A portfolio $\bar{\xi}$ is called self-financing if $\bar{\pi} \cdot \bar{\xi} = 0$, i.e., if the investor merely re-balances her wealth.*

It should be intuitively clear that in the absence of arbitrage opportunities any investment in risky assets which yields with positive probability a better result than investing the same amount in the risk-free asset must carry some downside risk. The following lemma confirms this intuition. Its proof is left as an exercise.

Lemma 1.2.3 *The following statements are equivalent.*

(i) The market model admits an arbitrage opportunity.

(ii) There is a vector $\xi \in \mathbb{R}^d$ such that

$$\mathbb{P}[\xi \cdot S \geq (1+r)\xi \cdot \pi] = 1 \quad \textit{and} \quad \mathbb{P}[\xi \cdot S > (1+r)\xi \cdot \pi] > 0.$$

(iii) There exists a self-financing arbitrage opportunity.

The assumption of no arbitrage is a condition imposed for *economic* reasons. In the following section we characterize arbitrage free models in a mathematically rigorous manner.

1.2.2 The Fundamental Theorem of Asset Pricing

In this section we link the "no free lunch" condition on a financial market model to the existence of equivalent martingale measures. To this end, let a financial market model along with a price system be given:

$$\bar{\pi} = (1, \pi), \quad \bar{S} = (1+r, S) \quad \text{where} \quad (S^i)_{i=1}^d \text{ are random variables on } (\Omega, \mathcal{F}, \mathbb{P}).$$

Definition 1.2.4 *The financial market model is called arbitrage-free if no free lunch exists, i.e, for all portfolios $\bar{\xi} \in \mathbb{R}^{d+1}$ that satisfy $\bar{\xi} \cdot \bar{\pi} \leq 0$ and $\bar{\xi} \cdot \bar{S} \geq 0$ almost surely, we have*

$$\bar{\xi} \cdot \bar{S} = 0 \quad \mathbb{P}\text{-}a.s.$$

The notion is risk-neutral or martingale measures is key in mathematical finance.

Definition 1.2.5 *A probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ is called a risk-neutral or martingale measure if asset prices at time $t = 0$ can be viewed as expected discounted future payoffs under $\mathbb{P}^*$, i.e., if*

$$\pi^i = \mathbb{E}^* \left[\frac{S^i}{1+r} \right].$$

The fundamental theorem of asset pricing states that the set of arbitrage free prices can be linked to the set of *equivalent* martingale measures

$$\mathcal{P} = \{\mathbb{P}^* \ : \ \mathbb{P}^* \text{ is a martingale measure and } \mathbb{P}^* \approx \mathbb{P}\}.$$

We notice that the characterization of arbitrage free prices does not take into account the preferences of the market participants. It is entirely based on the assumption of no arbitrage. Furthermore, recall that the real-world-measure $\mathbb{P}$ enters the definition of no-arbitrage only through its null sets. The assumption that any $\mathbb{P}^* \in \mathcal{P}$ is equivalent to $\mathbb{P}$, i.e., that

$$\mathbb{P}[A] = 0 \Leftrightarrow \mathbb{P}^*[A] = 0 \quad (A \in \mathcal{F})$$

guarantees that the null sets are the same.

In order to state and prove the fundamental theorem of asset pricing it will be convenient to use the following notation: we denote by

$$X^i = \frac{S^i}{1+r} \quad \text{and} \quad Y^i = X^i - \pi^i \quad (i = 0, 1, \ldots, d)$$

the discounted payoff of the i-th asset and the net gain from trading asset i, respectively. In terms of these quantities the no arbitrage condition reads

$$\mathbb{P}[\xi \cdot Y \geq 0] = 1 \Rightarrow \mathbb{P}[\xi \cdot Y = 0] = 1$$

and $\mathbb{P}^*$ is an EMM if and only if $\mathbb{E}^*[Y^i] = 0$ for every asset.

Theorem 1.2.6 *A market model is arbitrage free if and only if $\mathcal{P} \neq \varnothing$. In this case, there exists a $\mathbb{P}^* \in \mathcal{P}$ which has a bounded density with respect to $\mathbb{P}$.*

The following example that shows that the implication " ⇒" of the fundamental theorem of asset pricing that the absence of arbitrage implies the existence of an EMM may not hold in markets with infinitely many assets.

Example 1.2.7 *Let $\Omega = \{1, 2, \ldots\}$ and the risk free rate be zero and consider assets with initial prices and payoffs given by*

$$\pi^i = 1 \quad \textit{and} \quad S^i(\omega) = \begin{cases} 0 & \omega = i \\ 2 & \omega = i+1 \\ 1 & \textit{else} \end{cases} \qquad (i \geq 1)$$

We consider only portfolios $\bar{\xi} \in \ell^1$ that is, $\sum_{i=0}^{\infty} |\xi^i| < \infty$ and assume that there exists an EMM $\mathbb{P}^*$. In this case

$$1 = \pi^i = \mathbb{E}^*[S^i] = 2\mathbb{P}^*[\{i+1\}] + \sum_{k=1, k \neq i, i+1} \mathbb{P}^*[\{k\}] = 1 + \mathbb{P}^*[\{i+1\}] - \mathbb{P}^*[\{i\}]$$

so

$$\mathbb{P}^*[\{i\}] = \mathbb{P}^*[\{i+1\}] = \mathbb{P}^*[\{i+2\}] = \ldots$$

which is not possible. The model is, however, free of arbitrage. In fact, let $\bar{\xi} = (\xi^0) \in \ell^1$ be such that

$$\mathbb{P}[\bar{\xi} \cdot \bar{S} \geq 0, \, \bar{\xi} \cdot \bar{\pi} \leq 0] = 1$$

so $\xi^1 \leq 0$. For $\omega = 1$ this yields

$$0 \leq \bar{\xi} \cdot \bar{S}(1) = \xi^0 + \sum_{k=2}^{\infty} \xi^k = \bar{\pi} \cdot \bar{\xi} - \xi^1 \leq -\xi^1$$

and by analogy for $\omega > 1$:

$$0 \leq \bar{\xi} \cdot \bar{S}(\omega) \leq \xi^{i-1} - \xi^i.$$

Thus,

$$0 \geq \xi^1 \geq \xi^2 \geq \cdots$$

which is compatible with $\xi \in \ell^1$ only if $\xi^i = 0$ in which case

$$\xi \cdot S = 0 \quad \mathbb{P}\text{-}a.s.$$

Remark 1.2.8 *For the special case of a finite set $\Omega = \{\omega_1, \dots, \omega_n\}$ and a single risky asset equivalent martingale measures satisfy two simple linear equations. To see this, let $p_i = \mathbb{P}[\{\omega_i\}]$ and $s_i = S^1(\omega_i)$ and assume without loss of generality that*

$$p_i > 0 \quad \text{and} \quad s_1 < s_2 < \dots < s_n.$$

An equivalent martingale measure is then a vector $p^* = (p_1^*, \dots, p_n^*)$ with positive entries such that

$$\begin{aligned} s_1 p_1^* + \cdots + s_n p_n^* &= \pi^1(1+r) \\ p_1^* + \cdots + p_n^* &= 1. \end{aligned} \tag{1.2}$$

If a solution exists it will be unique if and only if $n = 2$. If there are more than two states of the world and just one asset, there will be infinitely many solutions.

Let us denote by $\mathcal{V}$ the linear space of all attainable payoffs:

$$\mathcal{V} := \{\bar{\xi} \cdot \bar{S} \ : \ \bar{\xi} \in \mathbb{R}^{d+1}\}. \tag{1.3}$$

The portfolio that generates $V \in \mathcal{V}$ is not necessarily unique but in an arbitrage free market the *law of one price* prevails:

$$\text{If } V \in \mathcal{V} \text{ can be written as } V = \bar{\xi} \cdot \bar{S} = \bar{\zeta} \cdot \bar{S} \text{ then } \bar{\pi} \cdot \bar{\xi} = \bar{\pi} \cdot \bar{\zeta}.$$

In particular, it makes sense to define the price of $V \in \mathcal{V}$ in terms of a linear form π on the finite-dimensional vector space $\mathcal{V}$. For any $\mathbb{P}^* \in \mathcal{P}$ we have

$$\pi(V) = \mathbb{E}^* \left[\frac{V}{1+r} \right].$$

For an attainable payoff V such that $\pi(V) \neq 0$ the *return* of V is defined by

$$R(V) = \frac{V - \pi(V)}{\pi(V)}.$$

For the special case of the risk free asset S^0 we have that

$$r = \frac{S^0 - \pi^0}{\pi^0}.$$

It turns out that in an arbitrage free market the expected return under any EMM equals the risk free rate.

Proposition 1.2.9 *Suppose that the market model is free of arbitrage and let $V \in \mathcal{V}$ be an attainable payoff with price $\pi(V) \neq 0$.*

(i) Under any $\mathbb{P}^* \in \mathcal{P}$ the expected return of V is equal to the risk free rate:

$$\mathbb{E}^*[R(V)] = r.$$

(ii) Under any measure $\mathbb{Q} \approx \mathbb{P}$ such that $\mathbb{E}_{\mathbb{Q}}[|\bar{S}|] < \infty$ the expected return is given by

$$\mathbb{E}_{\mathbb{Q}}[R(V)] = r - \mathrm{cov}_{\mathbb{Q}} \left(\frac{d\mathbb{P}^*}{d\mathbb{Q}}, R(V) \right)$$

where $\mathbb{P}^*$ is any martingale measure and $\mathrm{cov}_{\mathbb{Q}}$ is the covariance with respect to $\mathbb{Q}$.

Remark 1.2.10 *So far we considered only the (standard) case where the riskless bond is the numéraire, that is, where all prices were expressed in terms of shares of the bond. The numéraire can be changed and prices be quoted in terms of, for instance, the 1st asset (provided its price S^1 is almost surely strictly positive) without altering the main results of this section. In order to see this, let*

$$\tilde{\pi}^i = \frac{\pi^i}{\pi^1} \quad \text{and} \quad \tilde{S}^i = \frac{S^i}{S^1}.$$

The no arbitrage condition with respect to the new numéraire reads:

$$\tilde{\pi}^i \overset{!}{=} \tilde{\mathbb{E}}^*\left[\tilde{S}^i\right] = \tilde{\mathbb{E}}^*\left[\frac{S^i}{S^1}\right].$$

It is satisfied, for instance, for the measure $\tilde{\mathbb{P}}^*$ with density

$$\frac{d\tilde{\mathbb{P}}^*}{d\mathbb{P}^*} = \frac{S^1}{\mathbb{E}^*[S^1]} \quad (\mathbb{P}^* \in \mathcal{P}).$$

1.2.3 Derivative securities

In real financial markets not only primary but also a large variety of *derivative securities* such as options and futures are traded. A derivative's payoff depends in a possibly non-linear way on the primary assets $S^0, S^1, \ldots, S^n$.

Definition 1.2.11 *A contingent claim is a random variable C on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$0 \le C < \infty \quad \mathbb{P}\text{-a.s.}$$

A contingent claim is called a derivative *if*

$$C = f(S^0, S^1, \ldots, S^d)$$

for some non-negative measurable function f on $\mathbb{R}^{d+1}$.

Example 1.2.12 *(i) For a European Call Option on the first risky asset with strike K we have that*

$$f(S^1) = (S^1 - K)^+.$$

(ii) For a European Put Option on the first risky asset with strike K we have that

$$f(S^1) = (K - S^1)^+.$$

By the call-put-parity discussed in the introduction the price of a put option is determined by the price of the corresponding call option and vice versa.

(iii) For a forward contract *on the first risky asset with strike K we have that*

$$f(S^1) = S^1 - K.$$

(iv) *A straddle is a bet that a the price $\pi(V)$ of a portfolio with payoff V moves, no matter in what direction:*

$$C = |V - \pi(V)|.$$

A butterfly spread, by contrast is a bet that the price does not move much: for $0 \le a \le b$

$$C = (V-a)^+ + (V-b)^+ - 2(V-(a+b)/2)^+.$$

(v) *A reverse convertible bond pays interest that is higher than that paid by a riskless bond. However, at maturity the issuer has the right to convert the bond into a predetermined number of shares of a given asset S^i rather than paying the nominal value in cash. Suppose that the reverse convertible bond trades at \$1 at $t = 0$, that its nominal value at maturity $t = 1$ is $1 + \tilde{r}$ and that it con be converted into x shares of the i-th asset. The conversion will happen if*

$$S^i < K := \frac{1+\tilde{r}}{x}.$$

As a result, the purchase of the bond is equivalent to a risk-free investment of 1 with interest r and the sale of x put options with payoff $(K - S^i)$ for a unit price $(\tilde{r} - r)/(1+r)$.

Our goal is to identify those possible prices for C which do not generate arbitrage opportunities. To this end we observe that trading C at time 0 for a price π^C corresponds to introducing a new asset with prices

$$\pi^{d+1} := \pi^C \quad \text{and} \quad S^{d+1} = C.$$

We call π^C an *arbitrage free price* of C if the market model extended in this manner is free of arbitrage. The set of all arbitrage free prices is denoted $\Pi(C)$. It can be characterized in terms of the equivalent martingale measures.

Theorem 1.2.13 *Suppose that $\mathcal{P} \neq \emptyset$. Then the following holds:*

$$\Pi(C) = \left\{ \mathbb{E}^*\left[\frac{C}{1+r}\right] \ : \ \mathbb{P}^* \in \mathcal{P} \text{ such that } \mathbb{E}^*[C] < \infty. \right\} \neq \emptyset. \tag{1.4}$$

Arbitrage bounds and the super-hedging price

The preceding theorem yields a *unique* arbitrage free price of a contingent claim C if and only if there exists a unique EMM. For the special case of a finite set $\Omega = \{\omega_1, \dots, \omega_n\}$ and a single risky asset this the case only if Ω has at most two elements; see (1.2). In general $\Pi(C)$ is a convex open interval:

$$\Pi(C) = (\pi_{\min}(C), \pi_{\max}(C)).$$

Our next goal is thus to identify the *arbitrage bounds* $\pi_{\min}(C)$ and $\pi_{\max}(C)$. We first identify the upper bound $\pi_{max}(C)$.

Lemma 1.2.14 *For an arbitrage free market model the upper arbitrage bound is given by*

$$\begin{aligned} \pi_{\max}(C) &= \sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*\left[\frac{C}{1+r}\right] \\ &= \min\{m \in [0,\infty] \ : \ \exists \xi \in \mathbb{R}^d \ s.t. \ (1+r)m + \xi \cdot Y \ge C \ \ \mathbb{P}\text{-}a.s.\} \end{aligned} \tag{1.5}$$

Remark 1.2.15 *When calculating the arbitrage bounds we may as well take the inf and sup over the set of risk neutral measures that are merely absolutely continuous with respect to* $\mathbb{P}$*.*

The upper arbitrage bound $\pi_{max}(C)$ is the so-called *super-hedging price*. This is the minimal amount of money the writer of the option has to ask for in order to be able to buy a portfolio $\bar{\xi}$ that allows her to meet her obligations from selling the option in any state of the world. In many situations this prince is trivial. The writer of a Call option, for instance, can always hedge her risk from selling the option via a "buy-and-hold-strategy", that is, by buying the underlying asset S^i at π^i in $t = 0$ and holding it until maturity. The following example shows that π^i may in fact be the super-hedging price.

Example 1.2.16 *Consider a market model with a single risky asset* S^1 *and assume that under the real world measure* $\mathbb{P}$ *the random variable* S^1 *has a Poisson distribution:*

$$\mathbb{P}[S^1 = k] = \frac{e^{-1}}{k!} \quad (k = 0, 1, \ldots)$$

For $r = 0$ *and* $\pi = 1$ *the measure* $\mathbb{P}$ *is risk neutral*[1] *so the model is arbitrage free. By Remark 1.2.15 we may take the inf and sup over the set of risk neutral measures that are merely absolutely continuous with respect to* $\mathbb{P}$*. Let us then consider probability measures* $\tilde{\mathbb{P}}_n$ *with densities*

$$f_n(k) := \frac{1}{n} + \left(1 - \frac{1}{n}\right) \cdot g_n(k)$$

where

$$g_n(k) := \left(e - \frac{e}{n}\right) * \mathbf{1}_{\{0\}}(k) + (n-1)! * e * \mathbf{1}_{\{n\}}(k) \quad (k = 0, 1, 2, \ldots).$$

Then

$$\begin{aligned}
\int f_n(k) d\mathbb{P} &= \frac{1}{n} + \int \left(1 - \frac{1}{n}\right) g_n(k) d\mathbb{P} \\
&= \frac{1}{n} + \left(1 - \frac{1}{n}\right) \left\{ \left(e - \frac{e}{n}\right) e^{-1} + (n-1)! * e \frac{e^{-1}}{n!} \right\} \\
&= \frac{1}{n} + \left(1 - \frac{1}{n}\right) = 1
\end{aligned}$$

so that f_n *is indeed a density. It is now straightforward to check that*

$$\tilde{\mathbb{E}}_n[(S^1 - K)^+] = \frac{1}{n} \mathbb{E}[(S^1 - K)^+] + \left(1 - \frac{1}{n}\right) \left(1 - \frac{K}{n}\right)^+.$$

Letting $n \to \infty$ *we that the upper arbitrage bound is in fact attained:*

$$\pi_{\max}((S^1 - K)^+) = \lim_{n \to \infty} \left(1 - \frac{K}{n}\right)^+ = 1 = \pi.$$

The next example establishes upper and lower arbitrage bounds for derivatives whose payoff is a convex function of some underlying.

Example 1.2.17 *Let* V *be the payoff of some portfolio and* $f : [0, \infty) \to \mathbb{R}_+$ *convex such that*

$$\beta := \lim_{x \to \infty} \frac{f(x)}{x}$$

[1] Recall that the expected value of a standard Poisson random variable is 1.

exists and is finite. Convexity of f implies that

$$f(x) \leq \beta x + f(0)$$

so the derivative with payoff $C(V) = f(V)$ satisfies

$$C \leq \beta V + f(0).$$

Thus, any arbitrage free price π^C satisfies

$$\pi^C = \mathbb{E}^*\left[\frac{C}{1+r}\right] \leq \beta\pi(V) + f(0).$$

In order to obtain a lower bound, we apply Jensen's inequality to obtain

$$\pi^C \geq \frac{1}{1+r} f\left(\mathbb{E}^*[V]\right) = \frac{1}{1+r} f\left(\pi(V)(1+r)\right).$$

We close this section with a result on the lower arbitrage bound. The proof proceeds by analogy to that of Lemma 1.2.14.

Lemma 1.2.18 *For an arbitrage free market model the lower arbitrage bound is given by*

$$\begin{aligned} \inf \Pi(C) &= \inf_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*\left[\frac{C}{1+r}\right] \\ &= \max\{m \in [0,\infty] \;:\; \exists \xi \in \mathbb{R}^d \text{ s.t. } (1+r)m + \xi \cdot Y \leq C \quad \mathbb{P}\text{-a.s.}\} \end{aligned} \tag{1.6}$$

Attainable claims

For a portfolio $\bar{\xi}$ the resulting payoff $V = \bar{\xi} \cdot \bar{S}$, if positive, may be viewed as a contingent claim. Those claims that can be replicated by a suitable portfolio will play a special role in the sequel.

Definition 1.2.19 *A contingent claim C is called* attainable (replicable, redundant) *if $C = \bar{\xi} \cdot \bar{S}$ for some $\bar{\xi} \in \mathbb{R}^{d+1}$. The portfolio $\bar{\xi}$ is called a* replicating portfolio *for C.*

If a contingent claims is attainable the law of one price implies that the price of C equals the cost of its replicating portfolio. The following corollary shows that the attainable contingent claims are the only one that admit a *unique* arbitrage free price.

Corollary 1.2.20 *Suppose that the market model is free of arbitrage and that C is a contingent claim.*

(i) C is attainable if and only if it admits a unique arbitrage free price.

(ii) If C is not attainable, then $\pi_{\min}(C) < \pi_{\max}(C)$ and hence there is a non-trivial interval of possible arbitrage free prices.

Example 1.2.21 *(Portfolio Insurance) The idea of portfolio insurance is to enhance exposure to rising prices while reducing exposure to falling prices. For a portfolio with payoff $V \geq 0$ it is this natural to consider a payoff $h(V)$ for a convex increasing function h. Since a convex function is almost surely differentiable and h can be expressed*

$$h(x) = h(0) + \int_0^x h'(y)\,dy \quad \text{where } h' \text{ is the right-hand derivative.}$$

Furthermore $h'' > 0$ so h' is increasing. Hence h' can be represented as the distribution functions of a positive measure γ:

$$h'(x) = \gamma([0, x]).$$

Thus, Fubini's theorem yields

$$\begin{aligned} h(x) &= h(0) + \int_0^x \int_0^y \gamma(dz)\, dy \\ &= h(0) + \gamma(\{0\})x + \int_{(0,\infty)} \int_{\{y: z \le y \le x\}} dy\, \gamma(dz) \\ &= h(0) + \gamma(\{0\})x + \int_{(0,\infty)} (x - z)^+ \gamma(dz). \end{aligned}$$

Thus,

$$\begin{aligned} h(V) &= h(0) + \gamma(\{0\})V + \int_{(0,\infty)} (V - z)^+ \gamma(dz) \\ &= \textit{investment in the bond + direct investment in } V \textit{ + investment in call options.} \end{aligned}$$

If call options on V with arbitrary are available in the market the payoff $h(V)$ is replicable.

In general there is no reason to assume that a contingent claim is attainable. Typical examples are CAT- (catastrophic) bonds which are often written on non-tradable underlyings such as weather or climate phenomena.

Example 1.2.22 *A couple of years ago, the Swiss insurance company* Winterthur *issued a bond that paid a certain interest $\tilde{r} > r$ but only if - within a certain period of time - the number of cars insured by* Winterthur *and damaged due to hail-storms within a 24 hour period did not exceed some threshold level. This bond allowed* Winterthur *to transfer insurance related risks to the capital markets. Apparently its payoff cannot be replicated by investments in the financial markets alone.*

It turns out that for stochastic models in discrete time the paradigm of a complete market where all contingent claims admit a perfect hedge is the exception rather than the rule. The exception of a complete market is discussed in the following section.

1.2.4 Complete market models and perfect replication

In this section we characterize the more transparent situation in which all contingent claims are attainable and hence allow for a unique price.

Definition 1.2.23 *An arbitrage free market is called* complete *if every contingent claim is attainable. That is, for every claim C there exists a portfolio $\bar{\xi} \in \mathbb{R}^{d+1}$ such that*

$$C = \xi \cdot \bar{S} \quad \mathbb{P}\text{-}a.s.$$

In a complete arbitrage free market any claim is integrable with respect to any EMM. In particular, the set $\mathcal{V}$ of attainable claims defined in (1.3) satisfies

$$\mathcal{V} \subseteq L^1(\mathbb{P}^*) \subset L^0(\mathbb{P}^*) = L^0(\mathbb{P}).$$

In a complete market *any* claim $C \subset L^0(\mathbb{P})$ is attainable so the above inequalities are in fact equalities. Since $\mathcal{V} \subset \mathbb{R}^{d+1}$ is finite dimensional the same must be true for $L^0(\mathbb{P})$. As a result, $L^0(\mathbb{P})$ has at most $d+1$ *atoms*; an atom of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is an element $A \in \mathcal{F}$ that contains no measurable subset of positive measure.

Lemma 1.2.24 *For all $p \in [0, \infty]$ the dimension of L^p is given by*

$$\dim L^p = \sup\{n : \exists \text{ partition } A^1, \ldots, A^n \text{ of } \Omega \text{ with } \mathbb{P}[A^i] > 0\}.$$

We are now ready to show that a market model is complete if and only if there exists a unique EMM.

Theorem 1.2.25 *An arbitrage free market model is complete if and only if $|\mathcal{P}| = 1$. In this case* $\dim L^0 \leq d+1$.

For a model with finitely many states of the world and one risky asset the previous theorem implies that a market model is complete only when Ω consists of only two elements ω^+ and ω^-. With $p := \mathbb{P}[\{\omega^+\}]$ and $S(\omega^+) = b$ and $S(\omega^-) = a$ any risk neutral measure $(p^*, 1-p^*)$ must satisfy

$$\pi(1+r) = \mathbb{E}^*[S] = a(1-p^*) + bp^*.$$

Hence the risk neutral measure is given in terms of p^* by

$$p^* = \frac{\pi(1+r) - a}{b-a}.$$

The arbitrage free price for a call option $C = (S-K)^+$ with strike $K \in [a, b]$ is given by the expected discounted payoff under the risk neutral measure:

$$\pi(C) = \frac{b-K}{b-a} \cdot \pi - \frac{(b-K)a}{b-a} \cdot \frac{1}{1+r}.$$

Notice that the price does not depend on the real world probability p. In fact, we observed earlier that the real world measure enters the set of EMMs only through its null sets, a trivial condition if Ω is finite. Furthermore $\pi(C)$ is *increasing* in the risk-free rate while the classical discounted expectation with respect to the objective measure p is *decreasing* in r because

$$\mathbb{E}\left[\frac{C}{1+r}\right] = \frac{p(b-K)}{1+r}.$$

The central result of this section is that arbitrage free pricing requires the price of a contingent claim to be calculated as the discounted expected payoff with respect to an equivalent martingale measure rather than the objective, real-world measure.

1.2.5 Good deal bounds and super-deals

So far, the only condition we imposed on our financial market model was the assumption of no arbitrage. Ruling out arbitrage opportunities we ruled out "infinitely good deals". In this section we discuss a refinement of the no free lunch condition, due to Carr et al. (2001). More precisely, we consider a financial market model $(\bar{\pi}, \bar{S})$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In addition to the real world measure $\mathbb{P}$ we are also given *valuation measures*

$$\mathbb{Q}_i << \mathbb{P} \quad (i = 1, \ldots, n)$$

that satisfy $\mathbb{E}_{\mathbb{Q}_i}[|Y^1|] < \infty$. The valuation measures can be viewed as capturing *uncertainty* about the - perhaps unknown and/or misspecified "real world" measure $\mathbb{P}$. We put

$$\mathcal{Q} := \{\mathbb{Q}_1, \ldots, \mathbb{Q}_n\}$$

and assume that

$$\mathcal{Q} \approx \mathbb{P} \quad \text{in the sense that for all } A \in \mathcal{F} \text{ with } \mathbb{P}[A] > 0 \text{ there exists } i \text{ s.t. } \mathbb{Q}_i[A] > 0.$$

We are now ready to define the notion of a good deals and super deals.

Definition 1.2.26 *A portfolio ξ (or the associated net payoff $\xi \cdot Y$) is called a good deal if*

$$\mathbb{E}_{\mathbb{Q}_i}[\xi \cdot Y] \geq 0 \quad \textit{for all } i \in \{1, \ldots, n\}.$$

It is called a super deal if, in addition, $\mathbb{E}_{\mathbb{Q}_{i_0}}[\xi \cdot Y] > 0$ for at least one $i_0 \in \{1, \ldots, n\}$. The model is called $\mathcal{Q}$-normal if no super deals exist.

Thus, a portfolio is a good deal if it yields an non-negative expected payoff under all possible models. It is a super deal if it yields a strictly positive payoff under at least one possible model. The following remark shows that notion of $\mathcal{Q}$-normality is in fact a refinement of the no free lunch condition.

Remark 1.2.27 *Every free lunch is a super-deal.*

The notion of $\mathcal{Q}$-normal prices follows the definition of arbitrage-free prices.

Definition 1.2.28 *Let C be a contingent claim. A price π^C is called a $\mathcal{Q}$-normal price if the extended financial market model is $\mathcal{Q}$-normal. The set of all $\mathcal{Q}$-normal prices is denoted $\Pi^{\mathcal{Q}}(C)$.*

Our goal is now to characterize the set $\Pi^{\mathcal{Q}}(C)$ in terms of a subset of equivalent martingale measures. More specifically, our aim is a characterization of the form

$$\Pi^{\mathcal{Q}}(C) = \left\{\mathbb{E}^*\left[\frac{C}{1+r}\right] : \mathbb{P}^* \in \mathcal{P} \cap \mathcal{R}\right\} \tag{1.7}$$

for some set $\mathcal{R}$. It will turn out that $\mathcal{R}$ is given by the representative mixtures of the valuation measures so we define:

$$\mathcal{R} := \left\{\sum_{i=1}^{n} \lambda_i \mathbb{Q}_i : \lambda_i > 0, \sum_{i=1}^{n} \lambda_i = 1\right\}.$$

We notice that the valuation measures do *not* belong to $\mathcal{R}$ because all the λ_i are strictly positive. Furthermore, $\mathcal{R}$ is convex and each element from the class $\mathcal{R}$ is equivalent to $\mathbb{P}$.

Proposition 1.2.29 *The financial market model is $\mathcal{Q}$-normal if and only if $\mathcal{P} \cap \mathcal{R} \neq \emptyset$.*

We are now ready to state and prove the main result of this section.

Theorem 1.2.30 *If the financial market model is $\mathcal{Q}$-normal, then the characterization (1.7) of $\mathcal{Q}$-normal prices holds.*

In the sequel the following notation turns out to be useful. We denote by $L^1(\mathcal{Q})$ the class of all random variables that are integrable with respect to every valuation measure. For $X, Y \in L^1(\mathcal{Q})$ we write

$$X \geq_{\mathcal{Q}} Y \quad \text{iff} \quad \mathbb{E}_{\mathbb{Q}_i}[X] \geq \mathbb{E}_{\mathbb{Q}_i}[Y] \quad (i = 1, ..., n).$$

In terms of this notation X is a good deal if and only if

$$X \geq_{\mathcal{Q}} 0.$$

Let us now denote by $\pi^{\mathcal{Q}}_{\max}(C)$ the largest possible $\mathcal{Q}$-normal price of the claim C:

$$\pi^{\mathcal{Q}}_{\max}(C) = \sup_{\mathbb{Q} \in \mathcal{P} \cap \mathcal{R}} \mathbb{E}_{\mathbb{Q}}\left[\frac{C}{1+r}\right].$$

This quantity is finite because $\max_i \mathbb{E}_{\mathbb{Q}_i}[C] < \infty$. By analogy to the superhedgeing price $\pi^{\mathcal{Q}}_{\max}(C)$ can be viewed as the minimal cost such that the seller of C can generate a good deal.

Theorem 1.2.31 *Let $C \in L^1(\mathcal{Q})$. Then*

$$\pi^{\mathcal{Q}}_{\max}(C) = \min\left\{m : \exists\, \xi \in \mathbb{R}^d \,:\, m + \xi \cdot Y \geq_{\mathcal{Q}} \frac{C}{1+r}\right\}.$$

The proof of this theorem is based on the following two auxiliary results.

Lemma 1.2.32 *Let the financial market model be $\mathcal{Q}$-normal. For $X \in L^1(\mathcal{Q})$ the set*

$$\mathcal{C} := \left\{(\mathbb{E}_{Q_i}[\xi \cdot Y])_{i=1}^n + y \,:\, y \in \mathbb{R}^n_+,\, \xi \in \mathbb{R}^d\right\} \subset \mathbb{R}^n$$

is a closed convex cone that contains $\mathbb{R}_+$.

Lemma 1.2.33 *Let $\mathcal{A}$ be the class of all contingent claims which, when combined with a suitable portfolio, yield a good deal:*

$$\mathcal{A} := \{X \in L^1(\mathcal{Q}) \,:\, \exists\, \xi \,:\, X + \xi \cdot Y \geq_{\mathcal{Q}} 0\}.$$

Furthermore, let $\bar{\mathcal{R}}$ be the convex hull of $\mathcal{Q}$. Then $\mathcal{A} = \mathcal{A}^*$ where

$$\mathcal{A}^* := \{X \in L^1(\mathcal{Q}) \,:\, \mathbb{E}^*[X] \geq 0\, \forall\, \mathbb{P}^* \in \mathcal{P} \cap \bar{\mathcal{R}}\}.$$

We are now ready to prove the upper good deal bound.

Proof of Theorem 1.2.31: For a portfolio $\xi \in \mathbb{R}^d$ the following is equivalent, due to the receding lemma:

$$\begin{aligned} m + \xi \cdot Y \geq_{\mathcal{Q}} \frac{C}{1+r} \quad &\Leftrightarrow \quad m - \frac{C}{1+r} \in \mathcal{A} \\ &\Leftrightarrow \quad m - \frac{C}{1+r} \in \mathcal{A}^* \\ &\Leftrightarrow \quad m - \mathbb{E}^*\left[\frac{C}{1+r}\right] \geq 0 \quad \text{for all } \mathbb{P}^* \in \mathcal{P} \cap \mathcal{R} \\ &\Leftrightarrow \quad m \geq \sup_{\mathbb{P}^* \in \mathcal{P} \cap \mathcal{R}} \mathbb{E}^*\left[\frac{C}{1+r}\right]. \end{aligned}$$

□

1.2.6 Indifference valuation and risk sharing

In this section we discuss briefly the idea of indifference valuation and risk sharing in an exponential framework. Specifically, we consider agents with exponential utility function

$$U(X) = -\mathbb{E}[e^{-\alpha X}] \quad \text{where} \quad X \in L^\infty(\mathbb{P})$$

and $\alpha > 0$ denotes the agent's coefficient of risk aversion. For much of the subsequent analysis it will be convenient to consider instead the associate entropic utility functional

$$\varrho_\alpha(X) = -\frac{1}{\alpha} \ln \mathbb{E}[e^{-\alpha X}]$$

that describes the same preferences as U but has the additional translation invariance property

$$\varrho_\alpha(X + m) = \varrho_\alpha(X) + m \quad \text{for all} \quad m \in \mathbb{R}.$$

The preceding property we can easily identify the buyer's *indifference price* $\pi_B(C)$ of a claim C for an agent with initial endowment $H_B \in L^\infty(\mathbb{P})$ and preferences described by ϱ_{α_B}. It is defined as via

$$\varrho_{\alpha_B}(H_B + C - \pi_B(C)) = \varrho_{\alpha_B}(H_B),$$

so that

$$\pi_B(C) = \varrho_{\alpha_B}(H_B + C) - \varrho_{\alpha_B}(H_B).$$

By analogy the seller's indifference price is given by

$$\pi_S(C) = \varrho_{\alpha_S}(H_S + C) - \varrho_{\alpha_S}(H_S).$$

Thus, if $\pi_B(C) \leq \pi_S(C)$, then any price

$$\pi \in [\pi_B(C), \pi_S(C)] \subset \Pi(C)$$

is a possible transaction price.

Remark 1.2.34 *(i) Indifference prices are mere benchmarks beyond which no trade takes place, i.e., they are typically not openly quoted market prices.*

(ii) The concept of indifference prices yields a typically non-linear valuation scheme. That is, the maps

$$C \mapsto \pi_B(C), \quad C \mapsto \pi_S(C)$$

are typically non-linear, due to their dependence of (typically) concave utility functions and initial endowments.

We are now going to consider the problem of an insurance company or investment bank ("Agent A") that is initially exposed to a random income H of optimally transferring some of its risk to the financial market ("Agent B") subject to an individual rationality condition. More precisely, we consider the optimization problem

$$\sup_{F \in L^\infty, \pi \in \mathbb{R}} \varrho_A(H - F + \pi)$$

subject to

$$\varrho_B(F - \pi) \geq \varrho_B(0) \geq 0.$$

The latter equation yields $\pi(F) = \varrho_B(H)$ so this problem can be rewritten as

$$\sup_{F \in L^\infty} \{\varrho_A(H - F) + \varrho_B(F)\}.$$

The function

$$H \mapsto R(H) := \sup_{F \in L^\infty} \{\varrho_A(H - F) + \varrho_B(F)\}$$

can be viewed as the utility functional of some representative agent or social planer whose goal is to redistribute the total endowment H among the agent in such a way that the aggregate utility is maximal.

A similar optimization problem can be stated for the case of exponential utility functions and in this case the optimal claim F^* can be given in closed form. This result is known as Borch's theorem and shows that the agents redistribute the total endowment among them according to their relative risk aversions. More precisely, we have the following result.

Theorem 1.2.35 *(i) Up to a constant The solution F^* to the optimization problem*

$$\sup_{F,\pi} -\frac{1}{\alpha_A}\mathbb{E}[e^{-\alpha_A(H-F+\pi)}] \quad s.t. \quad -\frac{1}{\alpha_B}\mathbb{E}[e^{-\alpha_B(F-\pi)}] \geq -\frac{1}{\alpha_B}$$

is almost surely given by

$$F^* = \frac{\alpha_A}{\alpha_A + \alpha_B} H.$$

(ii) The utility function $R(H)$ is given by

$$R(H) = \varrho_{\alpha_C}(H) \quad with \quad \alpha_C^{-1} = \alpha_A^{-1} + \alpha_B^{-1}.$$

1.3 Dynamic hedging in discrete time

We are now going to develop a dynamic version of the arbitrage theory of the previous Chapter. Here we are in a multi-period setting, where the financial price fluctuations are described by a stochastic process. This section follows Chapter 5 of Föllmer & Schied (2004).

1.3.1 The multi-period market model

Throughout we consider a financial market in which $d+1$ assets are priced at time $t = 0, 1, \ldots, T$. The price of the ith asset at time t is modelled as a non-negative random variable S_t^i on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The random vector

$$\bar{S}_t = (S_t^0, S_t) = (S_t^0, \ldots, S_t^d)$$

is measurable with respect to a σ-field $\mathcal{F}_t \subset \mathcal{F}$. We think of $\mathcal{F}_t$ as the set of all events that are observable up to and including time t. It is hence natural to assume that

$$\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_T.$$

Such a family of σ-fields is called a *filtration* and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ is called a *filtered probability space*. To simplify the presentation we assume that

$$\mathcal{F}_0 = \{\varnothing, \Omega\} \quad \text{and} \quad \mathcal{F}_T = \mathcal{F}.$$

Definition 1.3.1 *Let $Y = (Y_t)_{t=0}^T$ be a stochastic process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$.*

(i) The process Y is called adapted *with respect to $(\mathcal{F}_t)_{t=0}^T$ if each Y_t is measurable with respect to $\mathcal{F}_t$.*

(ii) The process Y is called predictable *with respect to $(\mathcal{F}_t)_{t=0}^T$ if for $t \geq 1$ each Y_t is measurable with respect to $\mathcal{F}_{t-1}$.*

The asset price process $(\bar{S}_t)$ forms an adapted process with values in $\mathbb{R}^{d+1}$. That is to say that

$$S_t : (\Omega, \mathcal{F}_t) \to (\mathbb{R}^{d+1}, \mathcal{B}^{d+1})$$

where $\mathcal{B}$ denotes the Borel σ-field on the real line. By convention the 0-th asset is a riskless asset such as a government bond or a bank account and

$$S_0^0 = 1.$$

Its returns is denoted by $r \geq 0$ so

$$S_t^0 = (1+r)^t.$$

The *discount factor* is $(S_t^0)^{-1}$ and the *discounted price processes* will be denoted

$$X_t^i = \frac{S_t^i}{S_0^0}.$$

Then $X_t^0 \equiv 1$ and $X_t = (X_t^1, \ldots, X_t^d)$ expresses the value of the remaining assets in units of the *numeraire* S_t^0.

Definition 1.3.2 *A trading strategy is a predictable $\mathbb{R}^{d+1}$-valued process*

$$\bar{\xi} = (\xi^0, \xi) = (\xi_t^0, \xi_t^1, \ldots, \xi_t^d)_{t=1}^T$$

where ξ_t^i corresponds to the number of shares of the i-th asset held during the t-th trading period between $t-1$ and t.

Remark 1.3.3 *Economically, the fact that ξ_t^i is predictable, i.e., $\mathcal{F}_{t-1}$-measurable means that portfolio decisions at time t are based on the information available at time $(t-1)$ and that portfolios are kept until time t when new quotations become available.*

The total value of the portfolio $\bar{\xi}_t$ at time $t-1$ is

$$(\bar{\xi}_t \cdot \bar{S}_{t-1})(\omega) = \sum_{i=0}^{d} \xi_t^i(\omega) S_{t-1}^i(\omega).$$

By time t the value has changed to

$$(\bar{\xi}_t \cdot \bar{S}_t)(\omega) = \sum_{i=0}^{d} \xi_t^i(\omega) S_t^i(\omega).$$

If no funds are added or removed for consumption purposes the trading strategy is *self-financing*, that is,

$$\bar{\xi}_t \cdot \bar{S}_t = \bar{\xi}_{t+1} \cdot \bar{S}_t \quad \text{for} \quad t = 1, \ldots, T-1.$$

In particular the accumulated gains and losses resulting from the asset price fluctuations are the only source of variations of the portfolio value:

$$\bar{\xi}_t \cdot \bar{S}_t - \bar{\xi}_{t-1} \cdot \bar{S}_{t-1} = \bar{\xi}_t \cdot (\bar{S}_t - \bar{S}_{t-1}). \tag{1.8}$$

In fact, it is easy to see that a portfolio is self financing if and only if (1.8) holds.

Remark 1.3.4 *An important special case are the price dynamics of a money market account. Let r_k be the "short rate", i.e., the short term interest rate for the period $[k, k+1)$. Then*

$$S_t^0 = \prod_{k=1}^{t} (1 + r_k).$$

Typically, the short rate is predictable so (S_t^0) is predictable as well. Notice that in contrast to a savings account the short rate may change stochastically over time, though. For a savings account, one would usually assume that $r_k \equiv r$.

The *discounted value process* $V = (V_t)_{t=0}^T$ associated with a trading strategy $\bar{\xi}$ is given by

$$V_0 = \bar{\xi}_1 \cdot \bar{X}_0 \quad \text{and} \quad V_t = \bar{\xi}_t \cdot \bar{X}_t \quad \text{for} \quad t = 1, \ldots, T$$

while the associated *gains* process is defined by

$$G_0 = 0 \quad \text{and} \quad G_t = \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1}) \quad \text{for} \quad t = 1, \ldots, T.$$

The notion of a self-financing trading strategy can be expressed in terms of the value and gains processes as shown by the following proposition.

Proposition 1.3.5 *For a trading strategy $\bar{\xi}$ the following conditions are equivalent:*

(i) $\bar{\xi}$ is self-financing.

(ii) $\bar{\xi}_t \cdot \bar{X}_t = \bar{\xi}_{t+1} \cdot \bar{X}_t$ *for* $t = 1, \ldots, T-1$.

(iii) $V_t = V_0 + G_t$ *for all* t.

Remark 1.3.6 *The numéraire component of a self-financing portfolio* $\bar{\xi}$ *satisfies*

$$\xi^0_{t+1} - \xi^0_t = -(\xi_{t+1} - \xi_t) \cdot X_t.$$

Since $\xi^0_1 = V_0 - \xi_1 \cdot X_0$ *we see that the entire process* ξ^0 *is determined by the initial investment* V_0 *along with the process* ξ. *As a result, when* V_0 *and a d-dimensional predictable process* ξ *are given, we can define a predictable process* ξ^0 *such that* $\bar{\xi} = (\xi^0, \xi)$ *is a self-financing strategy. In dealing with self-financing strategies it is thus sufficient to focus on initial investments and holdings in the risky assets.*

1.3.2 Arbitrage opportunities and martingale measures

An arbitrage opportunity is an investment strategy that yields a positive profit with positive probability but without any downside risk.

Definition 1.3.7 *A self-financing trading strategy is called an* arbitrage opportunity *if the associated value process satisfies*

$$V_0 \le 0, \quad \mathbb{P}[V_T \ge 0] = 1 \quad \text{and} \quad \mathbb{P}[V_T > 0] > 0.$$

In this section we characterize arbitrage-free market models, i.e., those models that do not allow arbitrage opportunities. It will turn out that a model is free of arbitrage if and only if there exists a probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ such that the discounted asset prices are martingales with respect to $\mathbb{P}^*$.

Definition 1.3.8 *A stochastic process* $M = (M_t)$ *on a filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ *is called a* martingale *if* M *is adapted, satisfies* $\mathbb{E}_{\mathbb{Q}}[|M_t|] < \infty$ *for all* t *and*

$$M_t = \mathbb{E}_{\mathbb{Q}}[M_{t+1} | \mathcal{F}_t].$$

The process is called a sub- and super-martingale, respectively, if, respectively,

$$M_t \le \mathbb{E}_{\mathbb{Q}}[M_{t+1} | \mathcal{F}_t] \quad \text{and} \quad M_t \ge \mathbb{E}_{\mathbb{Q}}[M_{t+1} | \mathcal{F}_t].$$

A martingale can be regarded as the mathematical formalization of a fair game: for each time s and any time horizon $t - s > 0$, the conditional expectation of the future gain $M_t - M_s$ is zero, given the information available at time s. In our context of a finite time horizon a martingale M arises as a sequence of successive conditional expectations $\mathbb{E}_{\mathbb{Q}}[F|\mathcal{F}_t]$ for some $F \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q})$. Whether or not M is a martingale depends on the underlying probability measure and filtration. While the filtration will be fixed throughout this section, we shall deal with an array of different probability measures. To emphasize the dependence of the martingale on $\mathbb{Q}$ we say that M is a $\mathbb{Q}$-martingale or a martingale with respect to $\mathbb{Q}$.

Definition 1.3.9 *A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ is called a martingale measure if the discounted price process X is a d-dimensional martingale with respect to $\mathbb{Q}$. A martingale measure $\mathbb{P}^*$ is called an equivalent martingale measure (EMM) if it is equivalent to $\mathbb{P}$. The set of equivalent martingale measures will be denoted $\mathcal{P}$.*

The following result states that a fair game admits no realistic gambling system which produces a positive expected gain. Here Y^- denotes the negative part of a random variable Y.

Theorem 1.3.10 *For a probability measure $\mathbb{Q}$ the following conditions are equivalent:*

(i) $\mathbb{Q}$ *is a martingale measure.*

(ii) *If $\bar{\xi}$ is self-financing and ξ is bounded, then the value process associated with $\bar{\xi}$ is a $\mathbb{Q}$-martingale.*

(iii) *If $\bar{\xi}$ is self-financing and the associated value process V satisfies $\mathbb{E}_{\mathbb{Q}}[V_T^-] < \infty$, then V is a $\mathbb{Q}$-martingale.*

(iv) *If $\bar{\xi}$ is self-financing and the associated value process V satisfies $V_T \geq 0$ $\mathbb{Q}$-a.s. then $\mathbb{E}_{\mathbb{Q}}[V_T] = V_0$.*

Remark 1.3.11 *If the "real-world-measure" $\mathbb{P}$ is itself a martingale measure, the preceding theorem shows that there are no realistic self-financing strategies that would generate a positive expected gain. The assumption $\mathbb{P} \in \mathcal{P}$ is a version of the* efficient market hypothesis. *This hypothesis implies that risk-averse investors would not be attracted towards investing into risky assets if their expectations are consistent with $\mathbb{P}$.*

We are now ready to state the following dynamic version of the fundamental theorem of asset pricing which links the absence of arbitrage to the existence of equivalent martingale measures.

Theorem 1.3.12 *The market model is free of arbitrage if and only if $\mathcal{P} \neq \emptyset$. In this case there exists $\mathbb{P}^* \in \mathcal{P}$ with bounded density $d\mathbb{P}^*/d\mathbb{P}$.*

To prove the preceding result we first state a proposition that shows that a market model is arbitrage free if and only if there are no arbitrage opportunities for each single trading period. Its proof is left as an exercise.

Proposition 1.3.13 *The market model admits an arbitrage opportunity if and only if there exists $t \in \{1, 2, \ldots, T\}$ and a "trading strategy" $\eta \in L^0(\Omega, \mathcal{F}_{t-1}, \mathbb{P})$ such that*

$$\mathbb{P}[\eta \cdot (X_t - X_{t-1}) \geq 0] = 1 \quad \textit{and} \quad \mathbb{P}[\eta \cdot (X_t - X_{t-1}) > 0] > 0.$$

We are now ready to prove the main result of this section.

Proof of Theorem 1.3.12: It is clear that the existence of an EMM implies the absence of arbitrage opportunities so we only need to show the converse assertion. For this we define, for any $t \in \{1, 2, \ldots, T\}$ the set

$$\mathcal{X}_t := \{\eta \cdot (X_t - X_{t-1}) : \eta \in L^0(\Omega, \mathcal{F}_{t-1}, \mathbb{P})\}.$$

In view of the preceding proposition the market is arbitrage free if and only if

$$\mathcal{X}_t \cap L^0_+(\Omega, \mathcal{F}_t, \mathbb{P}) = \{0\}$$

for all t. Note that this condition depends on $\mathbb{P}$ only through its null sets. Applying the fundamental theorem of asset pricing for one-period models yields a probability measure $\mathbb{P}_T \approx \mathbb{P}$ with bounded density such that

$$\mathbb{E}_T[X_T - X_{T-1} | \mathcal{F}_{T-1}] = 0.$$

We now proceed by induction. For this suppose now that we already have a probability measure $\mathbb{P}_k \approx \mathbb{P}$ with bounded density such that

$$\mathbb{E}_k[X_k - X_{k-1} | \mathcal{F}_{T-1}] = 0 \quad \text{for} \quad t+1 \le k \le T. \tag{1.9}$$

Applying Theorem 1.2.6 to the t-th trading period yields a probability measure $\mathbb{P}_t \approx \mathbb{P}_{t+1}$ with bounded density Z_t such that

$$\mathbb{E}_t[X_t - X_{t-1} | \mathcal{F}_{t-1}] = 0.$$

Clearly, $\mathbb{P}_t$ is equivalent to $\mathbb{P}$ with bounded density

$$\frac{d\mathbb{P}_t}{d\mathbb{P}} = Z_t \frac{d\mathbb{P}_{t+1}}{d\mathbb{P}}.$$

Moreover, if $t+1 \le k \le T$ the $\mathcal{F}_t$ measurability of Z_t yields

$$\mathbb{E}_t[X_k - X_{k-1} | \mathcal{F}_{k-1}] = \frac{\mathbb{E}_{t+1}[(X_k - X_{k-1}) Z_t | \mathcal{F}_{k-1}]}{\mathbb{E}_{t+1}[Z_t | \mathcal{F}_{k-1}]} = \mathbb{E}_{t+1}[X_k - X_{k-1} | \mathcal{F}_{k-1}] = 0.$$

Hence (1.9) carries over from $\mathbb{P}_{t+1}$ to $\mathbb{P}_t$. We can repeat this recursion until finally $\mathbb{P}^* := \mathbb{P}_1$ yields the desired EMM. □

1.3.3 European options and attainable claims

A European contingent claim can be viewed as an asset which yields at *maturity* T a random amount C. Its payoff typically depends on the behavior of the primary assets $S^0, \ldots, S^d$.

Definition 1.3.14 *A non-negative random variable C on $(\Omega, \mathcal{F}_T, \mathbb{P})$ is called a European contingent claim. It is called a derivative of the underlyings $S^0, \ldots, S^d$ if it is measurable with respect to the σ-algebra generated by the price process $(\bar{S}_t)_{t=0}^T$.*

For a European call and put option, respectively, on the i-th asset with maturity T and strike K we have that

$$C^{call} = (S^i_T - K)^+ \quad \text{and} \quad C^{put} = (K - S^i_T)^+$$

Example 1.3.15 *(i) The payoff of an Asian option depends on the average price*

$$S^i_{av} := \frac{1}{T} \sum_{t=1}^{T} S^i_t$$

of the underlying. For instance, an average price call with strike K corresponds to a contingent claim

$$C^{call}_{av} = (S^i_{av} - K)^+$$

(ii) *The payoff of a barrier option depends on whether the price of the underlying reaches a certain level before maturity. Most barrier options are either* knock-ins *or* knock-outs*. For instance, a* down-and-in put *with strike K and barrier B pays*

$$C_{di}^{put} = \begin{cases} (K - S_T^i)^+ & \text{if } \min_{0\le t\le T} S_t^i \le B \\ 0 & \text{else} \end{cases} .$$

An up-and-out call *with strike K and barrier B corresponds to*

$$C_{uo}^{call} = \begin{cases} (S_T^i - K)^+ & \text{if } \max_{0\le t\le T} S_t^i < B \\ 0 & \text{else} \end{cases} .$$

(iii) *A* digital *or* binary option *is essentially a bet on whether or not the price of the underlying reaches a certain threshold B before maturity:*

$$C^{digital} = \begin{cases} 1 & \text{if } \max_{0\le t\le T} S_t^i \ge B \\ 0 & \text{else} \end{cases} .$$

From now on we will assume that our market model is free of arbitrage, i.e., that $\mathcal{P} \neq \emptyset$ and denote by H the discounted value of a European contingent claim C.

Definition 1.3.16 *A contingent claim C is* attainable (replicable, redundant) *if there exists a self-financing trading strategy $\bar{\xi}$ whose terminal portfolio value coincides almost surely with C, i.e.,*

$$C = \bar{\xi}_T \cdot \bar{S}_T \quad \mathbb{P}\text{-a.s.}$$

Such a trading strategy will be called a replicating strategy *for C.*

A contingent claim C is attainable if and only if the corresponding discounted claim H is of the form

$$H = V_0 + \sum_{t=1}^{T} \xi_t \cdot (X_t - X_{t-1})$$

for a self-financing strategy $\bar{\xi}$ with associated value process V. The following theorem shows that an *attainable* contingent claim is automatically integrable with respect to *any* equivalent martingale measure.

Theorem 1.3.17 *Any attainable discounted claim H is integrable with respect to each equivalent martingale measure. Moreover, for any $\mathbb{P}^* \in \mathcal{P}$ the value process of any replicating strategy satisfies $\mathbb{P}$-a.s.*

$$V_t = \mathbb{E}^*[H|\mathcal{F}_t] \quad \text{for} \quad t = 0, \dots T.$$

It follows from the preceding theorem that V_t is a version of the conditional expected payoff $\mathbb{E}^*[H|\mathcal{F}_t]$ under *any* equivalent martingale measure and that all replicating strategies have the same value process. Hence for an attainable claim H the (discounted) initial investment

$$V_0 = \mathbb{E}^*[H]$$

needed for the replication of H can be interpreted as the unique "fair value" of H. More generally, we formalize the idea of an arbitrage-free price of a general (not necessarily attainable) claim H as follows.

Definition 1.3.18 *A real number $\pi^H \geq 0$ is called an* arbitrage-free price *of a discounted claim H if there exists an adapted stochastic process X^{d+1} such that*

$$X_0^{d+1} = \pi^H, \quad X_t^{d+1} \geq 0 \quad \textit{for all } t = 1, \dots, T, \quad X_T^{d+1} = H,$$

and such that the market model with the price process $(X^0, \dots, X^d, X^{d+1})$ is free of arbitrage. The set of all arbitrage free prices is denoted by $\Pi(H)$. It's upper and lower bounds are

$$\pi_{inf}(H) := \inf \Pi(H) \quad \textit{and} \quad \pi_{sup}(H) := \sup \Pi(H).$$

Notice that when $\mathbb{P}^*$ is an EMM for the original model with respect to which the claim H has finite expectation and if we define the process (X_t^{d+1}) by

$$X_0^{d+1} = \mathbb{E}^*[H] \quad \text{and} \quad X_t^{d+1} = \mathbb{E}^*[H|\mathcal{F}_t]$$

then $\mathbb{P}^*$ is an EMM for the extended model. In this way every EMM of the original model can be viewed as an EMM for the extended model. At the same time, any EMM of the extended model is an EMM for the original model. With this we are now ready to state the following dynamic version of Theorem 1.2.13 above.

Theorem 1.3.19 *Suppose that $\mathcal{P} \neq \varnothing$. Then the set of arbitrage free prices of a discounted claim H is non-empty and given by*

$$\Pi(H) = \{\mathbb{E}^*[H] : \mathbb{P}^* \in \mathcal{P} \textit{ and } \mathbb{E}^*[H] < \infty\}. \tag{1.10}$$

Furthermore, the lower and upper arbitrage bounds are given by

$$\pi_{inf}(H) = \inf_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*[H] \quad \textit{and} \quad \pi_{sup}(H) = \sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*[H].$$

Notice that $\pi_{inf}(H) = \pi_{sup}(H)$ if H is attainable. In this case H can be priced unambiguously. The converse is also true, i.e., H is attainable if and only if $\pi_{inf}(H) = \pi_{sup}(H)$.

Theorem 1.3.20 *Let H be a discounted claim. Then the following holds:*

(i) If H is attainable, then $\Pi(H)$ contains a single element, namely $\mathbb{E}^*[H]$ where the expectation can be taken with any EMM $\mathbb{P}^*$.

(ii) If H is not attainable, then $\pi_{inf}(H) < \pi_{sup}(H)$ and $\Pi(H)$ is an interval:

$$\Pi(H) = (\pi_{inf}(H), \pi_{sup}(H)).$$

To proof Theorem 1.3.20 we need two lemmas and the Theorem of Hahn-Banach. Let

$$Y := S_i - S_0 \text{ with } S_i \in L^0(\mathcal{F}_i),\ Y \in \mathbb{R}^d,$$

and let $\mathcal{F}_o \subseteq \mathcal{F}_i$ be a σ-algebra, which is not necessarily trivial. We define

$$\begin{aligned}
\chi &:= \left\{\eta \cdot Y : \eta \in L^0(\mathcal{F}_0)\right\} \\
L^0_+ &:= L^0_+(\mathcal{F}_i) = \left\{X \in L^0(\mathcal{F}_i) \,:\, X \geq 0. \text{ a.s.}\right\} \\
\chi - L^0_+ &:= \left\{Z \in L^0 \text{ with } Z = \xi \cdot Y - U,\ U \in L^0_+\right\}.
\end{aligned}$$

Then we assume that $\xi \cdot Y = 0$, $\mathbb{P}$-a.s., implies $\xi = 0$, $\mathbb{P}$-a.s. (non-redundant market).

Lemma 1.3.21 *(randomized Bolzano-Weierstrass) Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *a probability space and* (ξ_n) *a sequence in* $L^0(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}^d)$ *with* $\liminf |\xi_n| < \infty$. *Then there exists a random variable* $\xi = (\xi)_{i=1}^d \in L^0$ *and a strictly increasing sequence* (σ_n) *with values in* $\mathbb{N}$ *and* $\sigma_n \in \mathcal{F}$ *such that it holds:*

$$\xi_{\sigma_n(\omega)}(\omega) \;\to\; \xi(\omega), \quad \mathbb{P}\text{-}a.s.$$

Lemma 1.3.22 *If the intersection of* χ *and* L^0_+ *only contains the null-element, then* $\chi - L^0_+$ *is closed in* L^0.

Theorem 1.3.23 *(Hahn-Banach) Let* B *and* C *be two non-empty, disjoint, and convex subsets of a locally convex space* E. *Then, if* B *compact and* C *closed, there exists a continuous linear functional* $l : E \to \mathbb{R}$ *such that*

$$\sup_{x \in C} l(x) < \inf_{y \in B} l(y).$$

For a European call option with strike K and maturity T the arbitrage free price is given by

$$\pi^{call} = \mathbb{E}^*\left[\frac{C^{call}}{S_T^0}\right] = \mathbb{E}^*\left[\left(X_T^1 - \frac{K}{S_T^0}\right)^+\right].$$

Due to the convexity of the function $x \mapsto x^+$ the quantity π^{call} can be bounded from below as follows:

$$\pi^{call} \geq \left(\mathbb{E}^*\left[X_T^1 - \frac{K}{S_T^0}\right]\right)^+ = \left(S_0^1 - \mathbb{E}^*\left[\frac{K}{S_T^0}\right]\right)^+ \geq (S_0^1 - K)^+.$$

Thus, the option value π^{call} is higher than its *intrinsic value* $(S_0^1 - K)^+$, i.e., the payoff if the option were exercised immediately.

1.3.4 Complete markets

We have seen that any attainable contingent claim in an arbitrage free market has a unique arbitrage free price. This situation becomes particularly transparent when *all* claims are attainable.

Definition 1.3.24 *An arbitrage-free market model is called complete if every contingent claim is attainable.*

Complete markets are appealing insofar as that every claim has a unique and unambiguous price. However, in discrete time, only a limited class of models enjoys this property as shown by the following theorem.

Theorem 1.3.25 *An arbitrage-free market model is complete if and only if there exists a unique EMM. In this case the number of atoms in* $(\Omega, \mathcal{F}, \mathbb{P})$ *is bounded above by* $(d+1)^T$.

It follows from the preceding theorem that in a one-period model the binary case where only two states of the world are possible is the only example of a complete market model. In discrete time the usual paradigm of complete markets is therefore the exception rather than the rule.

Theorem 1.3.26 *Let* $\mathcal{Q}$ *be the set of all martingale measures. For* $\mathbb{P}^* \in \mathcal{P}$ *the following conditions are equivalent:*

(i) $\mathcal{P} = \{\mathbb{P}^*\}$.

(ii) $\mathbb{P}^*$ *is an extreme point of* $\mathcal{P}$.

(iii) $\mathbb{P}^*$ *is an extreme point of* $\mathcal{Q}$.

(iv) Every $\mathbb{P}^*$*-martingale can be represented as a "stochastic integral" of a d-dimensional predictable process* ξ *:*

$$M_t = M_0 + \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1}).$$

In a complete market model the unique arbitrage-free price H_t of the a discounted claim H at time t is given by the conditional expected payoff under the EMM:

$$H_t = \mathbb{E}^*[H|\mathcal{F}_t].$$

In particular (H_t) is a $\mathbb{P}^*$-martingale. In view of part (iv) of the preceding theorem this martingale can be represented as a stochastic integral:

$$H_t = H_0 + \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1})$$

for some predictable process ξ. In economic terms $H_0 = \mathbb{E}^*[H]$ is the initial investment needed to implement the replication strategy $\bar{\xi} = (\xi^0, \xi)$. Here ξ^0 is the unique trading strategy in the bond market associated with H_0 and ξ. Hence the martingale representation theorem yields both the fair price of H and a replicating strategy in the stock and bond market.

1.4 Binomial trees and the Cox-Ross-Rubinstein Model

In view of Theorem 1.3.25 a complete market model with a single risky asset must have binary tree structure. In this section we consider a binary tree model that has originally been introduced by Cox, Ross & Rubinstein (1971). The underlying idea is that in any given period stock prices can only move up or down. More precisely, there is a riskless bond whose price at time time is given by $S_t^0 = (1+r)^t$ and a single risky asset whose price fluctuations satisfy

$$R_t = \frac{S_t^1 - S_{t-1}^1}{S_{t-1}^1} \in \{a, b\} \quad \text{for all } t = 1, 2, \ldots, T$$

and some $-1 < a < b$. We construct the model on the sample space

$$\Omega = \{-1, +1\}^T = \{\omega = (y_1, \ldots, y_T) : y_i \in \{-1, +1\}\}$$

and put

$$Y_t(\omega) = y_t.$$

Then

$$R_t(\omega) = a\frac{1 - Y_t(\omega)}{2} + b\frac{1 + Y_t(\omega)}{2}$$

and the price dynamics of the risk asset is modeled as

$$S_t = S_0 \prod_{k=1}^{t} (1 + R_k).$$

The discounted price process is again denoted (X_t). As the filtration we take

$$\mathcal{F}_t = \sigma(S_0, \dots, S_t) = \sigma(X_0, \dots, X_t) = \sigma(Y_1, \dots, Y_t) = \sigma(R_1, \dots, R_t).$$

Finally, we fix a probability measure $\mathbb{P}$ such that $\mathbb{P}[\{\omega\}] > 0$ for all $\omega \in \Omega$. We notice that we do not make any a-priori assumptions on the serial dependence of returns. It turns out though that in an arbitrage free model, the returns are independent of each other.

Proposition 1.4.1 *The model is free of arbitrage if and only if $a < r < b$. In this case there exists a unique MM $\mathbb{P}^*$ and the random variables are independent under $\mathbb{P}^*$ with*

$$\mathbb{P}^*[R_t = b] = \frac{r-a}{b-a}.$$

Let us now consider the problem of pricing a (possibly path dependent) derivative $H = h(S_0, ..., S_T)$ in a complete market model. We know already that the value (V_t) process associated with any replication strategy satisfies

$$V_t = \mathbb{E}^*[H|\mathcal{F}_t].$$

The following theorem further specifies the value process.

Theorem 1.4.2 *The value process is of the form*

$$V_t = v_t(S_0, ..., S_t)$$

where

$$v_t = \mathbb{E}^* \left[h\left(x_0, ..., x_t, x_t \frac{S_1}{S_0}, ..., x_t \frac{S_{T-t}}{S_0} \right) \right].$$

Lemma 1.4.3 *(Stochastic Fubini Theorem) For $i = 0, 1$ let $(E_i, \mathcal{E}_i)$ be measurable spaces and $U_i : (\Omega, \mathcal{F}) \to (E_i, \mathcal{E}_i)$ measurable. Let $\mathcal{F}_0 = \sigma(U_0)$ and U_1 independent of U_0. Then*

$$\mathbb{E}[f(U_0, U_1)|\mathcal{F}_0](\omega) = \mathbb{E}[f(U_0(\omega), U_1)] =: h(U_0(\omega))$$

for all non-negative measurable functions f on $E_0 \times E_1$.

1.4.1 Delta-Hedgeing in discrete time

Since the value process of a replicating strategy satisfies $V_T = H$ and $V_t = \mathbb{E}[V_{t+1}|\mathcal{F}_t]$ we obtain a recursive structure for the function v_t. In fact,

$$\begin{aligned} v_T(x_0, ..., x_T) &= h(x_0, ..., x_T) \\ v_t(x_0, ..., x_t) &= p^* v_{t+1}(x_0, ..., x_t, x_t(1+b)) + (1-p^*) v_{t+1}(x_0, ..., x_t, x_t(1+a)). \end{aligned}$$

It turns out that not only the value process, but also the hedgeing (replicating) strategy $\xi = (\xi^0, \xi)$ can be expressed in terms of v_t. To this end, we introduce the "option delta" Δ_t, i.e., a discrete derivative of the value function v_t with respect to possible changes in the stock price:

$$\Delta_t := (1+r) \frac{v_t(x_0, ..., x_{t-1}, x_{t-1}(1+b)) - v_{t-1}(x_0, ..., x_{t-1}, x_{t-1}(1+a))}{x_{t-1}(1+b) - x_{t-1}(1+a)}.$$

Proposition 1.4.4 *The delta yields a hedgeing strategy for the attainable claim H.*

For a discounted claim $H = h(S_T)$ whose payoff is an increasing function of the terminal price S_T (such as a European Call Option) the value function

$$v_t(x) = \mathbb{E}^*[h(xS_{T-t}/S_t)]$$

is increasing in x and hence the hedgeing strategy is non-negative, i.e., does not involve short sells:

$$\xi_t(\omega) = (1+r)^t \frac{v_t(S_{t-1}(\omega)(1+b)) - v_t(S_{t-1}(\omega)(1+a))}{S_{t-1}(\omega)(1+b) - S_{t-1}(\omega)(1+a)} \geq 0.$$

1.4.2 Exotic derivatives

The results of the preceding section can be used for numerical computation of the value process associated with a contingent claim H. In this section we focus on barrier and lookback options whose value depends on the maximum

$$\hat{M}_t := \max_{o \leq s \leq t} S_t$$

of the underlying stock price. The valuation formulas are based on the *reflection principle* for a symmetric random walk so from now on we work under the additional assumption that

$$\hat{a} = \frac{1}{\hat{b}} \quad \text{where} \quad \hat{a} = (1+a), \hat{b} = (1+b).$$

In this case the price process can be written as

$$S_t = S_0 \hat{b}^{Z_t} \quad \text{where} \quad Z_t = Y_1 + \ldots + Y_t.$$

If we denote by $\mathbb{P}$ the uniform distribution on Ω then (Y_t) is a sequence of iid random variables with $\mathbb{P}[Y_t = +1] = \frac{1}{2}$ and (Z_t) is a symmetric random walk. As a result,

$$\mathbb{P}[Z_t = k] = 2^{-t} \begin{pmatrix} t \\ \frac{t+k}{2} \end{pmatrix} \quad \text{if } t+k \text{ is even.}$$

The reflection principle

In order to calculate the value of an up-and-in barrier option we need the joint distribution of the terminal price and the running maximum of the underlying. This distribution can be given in closed form. To this end, it will be convenient to assume that the random walk (Z_t) is defined up to time $T+1$ and to put

$$M_t := \max_{o \leq s \leq t} Z_t.$$

We are now ready to state and prove the *reflection principle* for a symmetric random walk. It states that the joint distribution of the running maximum of a symmetric random walk and its terminal value can be expressed in terms of the distribution of the terminal value only.

Lemma 1.4.5 *For all $k \in \mathbb{N}$ and $l \in \mathbb{N}_0$ the following holds:*

(i)

$$\mathbb{P}[M_T \geq k,\, Z_T = k - l] = \mathbb{P}[Z_T = k + l].$$

(ii)

$$\mathbb{P}[M_T = k,\, Z_T = k - l] = 2\frac{k + l + 1}{T + 1}\mathbb{P}[Z_{T+1} = 1 + k + l].$$

In order to establish a pricing formula for exotic derivatives we need a reflection under the equivalent martingale measure $\mathbb{P}^*$ rather than $\mathbb{P}$. To this end, we first notice that the distribution of the random walk under $\mathbb{P}^*$ is given by

$$\mathbb{P}^*[Z_t = k] = (p^*)^{\frac{t+k}{2}}(1 - p^*)^{\frac{t-k}{2}}\begin{pmatrix} t \\ \frac{t+k}{2} \end{pmatrix} \quad \text{if } t + k \text{ is even.}$$

With this, one can show that the reflection principle under $\mathbb{P}^*$ reads:

$$\mathbb{P}^*[M_T \geq k,\, Z_T = k - l] = \left(\frac{1 - p^*}{p^*}\right)^l \mathbb{P}[Z_T = k + l] = \left(\frac{p^*}{1 - p^*}\right)^l \mathbb{P}[Z_T = -k - l]$$

and

$$\begin{aligned} \mathbb{P}[M_T = k,\, Z_T = k - l] &= \frac{1}{p^*}\left(\frac{1 - p^*}{p^*}\right)^l \frac{k + l + 1}{T + 1}\mathbb{P}[Z_{T+1} = 1 + k + l] \\ &= \frac{1}{1 - p^*}\left(\frac{p^*}{1 - p^*}\right)^l \frac{k + l + 1}{T + 1}\mathbb{P}[Z_{T+1} = -1 - k - l]. \end{aligned}$$

Valuation formulae for up-and-in call options

Based on the reflection principle we are now going to compute a closed form (though cumbersome) valuation formula for an up-and-in call option. To this end, we may with no loss of generality assume that the barrier lies within the range of asset prices, that is,

$$B = S_0\hat{b}^k \quad \text{for some } k \in \mathbb{N}.$$

Now we write

$$\begin{aligned} \pi &= \mathbb{E}^*[(S_T - K)^+; \hat{M}_t \geq B] \\ &= \mathbb{E}^*[(S_T - K)^+; S_T \geq B] + \mathbb{E}^*[(S_T - K)^+;\, \hat{M}_t \geq B,\, S_T < B]. \end{aligned}$$

The random variable Z_T takes values in $\{-T - T + 2, \ldots, T\}$, and so we get

$$\begin{aligned} \pi &= \sum_{l=0}^{T}\left(S_0\hat{b}^{2l-T}\right)^+ \mathbb{P}^*\left[M_T \geq k; Z_T = 2l - T\right] \\ &= \sum_{l=0}^{T}\left(S_0\hat{b}^{2l-T}\right)^+ 2^T (p^*)^l (1 - p^*)^{T-l}\, \mathbb{P}\left[M_T \geq k; Z_T = 2l - T\right]. \end{aligned}$$

Now we apply the reflection principle for $2l - T \leq k$ and set

$$l_k := \max\{l \in \mathbb{N} : 2l - T \leq k\}.$$

Then follows for $l \le l_k$ that

$$\mathbb{P}\left[M_T \ge k, Z_T = 2l - T\right] = \mathbb{P}\left[Z_T = 2(k-l) + T\right] = \begin{cases} 2^{-T}\binom{T}{T+k-l}, & l \ge k \\ 0, & \text{else} \end{cases}$$

and for $l > l_k$ we have

$$\mathbb{P}\left[M_T \ge k, Z_T = 2l - T\right] = \mathbb{P}\left[Z_T = 2l - T\right] = 2^{-T}\binom{T}{l}.$$

Together we obtain:

$$\begin{aligned} \pi &= \sum_{l=k}^{l_k} \left(S_0 \hat{b}^{2l-T}\right)^+ (p^*)^l (1-p^*)^{T-l} \binom{T}{T+k+l} \\ &\quad + \sum_{l=l_{k+1}}^{T} \left(S_0 \hat{b}^{2l-T}\right)^+ (p^*)^l (1-p^*)^{T-l} \binom{T}{l}. \end{aligned}$$

Valuation formulae for lookback options

A lookback put option corresponds to the contingent claim

$$C = \hat{M}_T - S_T.$$

The discounted arbitrage-free price is given by

$$\pi = \frac{1}{(1+r)^T} \mathbb{E}^*[\hat{M}_T] - S_0.$$

The expectation of the running maximum can be computed as

$$\mathbb{E}^*[\hat{M}_T] = S_0 \hat{b}^k \mathbb{P}^*[M_T = k].$$

By the reflection principle

$$\begin{aligned} \mathbb{P}^*[M_T = k] &= \sum_{l \ge 0} \mathbb{P}^*[M_T = k, Z_T = k - l] \\ &= \sum_{l \ge 0} \frac{1}{1-p^*} \left(\frac{p^*}{1-p^*}\right)^k \frac{k+l+1}{T+1} \mathbb{P}^*[Z_{T+1} = -1 - k - l] \\ &= \frac{1}{1-p^*} \left(\frac{p^*}{1-p^*}\right)^k \frac{1}{T+1} \mathbb{E}^*[-Z_{T+1}; Z_{T+1} \le -1 - k]. \end{aligned}$$

The latter formula can again be given in closed form.

1.4.3 Convergence to Black-Scholes Prices

In the preceding section we obtained closed form, though sometimes cumbersome valuation formula for European call and certain exotic options. In this section we consider a sequence of CRR models where the time lag between two consecutive trading dates tends to zero and study convergence properties of the associated sequence of options prices. Thus, for the remainder of this chapter T denotes the *terminal date* rather than the number of trading times. Instead we divide the time interval $[0, T]$ into

N equidistant time steps $t^1, ..., t^N$ where we interpret t^k as the k-th trading period. We assume that there is a single risky asset; in the N-th approximation we denote its price process by S^N. For the riskless bond we assume that

$$(1 + r_N)^N \to e^{rT}$$

as $N \to \infty$ for some constant $r > 0$. For every N the process S^N is defined on a probability space $(\Omega, \mathcal{F}^N, \mathbb{P}^N)$ where $\mathcal{F}^N$ is the σ-field generated by the asset prices and $\mathbb{P}^N$ denotes the unique EMM. Thus, the discounted price process X^N is an $\mathcal{F}^N$-martingale with respect to $\mathbb{P}^N$. The returns R_t^N take values in a set $\{a_N, b_N\}$ where $-1 < a_N < r_N < b_N$ and

$$\lim_{N \to \infty} a_N = \lim_{N \to \infty} b_N = 0.$$

Finally, we assume that the variances V_t^N of the return R_t^N under $\mathbb{P}^N$ satisfy

$$\sigma_N^2 := \frac{1}{T} \sum_{k=1}^{N} V_t^N \to \sigma^2 \in (0, \infty).$$

The preceding assumptions are satisfied if, for instance,

$$a_N = \frac{a}{\sqrt{N}}, \quad b_N = \frac{b}{\sqrt{N}}.$$

Example 1.4.6 *We consider the CRR-model with*

$$r_N = \frac{rT}{N}, \quad R_k^N \in \{a_N, b_N\},$$
$$1 + a_N = e^{-\sigma\sqrt{T}N}, \quad 1 + b_N = e^{\sigma\sqrt{T}N}.$$

Then the measure $\mathbb{P}_N^*$ is given by

$$\mathbb{P}_N^*\left[R_k^N = b_N\right] =: p_N^* = \frac{r_N - a_N}{b_N - a_N} \to \frac{1}{2}$$

since with Taylor we obtain $\sqrt{N}a_N \to -\sigma\sqrt{T}$ and $\sqrt{N}b_N \to \sigma\sqrt{T}$. Moreover we have $\mathbb{E}\left[R_k^N\right] = r_N$ and it follows that

$$\sum_{k=1}^{N} Var_N\left[R_k^N\right] = N\left(p_N^* b_N^2 + (1 - p^*)\, a_N^2 - r_N^2\right) \to \sigma^2 T$$

for $N \nearrow \infty$.

Remark 1.4.7 *a) A sequence of probabilities (μ_n) on $(\Omega, \mathcal{F})$ converges weakly to a probability μ, if*

$$\mu_n(f) := \int f d\mu_n \to \int f d\mu =: \mu(f)$$

for all f in C_0 (we write: $\mu_N \xrightarrow{w} \mu$).

b) A sequence of random variables (X_n) on $(\Omega, \mathcal{F}, \mathbb{P})$ converges in distribution, if $(\mathbb{P}_{X_n})$ converges weakly.

c) *Central limit theorem: For $N \in \mathbb{N}$ let $Y_1^{(N)}, \ldots, Y_N^{(N)}$ a sequence of independent random variables with*

$$(i)\ |Y_k^{(N)}| \le \gamma^{(N)}, \quad \mathbb{P}\text{-a.s. for } \gamma^{(N)} \in \mathbb{R},\ \gamma^{(N)} \to 0$$
$$(ii)\ \sum_{k=1}^{N} \mathbb{E}_N\left[Y_k^{(N)}\right] \to m$$
$$(iii)\ \sum_{k=1}^{N} \mathrm{Var}_N\left[Y_k^{(N)}\right] \to \sigma^2.$$

Then it holds:

$$Z_N := \sum_{k=1}^{N} Y_k^{(N)} \ \overset{w}{\to}\ N(m, \sigma^2).$$

This means that we can start with some benchmark CRR model and then derive a continuous time limit by suitably rescaling its dynamics in space and time. Before stating the main convergence result for asset prices, we recall that a random variable Y is called log-normally distributed if $\log Y$ follows a normal distribution. We also recall that a sequence of probability distributions $\{\mu_n\}$ converges weakly to a probability measure μ if the integrals of bounded continuous functions with respect to μ_n converge to the integral with respect to μ.

Theorem 1.4.8 *Under the above assumptions the distributions of the terminal prices S_N^N under $\mathbb{P}^N$ converge weakly to the distribution of*

$$S_T = S_0 \exp\left\{\sigma W_T + \left(r - \frac{1}{2}\sigma^2\right)T\right\}$$

where $W_T \sim N(0,T)$, i.e., S_T has a log-normal distribution.

We assume

$$C^{(N)} = f(S_T^{(N)})$$

for a bounded continuous function f. Then we have

$$\lim_{k \nearrow \infty} \mathbb{E}^*\left[\frac{C^{(N)}}{(1+r_N)^T}\right] \to e^{-rT}\mathbb{E}^*\left[f\left(S_0 \exp\left(\sigma\sqrt{T}W_T + rT - \frac{1}{2}\sigma^2 T\right)\right)\right]$$

and in particular for a put option

$$\lim_{k \nearrow \infty} \mathbb{E}^*\left[\frac{\left(S_T^{(N)} - K\right)^+}{(1+r_N)^T}\right] \to e^{-rT}\mathbb{E}^*\left[\left(S_0 \exp\left(\sigma\sqrt{T}W_T + rT - \frac{1}{2}\sigma^2 T\right) - K\right)^+\right].$$

With the put-call-parity we get

$$\mathbb{E}^*\left[\frac{\left(S_T^{(N)} - K\right)^+}{(1+r_N)^T}\right] = \mathbb{E}^*\left[\frac{\left(K - S_T^{(N)}\right)^+}{(1+r_N)^T}\right] + S_0 - \frac{K}{(1+r_N)^T}$$

and hence we obtain the convergence of call prices.

Example 1.4.9 *(Black-Scholes-price) The limit of* $C^{(N)}$ *is given by* $v(T,S_0)$ *where*

$$v(T,x) = e^{-rT}\mathbb{E}^*\left[\left(x\exp\left(\sigma\sqrt{T}W_T + rT - \frac{1}{2}\sigma^2 T\right) - K\right)^+\right]$$

and $W \sim N(0,1)$ *under* $\mathbb{P}^*$. *Let* $\tilde{K} := \frac{K}{x}e^{-rT}$ *and* $\tilde{\sigma} := \sigma\sqrt{T}$, *then it follows that*

$$v(T,x) = \frac{x}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(e^{\tilde{\sigma}y - \frac{\tilde{\sigma}^2}{2}} - \tilde{K}\right)^+ e^{-\frac{y^2}{2}}\,dy.$$

For

$$y \leq \frac{\log\tilde{K} + \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}} = -\frac{\log\frac{x}{K} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} =: -d_-(x,T) := -d_-(x,T)$$

the integrand vanishes. Furthermore, let

$$d_+ := d_+(x,T) := d_-(x,T) + \sigma\sqrt{T}$$

and $\Phi(z) := \mathbb{P}^*[w \leq z]$. *Then we get*

$$\begin{aligned} v(T,x) &= \frac{x}{\sqrt{2\pi}}\int_{-d_-}^{\infty} e^{-\frac{(y-\tilde{\sigma})^2}{2}}\,dy - x\tilde{K}(1 - \Phi(-d_-)) \\ &= x\Phi(d_+) - Ke^{-rT}\Phi(d_-), \end{aligned}$$

the Black-Scholes formula for the price of a European call option with strike K *and maturity* T.

One can show that the replicating strategy has the following form:

$$\begin{aligned} \xi_t^1 &.= \Delta_t(S_t, T-t) \\ \Delta_t &:= \frac{\partial}{\partial x}v(t,x) = \Phi(d_+(x,t)) \\ \xi_t^0 &:= v(S_t, T-t) - \xi_t^1 e^{-rt}S_t. \end{aligned}$$

Remark 1.4.10 *(The Greeks)*

- *Delta:* $\Delta_t = \frac{\partial}{\partial x}v(t,x) \geq 0$
- *Gamma:* $\Gamma_t = \frac{\partial^2}{\partial^2 x}v(t,x) \geq 0$
- *Theta:* $\Theta_t = \frac{\partial}{\partial t}v(t,x) \geq 0$
- *Vega:* $\mathcal{V}_t = \frac{\partial}{\partial \sigma}v(t,x) \geq 0$

Now we will show an alternative way to prove the convergence of call prices without using the put-call parity.

Proposition 1.4.11 *Let* $f : (0,\infty) \to \mathbb{R}$ *be a measurable and (a.e.) continuous function satisfying* $|f(x)| \leq c(1+|x|)^q$ *for* $c \geq 0$ *and* $0 \leq q < 2$. *Then it holds:*

$$\mathbb{E}_N^*\left[f\left(S_T^N\right)\right] \to \mathbb{E}^*\left[f(S_T)\right], \quad \textit{for } N \to +\infty.$$

To prove the Proposition 1.4.11 we use the following proposition.

Proposition 1.4.12 *Let (μ_N) a sequence of probability measures on $\mathbb{R}$ with $\mu_N \xrightarrow{w} \mu$. Let also f be a measurable and a μ-a.s. continuous function on $\mathbb{R}$ with*

$$\sup_N \int |f|^p d\mu_N < +\infty$$

for some $p > 1$. Then it holds: $\mu_N(f) \to \mu(f)$.

Proof of Theorem 1.4.11: We use $\log(1+x) = x - \frac{1}{2}x^2 + \rho(x)x^2$. Hence we have:

$$\begin{aligned}
\log \mathbb{E}\left[\left(S_N^N\right)^2\right] &= \log \prod_{k=1}^N \left(\mathrm{Var}_{\mathbb{P}_N^*}\left(1+R_k^N\right) + \mathbb{E}_N^*\left[1+R_k^N\right]^2\right) \\
&= \sum_{k=1}^N \log\left(\mathrm{Var}_{\mathbb{P}_N^*}\left(R_k^N\right) + (1+r_N)^2\right) \\
&\le \sigma_N^2 T + 2Nr_N + Nr_N^2 + \tilde{C}\sum_{k=1}^N \left(\mathrm{Var}\left(R_k^N\right) + 2|r_N| + r_N^2\right)^2
\end{aligned}$$

for some constant $\tilde{C} < +\infty$. In particular we have that $\sup \log \mathbb{E}^*\left[\left(S_N^N\right)^2\right] < +\infty$, if and only if $\sup \mathbb{E}^*\left[\left(S_N^N\right)^2\right] < +\infty$. Then it holds:

$$\sup_N \mathbb{E}_N^*\left[\left|f\left(S_N^N\right)\right|^p\right] \le C^p \sup_N \mathbb{E}\left[\left(1+S_N^N\right)^2\right] < +\infty$$

for $p = \frac{2}{q} > 1$. At last we use Proposition 1.4.12 on μ_N, which is the distribution of S_N^N under $\mathbb{P}_N^*$, and so the assertion holds. □

1.5 Introduction to optimal stopping and American options

This section contains a brief introduction into the theory of optimal stopping and American options. Our analysis follows Section 2 of the textbook by Lamberton & Lapeyre (1996). Throughout we assume that all stochastic processes are defined on a probabilistic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, that the sample space is finite and that $\mathcal{F}_0$ is the trivial σ-field.

1.5.1 Motivation and introduction

Let is consider an American put option on an underlying with price process (S_t) and assume that there exists a unique equivalent martingale measure $\mathbb{P}^*$. Since the option can be exercised at any time prior to maturity we shall define its value in terms of a positive sequence adapted (Z_t), where Z_t is the profit form immediate exercise at time t, that is,

$$Z_t = (K - S_t)^+.$$

If the option has not been exercised early, then its value U_T at time T is

$$U_T = Z_T.$$

At what price should one sell the option at time $T-1$? Straight exercise yields Z_{T-1}. At the same time, the seller must be prepared to meet his obligations at time T should the owner of the option choose not to exercise it early. Hence,

$$\begin{aligned} U_{T-1} &= \max\{Z_{T-1}, \frac{1}{1+r}\mathbb{E}^*[Z_T|F_{T-1}]\} \\ &= \max\{Z_{T-1}, \frac{1}{1+r}\mathbb{E}^*[U_T|F_{T-1}]\}. \end{aligned}$$

Similarly,

$$U_{t-1} = \max\{Z_{t-1}, \frac{1}{1+r}\mathbb{E}^*[U_t|F_{t-1}]\}.$$

In terms of the discounted sequences $\tilde{Z}_t := Z_t/(1+r)$ and $\tilde{U}_t := U_t/(1+r)$ we obtain that

$$\tilde{U}_{t-1} = \max\{\tilde{Z}_{t-1}, \mathbb{E}^*[\tilde{U}_t|F_{t-1}]\}.$$

The following proposition characterizes the value of an American put option in terms of a super-martingale property.

Proposition 1.5.1 *The sequence $(\tilde{U}_t)$ is the smallest $\mathbb{P}^*$ super-martingale that dominates $\tilde{Z}_t$.*

1.5.2 Stopping times

The buyer of an American option has the right to exercise the option at any time before maturity. The decision whether ot not to exercise at time t will be made according to the information available at that time. The exercise strategy is therefore defined by a *stopping time*.

Definition 1.5.2 *A random variable τ on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is called a stopping time if for any time t the event*

$$\{\tau = t\} \quad \textit{belongs to} \quad \mathcal{F}_t.$$

Remark 1.5.3 *If τ is stopping time, then*

$$\{\tau \le t\} = \{\tau = 1\} \cup \cdots \cup \{\tau = t\} \in \mathcal{F}_t.$$

On the other hand $\{\tau \le t\} \in \mathcal{F}_t$ for all $t \le T$ implies that

$$\{\tau = t\} = \{\tau \le t\} \backslash \{\tau \le t-1\} \in \mathcal{F}_t.$$

Thus, τ is a stopping time if and only if $\{\tau \le t\} \in \mathcal{F}_t$ for all $t = 1, \ldots, T$.

In order to analyze American options we need to study stochastic processes stopped at a stopping time. For a adapted process (X_t) and a stopping time τ we define the stopped process (X_t^τ) by

$$X_t^\tau(\omega) = X_{\tau(\omega)\wedge t}(\omega) = \begin{cases} X_t(\omega) & \text{if } \tau(\omega) \ge t \\ X_{\tau(\omega)}(\omega) & \text{else} \end{cases}.$$

The following proposition shows that martingales are stable with respect to stopping. This result is important because it states that under the (any) risk neutral measure the martingale property of stock prices, and hence the assumption of no arbitrage, is preserved when early exercise of options is allowed.

Proposition 1.5.4 *Let (X_t) be an adapted process and τ a stopping time. Then the following holds:*

(i) The stopped sequence (X_t^τ) is adapted.

(ii) If (X_t) is a (super-) martingale, then (X_t^τ) is a (super-) martingale.

1.5.3 The Snell envelope

We have already seen that the value process of an American put option is given by the smallest $\mathbb{P}^*$ super-martingale associated with the gain process from immediate exercise. Such processes are called *Snell envelopes*. The goal of this section is study the properties of Snell envelopes in greater detail.

Definition 1.5.5 *Let (Z_t) be an adapted process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. The associated snell envelope (U_n) is the smallest $\mathbb{P}$ super-martingale that dominates (Z_t), i.e.,*

$$\begin{cases} U_T = Z_T \\ U_t = \max\{Z_t, \mathbb{E}[U_{t+1}|\mathcal{F}_t]\} \quad \textit{for all } t \leq T_1 \end{cases}.$$

If $U_t > Z_t$, i.e., if early exercise of the option is not beneficial, then $U_t = \mathbb{E}[U_{t+1}|\mathcal{F}_t]$. The following proposition shows that, by stopping adequately, it is indeed possible to obtain a martingale.

Proposition 1.5.6 *The random variable*

$$\tau := \inf\{t : U_t = Z_t\} \tag{1.11}$$

is a stopping time and the stooped process (U_t^τ) is a martingale.

Our goal is now to establish an optimality property of the stopping time τ defined in (1.11). More precisely, if we think of Z_t as the gains of a gambler after t games, then the strategy τ maximizes the expected gains. In order to see this, we denote by $\mathcal{T}_{t,T}$ the set of all stopping time taking values in the set $\{t, t+1, ..., T\}$.

Proposition 1.5.7 *The stopping time τ satisfies*

$$U_0 = \mathbb{E}[Z_\tau] = \sup_{\tau' \in \mathcal{T}_{0,T}} \mathbb{E}[Z_{\tau'}].$$

Let us call a stopping time τ^* optimal for the adapted process (Z_t) if

$$Z_{\tau^*} = \sup_{\tau' \in \mathcal{T}_{0,T}} \mathbb{E}[Z_{\tau'}].$$

The preceding proposition stats that τ defined in (1.11) is optimal. The next theorem shows that among all the optimal stopping times, τ is minimal.

Theorem 1.5.8 *A stopping time τ^* is optimal for (Z_t) if and only if*

$$Z_{\tau^*} = U_{\tau^*} \quad \textit{and} \quad (U_t^{\tau^*}) \textit{ is a martingale.}$$

The next step is to characterize the largest optimal stopping time. This will be based on the Doob-decomposition of supermartingales.

1.5.4 Decomposition of supermartingales and pricing of American options

Supermartingales can written as the difference of a martingale and some non-decreasing predictable process as shown by the following proposition.

Proposition 1.5.9 *Let (U_t) be a supermartingale. There exists a unique decomposition*

$$U_t = M_t - A_t$$

where (M_t) is a martingale and (A_t) is a non-decreasing predictable process with $A_0 = 0$.

We are now ready to characterize the largest optimal stopping time of a process (Z_t) with associated Snell envelope (U_t) in terms of the Doob decomposition $U_t = M_t - A_t$.

Theorem 1.5.10 *The largest optimal stopping ν time for (Z_t) is given by $\nu = A_T$ if $A_T = 0$ and $\nu = \inf\{t : A_{t+1} \neq 0\}$.*

We are now going to apply the theory of optimal stopping to American options. We have already seen that the value process (U_t) of an American option described by process (Z_t) is given by the associated Snell envelope (after discounting). From our general theory we know that

$$\tilde{U}_t = \sup_{\tau' \in \mathcal{T}_{t,T}} \mathbb{E}^*[\tilde{Z}_{\tau'}|\mathcal{F}_t]$$

so

$$U_t = (1+r)^t \sup_{\tau' \in \mathcal{T}_{t,T}} \mathbb{E}^*[(1+r)^{-t} Z_{\tau'}|\mathcal{F}_t].$$

Furthermore,

$$\tilde{U}_t = \tilde{M}_t - \tilde{A}_t$$

for a $\mathbb{P}^*$-martingale $(\tilde{M}_t)$ and an increasing, predictable process $(\tilde{A}_t)$. Since the market is complete, there exists a self-financing strategy such that the associated discounted value process $(\tilde{V}_t)$ satisfies

$$\tilde{V}_T = \tilde{M}_T.$$

Since $(\tilde{V}_t)$ is a $\mathbb{P}^*$-martingale, $\tilde{V}_t = \tilde{M}_t$ and

$$\tilde{U}_t = \tilde{V}_t - \tilde{A}_t.$$

As a result, the writer of an option can hedge himself perfectly: he can generate a portfolio worth V_t at time t and may even withdraw some funds (A_t) for consumption. As for the buyer, there is no point executing the option after time

$$\nu = \inf\{j : A_{j+1} \geq 0\}.$$

Hence the optimal exercise time is given by a stopping time $\tau \leq \nu$ that makes U^τ a $\mathbb{P}^*$-martingale.

1.5.5 American options in the CRR model

We have seen above, a European call option has a non-negative time value. This suggests that an early exercise of a European call is not beneficial. In order to make this more precise, let is denote by (U_t) the discounted value process associated with an American option that is characterized by an adapted process (Z_t) and let (u_t) be the discounted price process of a European option defined by the terminal payoff Z_T. The supermartingale property of (U_t) and the martingale property of (u_t) under $\mathbb{P}^*$ implies that

$$U_t \geq \mathbb{E}^*[U_T|\mathcal{F}_t] = \mathbb{E}^*[\tilde{Z}_T|\mathcal{F}_t] = \mathbb{E}^*[u_T|\mathcal{F}_t] = u_t$$

for all t. If, at the same time $u_t \geq \tilde{Z}_t$ then (U_t) is a $\mathbb{P}^*$ martingale because (U_t) is the *smallest* $\mathbb{P}^*$-supermartingale that dominates $(\tilde{Z}_t)$. For a call option

$$Z_t = (S_t - K)^+$$

so the martingale property of the discounted price process (X_t) yields

$$\begin{aligned} u_t &= \frac{1}{(1+r)^T}\mathbb{E}^*[(S_T - K)^+|\mathcal{F}_t] \\ &\geq \mathbb{E}^*[X_T - K(1+r)^{-T}|\mathcal{F}_t] \\ &= X_t - K(1+r)^{-T} \\ &\geq \tilde{Z}_t \end{aligned}$$

provides the risk free rate is non-negative. This shows that it is not beneficial to exercise an American call option early. Unlike the European call option, the time value

$$W_t := (1+r)^t\mathbb{E}^*\left[\frac{(K - S_T)^+}{(1+r)^T}|\mathcal{F}_t\right] - (K - S_T)^+$$

of a European put option usually becomes negative at a certain time. This corresponds to an early exercise premium, which, in turn, is the surplus which an owner of an American option would have over the value of a European put.

We illustrate this effect in the context of the CRR model. To this end, we recall that

$$S_t = S_0\lambda_t \quad \text{where } \lambda_t := \prod_{j=1}^{t} R_j.$$

where the returns process (R_t) is given by a sequence of independent and identically distributed binary random variables:

$$R_t \in \{a, b\} \quad \text{with} \quad -1 < a < r < b$$

and

$$\mathbb{P}^*[R_t = b] = p^* = \frac{r-a}{b-a}.$$

We assume that $r > 0$ and $a < 0$ and denote by

$$\pi(x) := \sup_{\tau \in \mathcal{T}, \mathcal{T}} \mathbb{E}^*\left[\frac{(K - x\Lambda_\tau)^+}{(1+r)^\tau}\right]$$

the value of an American put as a function of the initial price $x = S_0$. If the option is far out of the money in the sense that

$$x \geq \frac{K}{(1+a)^T}$$

then $S_t \geq K$ for all t and the payoff of the put is always zero. In particular $\pi(x) = 0$. If

$$x \leq \frac{K}{(1+b)^T},$$

then $S_t \leq K$ for all t and the martingale property of asset prices yields

$$\pi(x) = \sup_{\tau \in \mathcal{T}_{,T}} \mathbb{E}^* \left[\frac{K - x\Lambda_\tau}{(1+r)^\tau} \right] = \sup_{\tau \in \mathcal{T}_{,T}} \mathbb{E}^* \left[\frac{K}{(1+r)^\tau} - x \right] = K - x.$$

In this case the price of the American put equals its intrinsic value at time $t = 0$ and immediate execution is optimal. Next, we consider the case

$$K \leq x < \frac{K}{(1+a)^T}$$

where the option is "at the money" (that is, $K \approx x$) or not too far out of the money. Under the risk neutral measure (R_t) is essentially a symmetric random walk. Thus

$$\mathbb{P}^*[\limsup_{t\to\infty} R_t = +\infty] = 1 \quad \text{and} \quad \mathbb{P}^*[\liminf_{t\to\infty} R_t = -\infty] = 1.$$

This implies that for a large enough t the probability of a non-zero payoff from immediate execution is strictly positive:

$$\mathbb{P}[Z_t > 0] > 0.$$

It follows that $\pi(x)$ is strictly positive while the intrinsic value $(K - x)^+$ vanishes. Hence immediate execution is not optimal. Thus, the map $x \mapsto \pi(x)$ has the following qualitative structure: there exists x^* with

$$\frac{K}{(1+b)^T} < x^* \leq K$$

such that

$$\begin{cases} \pi(x) = (K-x)^+ & \text{for } x \leq x^* \\ \pi(x) > (K-x)^+ & \text{for } x^* \leq x < K/(1+a)^T \\ \pi(x) = 0 & \text{else} \end{cases}.$$

Remark 1.5.11 *In the context of the CRR model asset prices, and hence discounted asset prices follow a time-homogeneous Markov chain. Furthermore, the value from immediate execution is given in term of a time-dependent function of the stock price:*

$$Z_t = h(t, S_t).$$

The general theory of Snell envelopes for Markoc chain implies that the value process is also given in terms of a time-dependent transformation of asset prices:

$$U_t = u(t, S_t)$$

where the functions $u(t,\cdot)$ can be determined recursively:

$$u(T,x) = h(T,x)$$

and

$$u(t,x) = \max\{h(t,x), u(t+1, x(1+b)p^* + u_{t+1}(x(1+a)(1-p^*)\}.$$

From this we see that the state space $[0,T]\times[0,\infty)$ can be decomposed into two regions, a stopping region

$$\mathcal{R}_s = \{(t,x) : u(t,x) = h(t,x)\}$$

and a continuation region

$$\mathcal{R}_c = \{(t,x) : u(t,x) > h(t,x)\}$$

and the minimal optimal stopping time can be viewed as the first exit time of the time-space process (t, S_t) (a homogeneous Markov chain) from $\mathcal{R}_c$.

1.5.6 Arbitrage-free prices

In this section we give up the assumption of market completeness and characterize the set of all arbitrage free prices of a discounted American claim H with (discounted) payoff H_t when exercised at time t. To this end, we identify, for a given exercise strategy (stopping time) τ the payoff H_τ at τ with a European claim with payoff H_τ at time T and recall that

$$\Pi(H_\tau) = \{\mathbb{E}^*[H_\tau] : \mathbb{P}^* \in \mathcal{P}^*, \mathbb{E}^*[\mathcal{H}_\tau] < \infty\}.$$

Definition 1.5.12 *A number $\pi \in \mathbb{R}$ is called an arbitrage free price for an American claim H if:*

- *there exists τ and $\pi' \in \Pi(H_\tau)$ such that $\pi \le \pi'$;*
- *there exists no τ' such that $\pi < \pi'$ for all $\pi' \in \Pi(H_\tau)$.*

The set of all arbitrage-free prices is denoted by $\Pi(H)$.

The first condition says that the claim is not too expensive from the buyer's perspective while the latter states that the claim is not too cheap from the seller's point of view. We see from the previous definition that when $\pi \in \Pi(H)$ then $\pi \in \Pi(H_\tau^*)$ for some τ^* and hence

$$\pi = \mathbb{E}^*[H_{\tau^*}] \quad \text{for some } \tau^*, \mathbb{P}^*.$$

Furthermore, $\pi \ge \inf_{\mathbb{P}^*} \mathbb{E}^*[H_\tau]$ for all stopping times, so we have the following arbitrage bounds: for all $\pi \in \Pi(H)$

$$\sup_\tau \inf_{\mathbb{P}^*} \mathbb{E}^*[H_\tau] \le \pi \le \sup_\tau \sup_{\mathbb{P}^*} \mathbb{E}^*[H_\tau]. \tag{1.12}$$

For the benchmark case of a complete market this yields:

$$\Pi(H) = \left\{\sup_\tau \mathbb{E}^*[H_\tau]\right\}.$$

If the market is incomplete, we shall work the additional assumption that

$$H_t \in L^1(\mathbb{P}^*) \quad \text{for all} \quad \mathbb{P}^* \in \mathcal{P}^*$$

so that $\sup_\tau \inf_{\mathbb{P}^*} \mathbb{E}^*[H_\tau] < \infty$ and denote by

$$U_t^{\mathbb{P}^*} = \text{esssup}_\tau \mathbb{E}^*[H_\tau \mid \mathcal{F}_t]$$

the Snell envelope of (H_t) with respect to $\mathbb{P}^* \in \mathcal{P}^*$.
For the remainder of this section we shall *assume* that we can interchange the inf and sup in (1.12) so that

$$\sup_\tau \inf_{\mathbb{P}^*} \mathbb{E}^*[H_\tau] = \inf_{\mathbb{P}^*} U_0^{\mathbb{P}^*} \quad \text{and} \quad \sup_\tau \sup_{\mathbb{P}^*} \mathbb{E}^*[H_\tau] = \sup_{\mathbb{P}^*} U_0^{\mathbb{P}^*}.$$

It will turn out that this assumption is indeed satisfied but we choose to postpone the proof to a later chapter.

Theorem 1.5.13 *Either $|\Pi(H)| = 1$ or $\Pi(H)$ is an interval with interval bounds*

$$\pi_{inf}(H) = \inf_{\mathbb{P}^*} U_0^{\mathbb{P}^*} \quad \textit{and} \quad \pi_{sup}(H) = \sup_{\mathbb{P}^*} U_0^{\mathbb{P}^*}.$$

Furthermore, $\pi_{sup}(H) \notin \Pi(H)$.

We close this section with a brief discussion of attainability of American claims.

Definition 1.5.14 *A discounted American claim H is called* attainable *if there exists a stopping time τ and a self-financing strategy ξ such that the associated value process (V_t) satisfies almost surely*

$$V_t \geq H_t \textit{ for all } t \textit{ and } V_\tau = H_\tau.$$

The following result is stated without proof.

Theorem 1.5.15 *For a discounted American claim H the following statements are equivalent:*

- *H is attainable.*
- *$|\Pi(H)| = 1$.*
- *$\pi_{sup}(H) \in \Pi(H)$.*

1.5.7 The lower Snell envelope

The goal of this section is to prove the Minimax theorem, i.e., to show that the interchange of the sup and inf in (refinfsup) was indeed justified. To this end, we establish an auxiliary result on Snell envelopes.

Proposition 1.5.16 *Let (H_t) be an adapted process on $L^1(\Omega, \mathcal{F}, Q)$, τ a stopping time and*

$$\mathbb{T}_\tau = \{\sigma : \sigma \textit{ stopping time }, \sigma \geq \tau \, Q\textit{-a.s.}\}$$

Then the Snell envelope (U_t) of (H_t) satisfies

$$U_\tau = \text{esssup}_{\sigma \in \mathbb{T}_\tau} \mathbb{E}_Q[U_\sigma \mid \mathcal{F}_\tau].$$

Stable sets

As a first step towards the proof of the Minimax theorem we are now going to introduce the notion of a pasting of probability measures at stopping times. More precisely, we consider two equivalent probability measures Q_1 and Q_2 on some probabilistic basis $(\Omega, \mathcal{F}(\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ with density $\frac{dQ_2}{dQ_1} = Z_T$ along with a stopping time τ. The *pasting* $\tilde{Q}$ of Q_1 and Q_2 at τ is given

$$\tilde{Q}[A] = \mathbb{E}_{Q_1}\left[\mathbb{E}_{Q_2}[\mathbf{1}_A \mid \mathcal{F}_\tau]\right].$$

Lemma 1.5.17 *The measure $\tilde{Q}$ satisfies the following conditions:*

(i) $\tilde{Q} = Q_1$ *on* $\mathcal{F}_\tau$.

(ii) $\frac{d\tilde{Q}}{dQ_1} = \frac{Z_T}{Z_t}$ *where* $Z_t = \frac{dQ_2}{dQ_1}|_{\mathcal{F}_t}$.

(iii) For all stopping times σ and $Y \geq 0$ we have

$$\mathbb{E}_{\tilde{Q}}[Y \mid \mathcal{F}_\sigma] = \mathbb{E}_{Q_1}\left[\mathbb{E}_{Q_2}[Y \mid \mathcal{F}_{\sigma\wedge\tau}] \mid \mathcal{F}_\tau\right].$$

We say that a family $\mathcal{Q}$ of probability measures is *stable* if for any stopping time τ any two measures Q_1, Q_2 from the set $\mathcal{Q}$, the pasting $\tilde{Q}$ also belongs to $\mathcal{Q}$. One example of a stable set is the set of equivalent martingale measures as shown by the following lemma.

Lemma 1.5.18 *The set $\mathcal{P}^*$ of equivalent martingale measures is stable.*

We are now ready to study lower Snell envelopes; they are key to our subsequent analysis. For this, let $\mathcal{Q}$ be a stable set and (H_t) and adapted process such that

$$H_t \in L^1(Q) \quad \text{for all} \quad Q \in \mathcal{Q}.$$

The *lower Snell envelope* associated with (H_t) is given by

$$\underline{U}_t := \text{essinf}_{Q\in\mathcal{Q}} U_t^Q = \text{essinf}_{Q\in\mathcal{Q}} \text{esssup}_{\tau\in\mathbb{T}_\tau} \mathbb{E}[H_\tau \mid \mathcal{F}_t].$$

In terms of the lower Snell envelope, our goal is to show that

$$\underline{U}_t = \text{esssup}_{\tau\in\mathbb{T}_\tau} \text{essinf}_{Q\in\mathcal{Q}} \mathbb{E}[H_\tau \mid \mathcal{F}_t].$$

This can be achieved by solving the following robust stopping problem:

$$\text{maximize } \inf_{Q\in\mathcal{Q}} \mathbb{E}_Q[H_\tau] \text{ over all stopping times } \tau. \tag{1.13}$$

Remark 1.5.19 *The stopping problem (1.13) can be viewed as a stopping game with respect to the non-additive expectation operator*

$$\inf_{Q\in\mathcal{Q}} \mathbb{E}_Q[\cdot].$$

This, in turn can be viewed as a optimal stopping game under model uncertainty where $\mathcal{Q}$ represents a set of possible financial market models.

The remainder of this section is devoted to the proof of the following theorem.

Theorem 1.5.20 *Let*

$$\tau_t := \inf\{u \geq t : \underline{U}_u = H_u\}.$$

Then it holds $\mathbb{P}$*-a.s. that*

$$\underline{U}_t = \text{essinf}_Q \mathbb{E}_Q[H_{\tau_t}|\mathcal{F}_t].$$

It follows from the preceding theorem that the Minimax result holds because

$$\begin{aligned}\underline{U}_t &= \inf_Q \sup_\tau \mathbb{E}[H_\tau] \\ &\geq \sup_\tau \inf_Q \mathbb{E}_Q[H_\tau] \\ &\geq \inf_Q \mathbb{E}_Q[H_{\tau_t}] \\ &= \underline{U}_t.\end{aligned}$$

The proof of Theorem 1.5.20 is based on two auxiliary results.

Lemma 1.5.21 *Let* $Q_1, Q_2 \in \mathcal{Q}$ *and* τ *a stopping time. Let* $B \in \mathcal{F}_\tau$ *and* $\tilde{Q}$ *the pasting at*

$$\sigma = \tau \mathbf{1}_B + T\mathbf{1}_{B^c}.$$

Then

$$U_\tau^{\tilde{Q}} = \mathbf{1}_B U_\tau^{Q_1} + \mathbf{1}_{B^c} U_\tau^{Q_2} \qquad \mathbb{P}\text{-}a.s.$$

The following lemma states that the lower Snell envelope can be approximated from above.

Lemma 1.5.22 *Let* $Q \in \mathcal{Q}$ *and* τ *a stopping time. There exists* $Q_k \in \mathcal{Q}$ *such that*

$$Q_k = Q \quad on \quad \mathcal{F}_\tau$$

and

$$U_\tau^{Q_k} \downarrow \text{esssup}_{\hat{Q} \in \mathcal{Q}} U_\tau^{\hat{Q}} = \underline{U}_\tau \quad as \quad k \to \infty.$$

We are now ready to prove the main result of this section.

Proof of Theorem 1.5.20: The super-martingale property of (U_t^Q) yields

$$U_t^Q \geq \mathbb{E}_Q[H_{\tau_t} \mid \mathcal{F}_\tau] \quad (Q \in \mathcal{Q})$$

so we only need to prove the converse inequality. To that end, let $Q, Q_k \in \mathcal{Q}$ such that

$$Q_k = Q \quad \text{on} \quad \mathcal{F}_{\tau_t}, \quad U_{\tau_t}^{Q_k} \downarrow \underline{U}_{\tau_t}.$$

Then

$$\begin{aligned}\mathbb{E}_Q[H_{\tau_t} \mid \mathcal{F}_t] &= \mathbb{E}_Q[\underline{U}_{\tau_t} \mid \mathcal{F}_t] \\ &= \mathbb{E}_Q[\lim_k U_{\tau_t}^{Q_k} \mid \mathcal{F}_t] \\ &= \lim_k \mathbb{E}_Q[U_{\tau_t}^{Q_k} \mid \mathcal{F}_t] \quad \text{(by monotone convergence)} \\ &= \lim_k \mathbb{E}_{Q_k}[U_{\tau_t}^{Q_k} \mid \mathcal{F}_t] \quad \text{(because } Q_k = Q \text{ on } \mathcal{F}_{\tau_t} \supset \mathcal{F}_t) \\ &= \lim_k U_t^{Q_k} \quad \text{(by the martingale property)} \\ &\geq \underline{U}_t.\end{aligned}$$

□

1.6 Introduction to risk measures

In this section we follow Chapter 4 of Föllmer & Schied (2004) in discussing the problem of quantifying the risk of a financial position X. In a probabilistic model specified by a set of scenarios and a probability measure on scenarios we could try to measure the risk in terms of moments and quantiles. A classical measure of risk is the *variance*. However, it does not capture a basic asymmetry in the financial interpretation of X. Here, the *downside risk* matters. This asymmetry is taken into account by measures such as *Value at Risk*. V@R, however, fails to satisfy some natural consistency requirements. Such observations have motivated an axiomatic approach to risk measures.

1.6.1 Risk measures and their acceptance sets

Let Ω be a set of scenarios. A financial position is described by a mapping $X : \Omega \to \mathbb{R}$ where $X(\omega)$ is the discounted net worth at the end of the trading period if $\omega \in \Omega$ realizes. We assume throughout that X belongs to a given class $\mathcal{X}$, where $\mathcal{X}$ is a linear space of bounded functions. The *risk* associated with X is quantified by some number $\varrho(X)$.

Definition 1.6.1 *A mapping $\varrho : \mathcal{X} \to \mathbb{R}$ is called a monetary measure of risk if it satisfies the following conditions for all $X, Y \in \mathcal{X}$:*

- *Monotonicity: $\varrho(X) \geq \varrho(Y)$ if $X \leq Y$.*
- *Translation (Cash) Invariance: If $m \in \mathbb{R}$, then $\varrho(X + m) = \varrho(X) - m$.*

Translation invariance is motivated by the interpretation of $\varrho(X)$ as a capital requirement. We view $\varrho(X)$ as the amount of money that should be added to the position X in order to make it acceptable from the point of view of a supervising agency. Translation invariance implies that

$$\varrho(X + \varrho(X)) = 0 \quad \text{and} \quad \varrho(m) = \varrho(0) - m \qquad (m \in \mathbb{R}).$$

Remark 1.6.2 *Monotonicity along with translation invariance implies that monetary risk measures are Lipschitz continuous with respect to the supremum norm:*

$$|\varrho(X) - \varrho(Y)| \leq \|X - Y\|.$$

From a practical point of view monetary risk measures should encourage diversification. The risk associated with a diversified portfolio should be no greater than the risk associated with a non-diversified portfolio. This property is captured by the notion of "convexity".

Definition 1.6.3 *A monetary risk measure $\varrho : \mathcal{X} \to \mathbb{R}$ is called a convex measure of risk is it satisfies*

- *Convexity: $\varrho(\lambda X + (1 - \lambda)Y) \leq \lambda\varrho(X) + (1 - \lambda\varrho(Y))$ for $0 \leq \lambda \leq 1$.*

A convex measure of risk is called a coherent measure of risk if it is positively homogeneous, i.e., if it satisfies

- *Positive Homogeneity: If* $\lambda \geq 0$ *then* $\varrho(\lambda X) = \lambda \varrho(X)$.

If a risk measure is positively homogeneous, it is *normalized*, i.e, $\varrho(0) = 0$. Under the assumption of positive homogeneity, convexity is equivalent to

- *Subadditivity:* $\varrho(X + Y) \leq \varrho(X) + \varrho(Y)$.

This property allows to decentralize the task of managing the risk arising from a collection of different positions. A risk manager could, for instance, assign different risk limits to different trading "desks" and the overall risk would be bounded by the sum of the individual risks. However, in many situations risk grows in non-linear manner; a prominent example is liquidity risk. We shall hence focus on convex risk measures.

Example 1.6.4 *(Entropic Risk Measure) Let* $\mathbb{P}$ *be a probability measure on* $(\Omega, \mathcal{F})$. *The entropic risk measure is given by*

$$\varrho(X) = \frac{1}{\beta} \log \mathbb{E}_{\mathbb{P}}[e^{-\beta X}].$$

The entropic risk measure is analytically convenient. It is primarily of academic interest, though, because it is closely related to exponential utility functions.

We denote the *acceptance set associated with* ϱ, i.e., the set of financial positions that are acceptable in the sense that they do not require any additional capital by

$$\mathcal{A}_\varrho := \{X \in \mathcal{X} : \varrho(X) \leq 0\}.$$

The following proposition summarizes the link between risk measures and their acceptance sets.

Proposition 1.6.5 *Suppose that* ϱ *is a monetary measure of risk with acceptance set* $\mathcal{A} = \mathcal{A}_\varrho$.

(i) $\mathcal{A} \neq \varnothing$ *and satisfies the following two conditions:*

$$\inf\{m \in \mathbb{R} : m \in \mathcal{A}\} > -\infty \quad \textit{and} \quad X \in \mathcal{A}, Y \in \mathcal{X}, X \geq Y \quad \Rightarrow \quad Y \in \mathcal{A}. \tag{1.14}$$

Moreover, $\mathcal{A}$ *is closed in the sense that for* $X \in \mathcal{A}$ *and* $Y \in \mathcal{X}$ *we have*

$$\{\lambda \in [0,1] : \lambda X + (1-\lambda)Y \in \mathcal{A}\} \textit{ is closed in } [0,1].$$

(ii) ϱ *can be recovered from* $\mathcal{A}$:

$$\varrho(X) = \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\}.$$

(iii) ϱ *is a convex measure of risk if and only if* $\mathcal{A}$ *is convex.*

(iv) ϱ *is positively homogeneous if and only if* $\mathcal{A}$ *is a cone.*

Conversely, we can take as given a set of acceptable positions $\mathcal{A} \subset \mathcal{X}$ and define $\varrho_{\mathcal{A}}(X)$ as the minimal capital requirement that makes X acceptable:

$$\varrho_{\mathcal{A}}(X) = \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\}.$$

Proposition 1.6.6 *Suppose that $\mathcal{A}$ is a non-empty subset of $\mathcal{X}$ and satisfies (1.14). Then the functional $\varrho_{\mathcal{A}}$ has the following properties:*

(i) $\varrho_{\mathcal{A}}$ is a monetary measure of risk.

(ii) If $\mathcal{A}$ is convex, then $\varrho_{\mathcal{A}}$ is a convex measure of risk.

(iii) If $\mathcal{A}$ is a cone, then $\varrho_{\mathcal{A}}$ is positively homogeneous.

Let us now consider some specific examples. We take $\mathcal{X}$ as the set of all bounded measurable functions on a measurable space $(\Omega, \mathcal{F})$ and denote by $\mathcal{M}_1$ the class of all probability measures on $(\Omega, \mathcal{F})$.

Example 1.6.7 *(Worst Case Measure) The worst case measure is defined by*

$$\varrho_{\max}(X) = -\inf_{\omega \in \Omega} X(\omega).$$

The value $\varrho_{\max}(X)$ is the least upper bound for the potential loss than can occur in any scenario. It is the most conservative risk measure in the sense that any other other measure ϱ satisfies

$$\varrho(X) \leq \varrho(\inf_{\omega} X(\Omega)) = \varrho_{\max}(X).$$

It can be represented as

$$\varrho_{\max}(X) = \sup_{Q \in \mathcal{M}_1} \mathbb{E}[-X].$$

Example 1.6.8 *(Floors) Let $\mathcal{Q} \subset \mathcal{M}_1$ and consider a mapping $\gamma : \mathcal{Q} \to \mathbb{R}$ with $\sup_{\mathcal{Q}} \gamma < \infty$ that specifies for any $Q \in \mathcal{Q}$ a "floor" $\gamma(Q)$. Suppose that a position is acceptable if*

$$\mathbb{E}_Q[X] \geq \gamma(Q).$$

The acceptance set is convex so the associated risk measure ϱ is convex. It takes the form

$$\varrho(X) = \sup_{Q \in \mathcal{Q}} \left(\gamma(Q) - \mathbb{E}_Q[X] \right).$$

Alternatively, we can represent ϱ in terms of the penalty function

$$\alpha(Q) = \begin{cases} -\gamma(Q) & \text{if } Q \in \mathcal{Q} \\ +\infty & \text{else} \end{cases}$$

as

$$\varrho(X) = \sup_{Q \in \mathcal{M}_1} \left(\mathbb{E}_Q[-X] - \alpha(Q) \right).$$

Example 1.6.9 *(V@R) Suppose that we have specified a model, i.e., a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. A position is often considered acceptable if the probability of a loss is bounded by a given level $\lambda \in (0,1)$, i.e., if*

$$\mathbb{P}[X < 0] \leq \lambda.$$

The corresponding monetary risk measure is called Value at Risk *at level λ. It is defined by*

$$V@R_{\lambda}(X) = \inf\{m \in \mathbb{R} : \mathbb{P}[m + X < 0] \leq \lambda\}.$$

V@R is positively homogeneous but typically not convex, i.e., it may penalize diversification. The reason is that it only looks at the probability that something is happening but not at "what happens if something happens".

In the preceding three examples we encountered risk measures that allowed for a robust representation. In the next section we show how such representations arise in a systematic manner.

1.6.2 Robust representations of risk measures

Throughout this section we assume that $\mathcal{X}$ denotes the set of bounded measurable functions $f : \Omega \to \mathbb{R}$ equipped with the sup-norm $\|\cdot\|$. We denote by $\mathcal{M}_1$ the class of all probability measures on $(\Omega, \mathcal{F})$ and by

$$\mathcal{M}_{1,f} = \mathcal{M}_{1,f}(\Omega, \mathcal{F})$$

the class of all finitely additive set functions $Q : \mathcal{F} \to [0,1]$ which are normalized to $Q[\Omega] = 1$.

Proposition 1.6.10 *A functional ϱ on $\mathcal{X}$ is a coherent measure of risk if and only if there exists a set $\mathcal{Q} \subset \mathcal{M}_{1,f}$ such that*

$$\varrho(X) = \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[-X].$$

Moreover, $\mathcal{Q}$ can be chosen as a convex set for which the supremum is attained:

$$\varrho(X) = \max_{Q \in \mathcal{Q}} \mathbb{E}_Q[-X].$$

Our goal is now twofold; (i) we would like to have a similar result for convex risk measures; (ii) it is an undesirable feature of the Riesz representation theorem that we only obtain a representation in terms of finitely additive set functions rather than probability measures. It turns out that (i) can be achieved by introducing a penalization; (ii) requires additional continuity or tightness conditions.

Robust representations of convex risk measures

Let $\alpha : \mathcal{M}_{1,f} \to \mathbb{R} \cup \{\infty\}$ be any function such that

$$\inf_{Q \in \mathcal{M}_{1,f}} \alpha(Q) \in \mathbb{R}.$$

For each $Q \in \mathcal{M}_{1,f}$ the function $X \mapsto \mathbb{E}_Q[-X] - \alpha(Q)$ is convex, monotone and translation invariant. These properties are preserved when taking the supremum over the set of additive set functions. Hence

$$\varrho(X) := \sup_{Q \in \mathcal{M}_{1,f}} (\mathbb{E}_Q[-X] - \alpha(Q))$$

defines a convex measure of risk such that

$$\varrho(0) = -\inf_{Q \in \mathcal{M}_{1,f}} \alpha(Q). \tag{1.15}$$

The functional α will be called a *penalty function* for ϱ on $\mathcal{M}_{1,f}$ and we say that ϱ is represented by α on $\mathcal{M}_{1,f}$.

Theorem 1.6.11 *Any convex measure of risk ϱ on $\mathcal{X}$ is of the form*

$$\varrho(X) = \max_{Q \in \mathcal{M}_{1,f}} (\mathbb{E}_Q[-X] - \alpha_{\min}(Q))$$

where the penalty function $\alpha_{\min}$ is given by

$$\alpha_{\min}(Q) := \sup_{X \in \mathcal{A}_\varrho} \mathbb{E}_Q[-X].$$

Moreover, $\alpha_{\min}$ is the minimal penalty function that represents ϱ.

The penalty function arising in (1.15) is not necessarily unique. For a coherent risk measure we already have a representation via some set $\mathcal{Q} \subset \mathcal{M}_{1,f}$ that corresponds to the penalty function

$$\alpha(Q) = \begin{cases} 0 & \text{if } Q \in \mathcal{Q} \\ +\infty & \text{else} \end{cases}.$$

The following corollary shows that the minimal penalty function of a coherent risk measure is always of this form.

Corollary 1.6.12 *The minimal penalty function of a coherent risk measure takes only the values* 0 *and* $+\infty$*.*

Robust representations in terms of probability measures

In the sequel we are interested in those convex measures of risk that admit a representation in terms of σ-additive probability measure. Such measures can be represented by penalty functions that are infinite outside the set $\mathcal{M}_1$ of probability measures on $(\Omega, \mathcal{F})$:

$$\varrho(X) := \sup_{Q \in \mathcal{M}_1} \left(\mathbb{E}_Q[-X] - \alpha(Q)\right). \tag{1.16}$$

This representation is closely related to continuity properties of ϱ. We say that a risk measure is continuous from above if

$$X_n \downarrow X \quad \text{implies} \quad \varrho(X_n) \uparrow \varrho(X).$$

The following lemma shows that continuity from above is equivalent to lower-semicontinuity with respect to bounded pointwise convergence:

$$X_n \to X \in \mathcal{X} \quad \text{and} \quad \sup \|X_n\|_\infty < \infty$$

implies

$$\varrho(X) \le \liminf_{n\to\infty} \varrho(X_n). \tag{1.17}$$

This property is usually referred to as the Fatou property.

Lemma 1.6.13 *Continuity from above is equivalent to the Fatou property.*

We are now ready to show that a convex risk measure that admits a representation in term of probability measures is continuous from above.

Lemma 1.6.14 *A convex measure of risk ϱ which admits a representation of the form (1.16) is continuous from above.*

While "Continuity from Above" is necessary for (1.16) to hold, "Continuity from Below" is sufficient.

Proposition 1.6.15 *Let ϱ be a convex measure of risk that is continuous from below, i.e.,*

$$X_n \uparrow X \Rightarrow \varrho(X_n) \downarrow \varrho(X)$$

and let α be a representing penalty function. Then

$$\alpha(Q) < \infty \Rightarrow Q \in \mathcal{M}_1.$$

We close this section with three prominent examples of risk measures that have certain continuity properties: the entropic risk measure, shortfall risk and average value at risk. All these measures require an a-priori model, i.e., a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$.

Example 1.6.16 *(Entropic Risk Measure) Consider the penalty function $\alpha : \mathcal{M}_1 \to (0, \infty]$ that is defined by*

$$\alpha(Q) = \frac{1}{\beta} H(Q|P) \quad \text{where} \quad H(Q|P) = \mathbb{E}_Q[\log \frac{dQ}{d\mathbb{P}}]$$

where $H(Q|P)$ denotes the relative entropy of $Q \in \mathcal{M}_1$ with respect to $\mathbb{P}$. The entropic risk measure is given by

$$\varrho(X) = \sup_{Q \in \mathcal{M}_1} \left\{ \mathbb{E}_Q[-X] - \frac{1}{\beta} H(Q|\mathbb{P}) \right\}$$

and one can show that

$$\varrho(X) = \frac{1}{\beta} \log \mathbb{E}_{\mathbb{P}}[e^{-\beta X}].$$

Example 1.6.17 *(Shortfall Risk) Let $l : \mathbb{R} \to \mathbb{R}$ be a convex loss function, i.e., a convex increasing function that is not identically zero and suppose that a position X is acceptable if the expected loss $\mathbb{E}_{\mathbb{P}}[l(-X)]$ is bounded from above by some interior point x_0 in the range of l. Thus the acceptance set takes the form*

$$\mathcal{A} = \{X \in \mathcal{X} : \mathbb{E}_{\mathbb{P}}[l(-X)] \le x_0\}.$$

The associated convex risk measure $\varrho_{\mathcal{A}}$ is continuous from below; its penalty function can be given in closed form. The special case of shortfall risk is given by the acceptance set

$$\mathcal{A} = \{X \in \mathcal{X} : \mathbb{E}_{\mathbb{P}}[l(X^-)] \le x_0\}.$$

Example 1.6.18 *(AV@R) The Average Value at Risk at level $\lambda \in (0,1)$ of a position X is given by*

$$AV@R_\lambda(X) = \frac{1}{\lambda} \int_0^\lambda V@R_\gamma(X) d\gamma.$$

Unlike V@R that fails to be convex, $AV@R_\lambda$ is coherent. It is also continuous form below.

In many cases continuity from below (i.e. continuity) is too restrictive. One can show that certain tightness conditions are also sufficient for (1.16).

1.6.3 V@R, AV@R and shortfall risk

In this section we briefly discuss the current industry standard Value at Risk and its generalization Average Value at Risk which, in contrast to V@R is convex.

V@R

The V@R at level λ can be expressed in terms of λ-quantiles. The λ-quantile of a random variable X is defined as any $q \in \mathbb{R}$ that satisfies

$$\mathbb{P}[X \leq q] \geq \lambda \quad \text{and} \quad \mathbb{P}[X < q] \leq \lambda.$$

The set of all such quantiles is an interval

$$[q_X^-(\lambda), q_X^+(\lambda)]$$

and

$$V@R_\lambda(X) = -q_X^+(\lambda) = q_X^-(1-\lambda).$$

V@R is a positively homogeneous but in general not convex. Our goal is thus to look for convex risk measures that come close to V@R. A first guess is to take the smallest convex risk measure on $L^\infty(\mathbb{P})$, continuous from above that dominates V@R. Such a risk measure, however, does not exist as shown by the following proposition.

Proposition 1.6.19 *For any $X \in L^\infty$ and each $\lambda \in (0,1)$ we have that*

$$V@R_\lambda(X) = \min\{\varrho(X) : \varrho \textit{ convex, continuous from above, dominates } V@R\}.$$

AV@R

A risk measure which is defined in terms of V@R but satisfies the axioms of a coherent risk measure is Average Value at Risk:

$$AV@R_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda V@R_\gamma(X)d\gamma.$$

Remark 1.6.20 *Under mild technical assumptions on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ one can show that AV@R at level λ coincides with the worst conditional expectation at level λ. The latter risk measure is defined as*

$$WCE_\lambda(X) = \sup\{\mathbb{E}[-X|A] : \mathbb{P}[A] > \lambda\}.$$

The following theorem yields a robust representation of AV@R in terms of bounded densities.

Theorem 1.6.21 *Let $\lambda \in (0,1)$. $AV@R_\lambda$ is a coherent risk measure that is continuous from below. It has the representation*

$$AV@R_\lambda(X) = \max_{Q \in \mathcal{Q}_\lambda} \mathbb{E}_Q[-X]$$

where $\mathcal{Q}_\lambda$ is the set of all probability measures that are absolutely continuous with respect to $\mathbb{P}$ whose density is bounded by λ^{-1}. The maximum is attained by the measure Q^* with density

$$\frac{dQ^*}{d\mathbb{P}} = \frac{1}{\lambda}\left(\mathbf{1}_{\{X<q\}} + \kappa \mathbf{1}_{\{X=q\}}\right) \quad \textit{for some } \kappa.$$

If $A \in \mathcal{F}$ with $\mathbb{P}[A] \geq \lambda$, then the density of $\mathbb{P}[\cdot|A]$ with respect to $\mathbb{P}$ is bounded from above by $\frac{1}{\lambda}$. Hence the preceding theorem shows that AV@R dominates WCE. Furthermore,

$$\mathbb{P}[-X \geq V@R_\lambda(X) - \varepsilon] > \lambda$$

so

$$WCE_\lambda(X) \geq \mathbb{E}[-X| - X \geq V@R_\lambda(X) - \varepsilon]$$

and for $\varepsilon \to 0$ we obtain that

$$WCE_\lambda(X) \geq \mathbb{E}[-X| - X \geq V@R_\lambda(X)] \geq V@R_\lambda(X).$$

In particular, we have shown that Average Value at Risk dominates the Worst Conditional Expectation.

Shortfall risk

We close this section with a brief discussion of the shortfall risk. To this end, we introduce a convex, increasing and not identically constant *loss function* $l : \mathbb{R} \to \mathbb{R}$ and define, for an interior point x_0 of its range, an acceptance set

$$\mathcal{A} := \{X \in \mathcal{X} : \mathbb{E}[l(-X)] \leq x_0\}.$$

The associated convex risk measure ϱ is called *shortfall risk*. It's minimal penalty function is concentrated on the set of probability measures on $(\Omega, \mathcal{F})$ and can be given in closed form.

Theorem 1.6.22 *The minimal penalty function of shortfall is given by*

$$\alpha_{\min}(Q) = \inf_{\lambda>0} \frac{1}{\lambda} \left(x_0 + \mathbb{E}[l^*(\lambda dQ/d\mathbb{P})]\right) \quad (Q \in \mathcal{M}_1)$$

where l^* is the Lengendre-Fenchel transform of l, i.e.,

$$l^*(z) := \sup_{x\in\mathbb{R}}(zx - l(x)).$$

The special choice $l(x) = e^{\beta x}$ corresponds to the entropic risk measure. The case

$$l(x) = \frac{1}{p}x^p \quad (p > 1,\, x \geq 0)$$

yields

$$l^*(z) = \frac{1}{q}z^q \text{ if } z \geq 0 \quad \text{and} \quad l^*(z) = +\infty \text{ otherwise}$$

where $q = \frac{p}{p-1}$. If $Q << \mathbb{P}$ with density φ, then $\alpha_{\min}(Q) = \infty$ if $\varphi \notin L^q(\mathbb{P})$. Otherwise the infimum in the representation of $\alpha_{\min}$ is attained for

$$\alpha_{\min}(Q) = (px_0)^{1/p}\mathbb{E}\left[\left(\frac{dQ}{d\mathbb{P}}\right)^q\right]^{1/q}.$$

1.6.4 Law invariance

We argued above that there is no smallest convex risk measure that dominates V@R. The situation is different if we restrict ourselves to convex risk measures that dominate V@R and only depend on the distribution of a financial position. Risk measures that depend only the on the law of a random variable are called law invariant.

Definition 1.6.23 *A risk measure ϱ on $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is called* law invariant *if $\varrho(X) = \varrho(Y)$ if X and Y have the same distribution under $\mathbb{P}$.*

The following theorem shows that AV@R can be viewed as a basis for the set of law invariant risk measures.

Theorem 1.6.24 *A convex risk measure ϱ is law invariant and continuous from above if and only if*

$$\varrho(X) = \sup_{\mu \in \mathcal{M}_1((0,1])} \left\{ \int_0^1 AV@R_\lambda(X)\mu(d\lambda) - \beta_{\min}(\mu) \right\}$$

where

$$\beta_{\min}(\mu) = \sup_{X \in \mathcal{A}_\varrho} \int_0^1 AV@R_\lambda(X)\mu(d\lambda).$$

For a coherent risk measure the preceding theorem takes the following form:

Corollary 1.6.25 *A coherent risk measure ϱ is law invariant and continuous from above if and only if*

$$\varrho(X) = \sup_{\mu \in \mathcal{M}_1((0,1])} \int_0^1 AV@R_\lambda(X)\mu(d\lambda).$$

We proved above that a smallest convex risk measure that dominates V@R does not exist. The following theorem shows that within the class of law invariant risk measures a smallest convex risk measure dominating V@R exists: AV@R.

Theorem 1.6.26 *Suppose that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has no atoms. Then AV@R is the smallest law invariant convex risk measure that is continuous from above and dominates V@R.*

Chapter 2

Continuous Time Finance

2.1 Introduction to continuous time models and Pathwise Itô calculus

In this section we give the definition of stochastic processes in continuous time. We also introduce the Brownian motion or Wiener process that is the core of most financial market models. We show that typical Brownian sample paths are of unbounded variation and nowhere differentiable and give the joint distribution of Brownian motion and its running maximum. We will need this distribution to calculate the fair value of barrier options.

Suggested Reading: Shreve (2005); Chapter 3.

2.1.1 Basic notions and classes of stochastic processes

Throughout we work on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. As usual the filtration $(\mathcal{F}_t)$ models the flow of information and $\mathcal{F}_t$ is interpreted as the set of event which are observable up to time $t \in [0, \infty)$. Unless stated otherwise we always assume that $(\mathcal{F}_t)$ satisfies the usual conditions of right continuity and completeness. This means that $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$. A real-valued stochastic process $X = (X_t)_{t\geq 0}$ is a family of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $\mathbb{R}$. The process is called *adapted* to the filtration $(\mathcal{F}_t)$ if each random variable X_t is $\mathcal{F}_t$-measurable. For every state of the world $\omega \in \Omega$ the mapping

$$X_\cdot(\omega) : [0, \infty) \to \mathbb{R}, \quad t \mapsto X_t(\omega)$$

is called the *trajectory* or *sample path* of X. A stochastic process can hence be viewed as a random draw of sample paths.

Two processes X and Y defined on the same probability space are *modifications* of each other if $\mathbb{P}[X_t = Y_t] = 1$ for every $t \geq 0$. The processes are called *indistinguishable* if almost all their sample paths agree, i.e., $\mathbb{P}[X_t = Y_t \text{ for all } t \geq 0] = 1$. Evidently, if two processes are indistinguishable, then they are modifications of each other. The following lemma establishes a partial converse.

Lemma 2.1.1 *If two processes X and Y are modifications of each other and have almost surely rightcontinuous sample paths, then they are indistinguishable.*

In this course will be particularly interested in processes with continuous sample paths such as Brownian motion or sample paths that are right continuous with left limits (cadlag) such as Poisson processes. An adapted process that satisfied $X_t \in L^1(\mathbb{P})$ for all t is called a

- *submartingale* if $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$ for all $t \geq s$;
- *supermartingale* if $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$ for all $t \geq s$;
- *martingale* if $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ for all $t \geq s$.

An adapted process X is called a *Markov process* if for all $t, s > 0$ and every bounded real-valued function f on $\mathbb{R}$ we have

$$\mathbb{E}[f(X_{t+s})|\mathcal{F}_s] = \mathbb{E}[f(X_{t+s})|\sigma(X_s)].$$

Here $\sigma(X_t)$ denotes the σ-field generated by the random variable X_t. Intuitively, X has the Markov property if the distribution of future states is completely determined by the present state. If X models the price fluctuations of some risky asset the Markov property is equivalent to saying that the market satisfies a weak form of the efficient market hypothesis: all available information is contained in the current price.

2.1.2 Brownian motion

The Brownian motion or Wiener process is the most important building block of continuous-time asset pricing models.

Definition 2.1.2 *A stochastic process $W = (W_t)_{t\geq 0}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a standard one-dimensional Brownian motion if the following conditions are satisfied:*

(i) $W_0 = 0$ $\mathbb{P}$*-a.s.*

(ii) W has independent increments; that is for all $t, s > 0$ we have that

$$W_{t+s} - W_s \quad \text{is independent of} \quad (W_u)_{0\leq u\leq s}.$$

(iii) The increments are stationary and normally distributed:

$$W_{t+s} - W_t \sim N(0, s).$$

(iv) W has almost surely continuous sample paths.

Notice that the increments of a Brownian motion over a time period $[t, t+s]$ are normally distributed with mean zero and variance equal to the length s of the time interval. In particular

$$W_t \sim N(0, t).$$

As for the covariance of the random variables W_t and W_s $(t \geq s)$ observe that

$$\begin{aligned} Cov(W_t, W_s) &= \mathbb{E}[W_t \cdot W_s] - \mathbb{E}[W_t] \cdot \mathbb{E}[W_s] \\ &= \mathbb{E}[(W_t - W_s)W_s] + \mathbb{E}[W_s^2] \\ &\overset{(ii)}{=} \mathbb{E}[(W_t - W_s)] \cdot \mathbb{E}[W_s] + \mathbb{E}[W_s^2] \\ &\overset{(iii)}{=} (t-s) \cdot 0 + s \\ &= s. \end{aligned}$$

In other words, for all $t, s \in \mathbb{R}_+$ we have that

$$Cov(W_t, W_s) = \min\{s, t\} \tag{2.1}$$

and all the finite-dimensional distributions of W are normally distributed with mean zero and covariance function (2.1). A construction of a Wiener process is beyond the scope of this course so we only state an existence result without proof.

Theorem 2.1.3 *A standard Brownian motion exists.*

For a standard Wiener process W let $(\mathcal{F}_t)$ be its canonical filtration, that is $\mathcal{F}_t = \sigma(W_s, s \leq t)$. We will need the following important property.

Proposition 2.1.4 *Let W be a standard BM. Then (W_t) and $(W_t^2 - t)$ are martingales with respect to the canonical filtration.*

A famous characterization result of Brownian motion due to Paul Lévy states that the converse of the preceding proposition is also true. We state the result without further proof.

Theorem 2.1.5 *A continuous real-valued process (X_t) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and starting in zero is a Brownian motion if and only if both (X_t) and $(X_t^2 - t)$ are martingales.*

2.1.3 Quadratic Variation

Since Brownian sample paths are typically of unbounded variation as we will see in this section, integrals of the form $\int W_t(\omega) dW_t(\omega)$ cannot be defined in the usual Riemann-Stieltjes sense. We illustrate this by the following example where we approximate a Brownian path in two "reasonable" ways though the resulting integrals turn out to the be quite different.

Example 2.1.6 *Let W be a standard Wiener process. Both*

$$\phi_1(t, \omega) = \sum_j W_{t_j}(\omega) \mathbf{1}_{[t_j, t_{j+1})}(t)$$

and

$$\phi_2(t, \omega) = \sum_j W_{t_{j+1}}(\omega) \mathbf{1}_{[t_j, t_{j+1})}(t)$$

"reasonably" approximate the sample path $t \to W_t(\omega)$. Since ϕ_1 and ϕ_2 are piecewise constant the integrals

$$\int_0^T \phi_i(t,\omega)dW_t(\omega) \quad (i = 1,2)$$

make sense if we define $\int_s^u dW_t = W_u - W_s$. However,

$$\mathbb{E}\left[\int_0^T \phi_1(t,\omega)dW_t(\omega)\right] = \sum_j \mathbb{E}\left[W_{t_j}(W_{t_{j+1}} - W_{t_j})\right] = 0$$

while

$$\begin{aligned}\mathbb{E}\left[\int_0^T \phi_2(t,\omega)dW_t(\omega)\right] &= \sum_j \mathbb{E}\left[W_{t_{j+1}}(W_{t_{j+1}} - W_{t_j})\right] \\ &= \sum_j \mathbb{E}\left[(W_{t_{j+1}} - W_{t_j})^2\right] \\ &= T.\end{aligned}$$

Remark 2.1.7 *Notice that the integrant ϕ_2 of the preceding example is not adapted to the filtration generated by the Brownian motion W.*

Our goal is to now give a first definition of integrals of smooth functions of a Wiener process with respect to Brownian motion. More generally, we shall define integrals of smooth transformations of continuous functions $X : [0,T] \to \mathbb{R}^n$ of continuous quadratic variation w.r.t X. It turns out that such integrals can be defined pathwise. We denote the components of X by X^i $(i = 1,...,n)$.

Definition 2.1.8 *A partition Π of the time interval $[0,T]$ is a set of points $0 = t_0 < t_1 < \cdots t_n = T$. The mesh is*

$$|\Pi| := \max_i |t_i - t_{i-1}|.$$

Definition 2.1.9 *Let $X : [0,T] \to \mathbb{R}^n$ be given. The variation-process associated with $X^i : [0,T] \to \mathbb{R}$ is defined by*

$$V(X^i) := \sup\left\{\sum_j |X^i(t_j) - X^i(t_{j-1})| : \Pi = (t_0,\ldots,t_n) \text{ is a partition}\right\}$$

If $V(X^i) < \infty$, then X^i is of finite variation. If X is continuous then we say that it is of continuous quadratic variation if the limit

$$< X^j, X^k >_t = \lim_{n\to\infty} \sum_{t_i^n \in \Pi^n, t_i^n \le t} \left(X^j(t_i^n) - X^j(t_{i-1}^n)\right)\left(X^k(t_i^n) - X^k(t_{i-1}^n)\right)$$

exists for all $j,k = 1,...,n$ along a sequence of partitions (Π_n) with $|\Pi_n| \to 0$ as $n \to \infty$ and the map $t \mapsto < X,Y >_t$ is continuous.

Remark 2.1.10 *Notice that we require the quadratic variation to exist only along a given sequence of partition. For the special case of Brownian sample paths we showed that the choice of partition is irrelevant as long as the mesh sizes converge to zero sufficiently fast. In general, this need not be the case, though.*

The quadratic variation process $< X >$ of a real-valued continuous function $X : [0,T] \to \mathbb{R}$ is defined by $< X > := < X, X >$. It is increasing and satisfies $< X >_0 = 0$. If $X \in \mathcal{C}^1$ with $X_0 = 0$, then

$$X_t = \int_0^t X_s' ds \quad \text{and} \quad V(X) = \int_0^t |X_s'| ds.$$

The next proposition shows that X has quadratic variation zero if it is of bounded variation. We deduce that all monotone processes are of trivial quadratic variation.

Proposition 2.1.11 *If $X : [0,T] \to \mathbb{R}$ is continuous and of bounded variation, then its quadratic variation process is trivial, i.e., identically equal to zero*

We recall form our standard calculus course that an integral $\int f dX$ with respect to X can be defined in the usual Riemann-Stieltjes sense only if X is of bounded variation. In particular, the integral cannot be defined in the standard sense when X has a non trivial quadratic variation process. For a typical path of a Brownian motion it turns out that

$$< W(\omega) >_t = t$$

for all times so an integral with respect to Brownian motion cannot be defined as a Riemann-Stieltjes integral giving an explaination for the different outcomes of Example 2.1.6.

Theorem 2.1.12 *Let (Π_n) be a sequence of partitions of $[0,T]$ such that $\sum_n |\Pi_n| < \infty$. Then*

$$\lim_{n\to\infty} V_t^2(W(\omega), \Pi_n) = t \quad \text{for all } t \in [0,T] \quad \mathbb{P}\text{-a.s.}$$

The proof of the preceding theorem used the following result, known as the Borell-Cantelli-Lemma.

Lemma 2.1.13 *(Borell-Cantelli) Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of events defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.*

(i) If $\sum_n \mathbb{P}[A_n] < \infty$, only finitely many A_n occur: $\mathbb{P}[\text{only finitely many } A_n \text{ occur}] = 1$.

(ii) If $\sum_n \mathbb{P}[A_n] = \infty$ and the events $A_1, A_2, \ldots$ are independent, infinitely many A_n occur:

$$\mathbb{P}[\text{infinitely many } A_n \text{ occur}] = 1.$$

Going back to our study of quadratic variationprocesses, the following properties are easily verified.

Proposition 2.1.14 *Let $X, Y : [0,T] \to \mathbb{R}$ be continuous and of continuous quadratic variation. Then the following holds:*

(i) The quadratic variation $< X, Y >$ exists and is continuous if and only if $< X + Y >$ and $X + Y$ exist and are continuous.

(ii) $< \cdot, \cdot >$ defines a (not necessarily positive semi-definite) bilinear form.

(iii) The polarization identity holds:

$$< X, Y > = \frac{1}{2} \left(< X + Y > - < X > - < Y > \right).$$

(iv) The Chauchy-Schwartz inequality holds:

$$|<X,Y>| \leq \sqrt{<X><Y>}.$$

The polarization identity along with the Chauchy-Schwartz inequality yields the following corollary.

Corollary 2.1.15 *Let X be continuous with continuous variation and A be of bounded variation. Then*

$$<X+A>=<X>.$$

Remark 2.1.16 *We notice that the quadratic variation process $<X^i>$ is non-decreasing and hence of bounded variation. By the polarization identity, the co-variation processes $<X^k, X^j>$ for $k \neq j$ can be expressed as the difference of two bounded variation functions and hence $<X^k, X^j>$ is also of bounded variation. As a result, integrals with respect to $d<X^i>$ and $d<X^k, X^j>$ can be defined in the usual way.*

2.1.4 The basic Itô formula

For a given partition $\Pi = (t_i)$ of the interval $[0,T]$ and a given continuous function $X : [0,T] \to \mathbb{R}^n$ of continuous quadratic variation we denote by

$$\Delta_i X^j := X^j_{t_{i+1}} - X^j_{t_i}$$

the increments of X^j along Π. If X were of bounded variation, then for every continuously differentiable function f on $\mathbb{R}^n$ we would have

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s.$$

If $<X> \neq 0$, i.e., X is of unbounded variation, then we need a generalization that accounts for the "many small fluctuations responsible for the quadratic variation". The Itô formula provides such a generalization. For its proof we need the following lemma.

Lemma 2.1.17 *Let (Π_n) be a sequence of partitions of $[0,T]$ with $|\Pi_n| \to 0$ and $g : [0,T] \to \mathbb{R}$ continuous. Then*

$$\lim_{n\to\infty} \sum_{t_i \in \Pi_n, t_i \leq t} g(t_i) \Delta_i X^j \Delta_i X^k = \int_0^t g(s) d<X^j, X^k>_s \quad \text{for all} \quad t \in [0,\infty)$$

We are now ready to state and prove a first version of Itô's formula.

Theorem 2.1.18 *(Itô formula) Let $X : [0,T] \to \mathbb{R}^n$ be continuous with continuous quadratic variation $<X>$ and let $A : [0,T] \to \mathbb{R}^m$ be continuous and of bounded variation. Let $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be of class $\mathcal{C}^{1,2}$. Then*

$$\begin{aligned} F(A_t, X_t) = F(A_0, X_0) &+ \int_0^t \nabla_a F(A_s, X_s) dA_s + \int_0^t \nabla_x F(A_s, X_s) dX_s \\ &+ \frac{1}{2} \sum_{j,k=1}^n \int_0^t \frac{\partial^2}{\partial x^j \partial x^k} F(A_s, X_s) d<X^j, X^k>_s \end{aligned} \tag{2.2}$$

with

$$\int_0^t \nabla_x F(X_s) dX_s := \lim_n \sum_{t_i} \nabla_x F(X_{t_i}) \Delta_i X.$$

A special case of the preceding formula is $A_t = t$. In this case the mapping f is of the form $f(t, X_t)$.

Corollary 2.1.19 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be of class $C^{1,2}$ and $X : [0,T] \to \mathbb{R}$ be continuous with continuous quadratic variation $< X >$. Then*

$$f(t, X_t) - f(0, X_0) = \int_0^t f_t(s, X_s) ds + \int_0^t f_x(s, X_s) dX_s + \frac{1}{2} \int_0^t f_{xx}(s, X_s) d < X >_s .$$

Another immediate result is the product rule which is obtained by applying the Itô formula to the function $f(x, y) = xy$.

Corollary 2.1.20 *(Product Rule) Let (X, Y) be continuous with continuous quadratic variation. Then*

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s dY_s + \int_0^t Y_s dX_s + < X, Y >_t .$$

2.1.5 The quadratic variation of Itô integrals

If $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $G' = g$, then it follows from the Itô formula that the Itô integral

$$Y_t := \int_0^t g(X_s) dX_s$$

is well defined and that the mapping $t \to Y_t$ is continuous. For many applications it will be important to identify the quadratic variation of Y. To this end, we first prove the following result.

Proposition 2.1.21 *Let X be real-valued, continuous and of continuous quadratic variation and $g \in \mathcal{C}^1$. Then the mapping $t \mapsto g(X_t)$ is of continuous quadratic variation and*

$$< g(X) >_t = \int_0^t (g'(X_s))^2 \, d < X >_s .$$

As a corollary to the preceding proposition we obtain a formula for the quadratic variation of Itô integrals.

Corollary 2.1.22 *Let X be real-valued, continuous and of continuous quadratic variation and $g \in \mathcal{C}^1$. Then the process*

$$Y_t := \int_0^t g(X_s) dX_s$$

has quadratic variation

$$< Y >_t = \int_0^t g^2(X_s) d < X >_s .$$

The Itô formula also allows us to evaluate the integral of smooth functions of BM w.r.t. the Wiener process.

Example 2.1.23 *Let W be a standard Brownian motion and $F : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Then*

$$F(W_t) = F(0) + \int_0^t F'(W_s) dW_s + \frac{1}{2} \int_0^t F''(W_s) ds. \tag{2.3}$$

If $F(x) = x^n$, then

$$X_t^n = X_0^n + n\int_0^t X_s^{n-1} dX_s + \frac{n(n-1)}{2} X_s^{n-2} d < X >_s .$$

In particular, we obtain a closed form solution for the integral of a Brownian path with respect to itself:

$$W_t^2 = W_0^2 + 2\int_0^t W_s dW_2 + \int_0^t d < W >_s = 2\int_0^t W_s dW_s + t.$$

The following example is of particular interest in mathematical finance.

Example 2.1.24 *(Geometric Brownian Motion) Given a BM W, and constants $\mu \in \mathbb{R}$ and $\sigma > 0$ the geometric Brownian motion is given by the process*

$$S_t = S_0 \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right).$$

In order to write this process in differential form we apply Itô's formula to the exponential function $x \mapsto e^x$ and to the process

$$X_t = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t.$$

In view of Corollary 2.1.15 we have that $< X >_t = \sigma^2 t$ so Itô's formula yields

$$\begin{aligned} S_t = S_0 \exp(X_t) &= S_0 + \int_0^t S_u dX_u + \frac{1}{2}\int_0^t S_u d < X >_u \\ &= S_0 + \int_0^t S_u \left[\left(\mu - \frac{1}{2}\sigma^2\right) du + \sigma dW_u\right] + \frac{1}{2}\int_0^t \sigma^2 S_u du \\ &= S_0 + \int_0^t \mu S_u du + \int_0^t \sigma S_u dW_u. \end{aligned}$$

In differential form the geometric Brownian motion can be written as

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

Thus, if we assume that a stock price grows at an average rate μ and that the growth rates fluctuate around their average rate according to σW_t, then the price dynamics follows a geometric Brownian motion. This is the assumption underlying the Black-Scholes option pricing model.

2.1.6 Application: The Bachelier Model

Let us denote by X_t the price of some risky asset ("stock") at time $t \geq 0$ and assume that the asset pays no dividends and that there is no storage cost. In this section we discuss the problem of pricing and hedging derivative securities in X in two benchmark models of continuous time where $r = 0$, i.e., all prices are in discounted units. We assume that $t \mapsto X_t(\omega)$ is continuous and of continuous quadratic variation $\langle X(\omega)\rangle_t = \int_0^t (\sigma(s, X_s(\omega)))^2 ds$, where $\sigma(t,x)$ is the "volatility profile". Two classical examples that fit in this framework are the Bachelier and the Black-Scholes model.

Example 2.1.25 (Bachelier) *The Bachelier model assumes that the stock price process is given by*

$$X_t = X_0 + mt + \sigma W_t, \tag{2.4}$$

where $m \in \mathbb{R}$ is the drift parameter, $\sigma > 0$ is the volatility parameter and W is a Brownian motion. From (2.4), we see that $\sigma(t,x) = \sigma$:

$$\langle X\rangle_t = \langle \sigma W\rangle_t = \sigma^2 t.$$

Example 2.1.26 (Black-Scholes) *In the Black-Scholes model the stock price process is given by*

$$X_t = X_0 \exp\left(\left(m - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right), \tag{2.5}$$

which is a geometric Brownian motion (suggested by P. Samuelson as model for asset prices). The process X in (2.5) is the solution to the SDE

$$dX_t = X_t\,(mdt + \sigma dW_t) \qquad (X_0 > 0)$$

from which we obtain

$$\langle X\rangle_t = \int_0^t \sigma^2 X_t^2 dt,$$

that is $\sigma(t,x) = \sigma x$.

We now discuss the problem of valuation and hedging of a derivative security with payoff profile

$$H(\omega) = f(X_T(\omega)).$$

The idea is to determine the fair price as the cost of a (dynamic) replication strategy. The ansatz is to solve the partial differential equation (PDE)

$$\begin{cases} \frac{1}{2}\sigma^2 F_{xx} + F_t &= 0 \qquad (t,x) \in [0,T) \times \mathbb{R} \\ \quad F(T,x) &= f(x) \end{cases}. \tag{2.6}$$

If this PDE admits a classical solution F, i.e. a solution of class $\mathcal{C}^{1,2}$, then F satisfies the assumption of Corollary 2.1.19 and a *replicating strategy* for H can be derived by invoking Itô's formula.

Lemma 2.1.27 *Let $F \in \mathcal{C}^{1,2}$ solve the terminal value problem (2.6). Then it holds for all $\omega \in \Omega$ that*

$$H(\omega) := f(X_T(\omega)) = F(0, X_0) + \int_0^T F_x(s, X_s(\omega))dX_s(\omega). \tag{2.7}$$

The preceding result states that by following the trading strategy $F_x(t, X_t)$ the writer of an option can eliminate all the risk associated with issuing H. Implementing this strategy requires the initial investment $F(0, X_0)$, the fair (arbitrage-free) price of the contingent claim H. In the sequel we are going to show how to solve the pricing and hedging problem for the Bachelier model (Example 2.1.25); an analysis of the Black-Scholes model is postponed to later chapters. We discuss both a PDE and a probabilistic approach.

The PDE Approach. For the Bachelier model the PDE is given by

$$\begin{cases} \frac{1}{2}\sigma^2 F_{xx} + F_t &= 0 \qquad (t,x) \in [0,T) \times \mathbb{R} \\ \quad F(T,x) &= f(x). \end{cases} \tag{2.8}$$

This PDE is very similar to the heat equation:

$$\frac{1}{2}F_{xx} - F_t = 0 \qquad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{2.9}$$

It is easy to check that

$$P(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \tag{2.10}$$

solves the heat equation. The function is called a *fundamental solution*. It has the following important properties.

Proposition 2.1.28 (Smoothing Properties of the Heat Kernel) *Let $P(t,x)$ be the fundamental solution to the heat equation and $g \in C_b(\mathbb{R})$. Then the function*

$$u(t,x) := \int_{\mathbb{R}} g(y) P(t, x-y) dy$$

satisfies the following properties:

(i) It belongs to the class $C^\infty((0,\infty) \times \mathbb{R})$.

(ii) It satisfies the heat equation on $\mathbb{R} \times (0,\infty)$.

(iii) $\lim_{(t,x)\to(0,y),\, t>0} u(t,x) = g(y)$.

With minor modifications, the proof of the preceding theorem can be extended to measurable functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the growth condition

$$|f(x)| \le \alpha \left(1 + e^{C|x|}\right)^2 \tag{2.11}$$

for some $C, \alpha > 0$. Reversing the time and scaling it by σ^2 therefore yields the following result (the proof is left as an exercise).

Theorem 2.1.29 *Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies (2.11). Then the function*

$$F(t,x) := \int_{\mathbb{R}} f(y) P_{\sigma^2(T-t)}(x,y) dy, \qquad (t,x) \in [0,T) \times \mathbb{R} \tag{2.12}$$

belongs to $C^\infty(\mathbb{R} \times [0,T))$, satisfies $\lim_{t\to T} F(t,x) = f(x)$ and hence solves the terminal value problem (2.8).

To conclude, we obtained the no-arbitrage-price $F(t,x)$ and the replication strategy $F_x(t,x)$ in terms of one-dimensional integrals. The function F in (2.12) is given by

$$\begin{aligned} F(t,x) &= \int_{\mathbb{R}} f(y) \frac{1}{\sigma\sqrt{2\pi(T-t)}} \exp\left(-\frac{(x-y)^2}{2\sigma^2(T-t)}\right) dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x + \sigma z\sqrt{T-t}) \exp\left(-\frac{z^2}{2}\right) dz \\ &= \tilde{\mathbb{E}}\left[f(x + \sigma\sqrt{T-t}Z)\right], \end{aligned} \tag{2.13}$$

where $Z \sim \mathcal{N}(0,1)$ under some probability measure $\tilde{\mathbb{P}}$; for specific payoffs $f(.)$, this integral expression can be solved in closed form (see exercises sheet). It is important to notice that the representation of F involves the volatility parameter σ, but not the drift parameter m. We shall further comment on this below.

The probabilistic approach. Thus far, we have solved the pricing and hedging problem invoking only PDE methods; no reference to probability was made. In order to take a more probabilistic approach let us assume that $\sigma \ne 0$ and rewrite the process (X_t) as

$$X_t = X_0 + \sigma W_t^*, \tag{2.14}$$

where $W_t^* = W_t + \frac{m}{\sigma} t$. The process W is a Brownian motion with respect to $\mathbb{P}$ and the filtration $(\mathcal{F}_t)$. We are now going to show that W^* is a Brownian motion with respect to the same filtration $(\mathcal{F}_t)$ but a different (equivalent) probability measure $\mathbb{P}^* \sim \mathbb{P}$. For this, we recall the following lemma.

Lemma 2.1.30 (Bayes' Formula) *Let $\tilde{\mathbb{P}} \sim \mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with density $Z_t := \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\Big| > 0$. Then,*

$$H_t := \tilde{\mathbb{E}}_t[H] = \frac{\mathbb{E}_t[HZ]}{\mathbb{E}_t[Z]} \qquad \text{for} \qquad H \geq 0 \text{ or } HZ \in L^1(\mathbb{P}), \tag{2.15}$$

where $\mathbb{E}_t[.] = \mathbb{E}[.|\mathcal{F}_t]$ denotes the conditional expectation given $\mathcal{F}_t$.

We are now ready to state and prove a first version of the change of measure formula on the Wiener space, known as the Cameron-Martin-Girsanov formula.

Theorem 2.1.31 (Cameron-Martin-Girsanov) *The following statements hold:*

i) There exists an equivalent probability measure $\mathbb{P}^* \sim \mathbb{P}$ such that $W^* = W + \frac{m}{\sigma}t$ is a $\mathbb{P}^*$-Brownian motion.

ii) The measure $\mathbb{P}^*$ has the density $Z := \frac{dP^*}{dP}\Big|_{\mathcal{F}_T} = \exp\left(\alpha W_T - \frac{1}{2}\alpha^2 T\right)$ with respect to $\mathbb{P}$ where $\alpha = -\frac{m}{\sigma}$.

In view of the preceding theorem we can write the NA-price in (2.13) in the Bachelier model as

$$F(t,x) = \mathbb{E}_t^*\left[f\left(x + \sigma(W_T^* - W_t^*)\right)\right].$$

That is

$$\begin{aligned} F(t, X_t(\omega)) &= \mathbb{E}^*\left[f\left(x + \sigma(W_T^* - W_t^*)\right)\right]\big|_{x=X_t(\omega)} \\ &= \mathbb{E}_t^*\left[f\left(X_t + \sigma(W_T^* - W_t^*)\right)\right] \\ &= \mathbb{E}_t^*\left[f\left(X_t + (X_T - X_t)\right)\right] \\ &= \mathbb{E}_t^*\left[f\left(X_T\right)\right], \end{aligned} \tag{2.16}$$

just as in the classical actuarial approach, but under the "risk neutral" probability measure $\mathbb{P}^*$; in general $\mathbb{P}^* \neq \mathbb{P}$, unless $m = 0$. In view (2.14) and (2.16) the price process (X_t) of the underlying and the derivative price process $(F(t, X_t))$ are martingales under $\mathbb{P}^*$ with respect to the original filtration.

2.2 Stochastic Calculus for Brownian Motion

The Itô formula allows us to calculate integrals of smooth functions of Brownian motion W with respect to Brownian motion in terms of ordinary Riemann integrals. For more general integrands, i.e., for integrants which are measurable with respect to the filtration generated by W one needs a more advanced machinery, known as Itô calculus.

Suggested Reading: Oksendal (2003), Chapter provides a good introduction into stochastic integration with respect to BM. For a more general theory of stochastic integration see also the books of Karatzas & Shreve (1998), Chapter 3, and Protter (2005), Chapter 2.

2.2.1 The Itô Integral

For our financial market models we will need to approximate integrals of the form

$$\sum_{t_i} \phi_{t_{i-1}}(\omega)\left(W_{t_i}(\omega) - W_{t_{i-1}}(\omega)\right) \tag{2.17}$$

so the question is for what functions (trading strategies) ϕ a limit in (2.17) can be taken and if so in what sense, i.e., in the almost sure sense, in probability, or in L^2. To this end, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ carrying a standard Brownian motion W and denote by $\mathcal{B}$ the Borel σ-field on $\mathbb{R}_+$.

Definition 2.2.1 *Let $\mathcal{V}$ be the class of all functions $f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$ such that*

(i) The map $(t,\omega) \mapsto f(t,\omega)$ is $\mathcal{B} \otimes \mathcal{F}$measurable.

(ii) The random variable $f(t,\cdot)$ is $\mathcal{F}_t$-adapted.

(iii) The map f is square integrable, that is

$$\mathbb{E}\left[\int_0^T f^2(t,\omega)dt\right] < \infty.$$

Notice that the functions ϕ_1 of Example 2.1.6 belongs to $\mathcal{V}$ while ϕ_2 does not. For a function $f \in \mathcal{V}$ we shall define the *Itô-integral* (up to some time T)

$$I[f](\omega) := \int_0^T f(t,\omega)dW_t(\omega).$$

The definition is obvious for *elementary functions*, i.e., functions of the form

$$\phi(t,\omega) = \sum_j e_j(\omega)\mathbf{1}_{[t_j,t_{j+1})}(t)$$

where $0 \le t_0 < t_1 < \cdots < t_n \le T$ and e_j is $\mathcal{F}_{t_j}$-measurable. For any such function we put

$$\int_0^T \phi(t,\omega)dW_t := \sum_j e_j(\omega)\left[W_{t_{j+1}} - W_{t_j}\right](\omega).$$

It was Itô's fundamental insight that one should *not* proceed in a pathwise way to define stochastic integrals. Instead, one should take a functional-analytic point of view, applying Hilbert spaces. The key insight is that the norm of an elementary function equals the norm of the stochastic integral. More precisely, the extension of the class of integrands from the set of elementary functions to $\mathcal{V}$ is based on the so-called *Itô isometry*.

Lemma 2.2.2 *(Itô isometry) Let ϕ be a bounded elementary function. Then*

$$\mathbb{E}\left[\left(\int_0^T \phi(t,\omega)dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T \phi^2(t,\omega)dt\right].$$

An extension of the stochastic integral to function belonging to $\mathcal{V}$ proceeds in three steps; for the details we refer to Chapter 3.1 of Oksendal (2003).

(i) For any bounded $g \in \mathcal{V}$ such that $g(\cdot,\omega)$ is continuous there exists elementary functions ϕ_n such that the following approximation in L^2 holds true:

$$\mathbb{E}\left[\int_0^T (g-\phi_n)^2 dt\right] \to 0.$$

The approximating functions are defined as follows:

$$\phi_n(t,\omega) = \sum_j g(t_j,\omega)\mathbf{1}_{[t_j,t_{j+1})}(t).$$

It follows from the Itô isometry that that the sequence $\{\int_0^T \phi_n dW\}_{n\in\mathbb{N}}$ forms a Cauchy sequence in L^2:

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^T \phi_m(t,\cdot)dW_t - \int_0^T \phi_n(t,\cdot)dW_t\right)^2\right] = \lim_{n\to\infty} \mathbb{E}\left[\int_0^T (\phi_m(t,\cdot)-\phi_n(t,\cdot))^2\, dt\right] = 0$$

because g is bounded and continuous in the time variable. Since the L^2 is complete the sequence $\{\int_0^T \phi_n dW\}_{n\in\mathbb{N}}$ converges. We define the integral of g with respect to BM as the unique limit $\int_0^T g dW$.

(ii) We prove in Lemma 2.2.3 below that for any bounded $h \in \mathcal{V}$, there exists functions $g_n \in \mathcal{V}$ such that $g_n(\cdot,\omega)$ is continuous and such that

$$\mathbb{E}\left[\int_0^T (h-g_n)^2 dt\right] \to 0.$$

(iii) For any $f \in \mathcal{V}$ there exists bounded functions $h_n \in \mathcal{V}$ such that

$$\mathbb{E}\left[\int_0^T (f-h_n)^2 dt\right] \to 0.$$

Here, the approximating functions are defined as follows:

$$h_n(t,\omega) = \begin{cases} -n & \text{if } f(t,\omega) < -n \\ f(t,\omega) & \text{if } -n \le f(t,\omega) \le n \\ n & \text{if } f(t,\omega) > n \end{cases}.$$

Lemma 2.2.3 *For any bounded $h \in \mathcal{V}$, there exists functions $g_n \in \mathcal{V}$ such that $g_n(\cdot,\omega)$ is continuous and such that*

$$\mathbb{E}\left[\int_0^T (h-g_n)^2 dt\right] \to 0.$$

Hence for any $f \in \mathcal{V}$ the Itô integral

$$I[f](\omega) := \int_0^T f(t,\omega)dW_t$$

is defined as an L^2-limit of stochastic integrals of elementary functions with respect to Brownian motion.

Remark 2.2.4 *Let $\{\phi_n\}$ and $\{\psi_n\}$ be two sequences approximating f in the L^2-sense. Then the isometry for elementary functions yields*

$$\mathbb{E}\left[\left(\int_0^T \phi_n dW_t - \int_0^T \psi_n dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T (\phi_n - \psi_n)^2\, dt\right] \to 0$$

so the definition of $I[f]$ does not depend on the approximating sequence.

As an immediate corollary from the construction of the stochastic integral we obtain an isometry for functions belonging to the class $\mathcal{V}$.

Corollary 2.2.5 *(Itô isometry) Let $f \in \mathcal{V}$. Then*

$$\mathbb{E}\left[\left(\int_0^T f(t,\omega)dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T f^2(t,\omega)dt\right].$$

Example 2.2.6 *(i) If follows from the construction of the Itô integral that*

$$\int_0^t s dW_s = tW_t - \int_0^t W_s ds.$$

Indeed, let us define the simple functions

$$\phi_n(s) = \sum_j \frac{j}{n} \mathbf{1}_{[\frac{j}{n}, \frac{j+1}{n})}(s)$$

approximating the identity function and put $s_j^n = \frac{j}{n}$ on $[\frac{j}{n}, \frac{j+1}{n})$ and $s_j^n = 0$ elsewhere. We have that

$$\int_0^t s dW_s = \lim_{n\to\infty} \int_0^t \phi_n dW = \lim_{n\to\infty} \sum_j s_j^n \Delta W_j.$$

Furthermore

$$\sum_j s_j^n \Delta W_j = \sum_j \Delta(s_j^n W_j) - \sum_j W_{\frac{j+1}{n}} \Delta s_j^n$$

The first term on the right hand side of the preceding equation converges to tW_t while the second term is an ordinary Riemann sum that approximates the integral $\int_0^t W_s ds$.

(ii) The integral $\int W_s dW_s$ has finite moments because

$$\mathbb{E}\left[\int_0^t W_s dW_s\right] = 0 \quad \textit{and} \quad \mathbb{E}\left[\left(\int_0^t W_s dW_s\right)^2\right] = \int_0^t \mathbb{E}[W_s^2]ds = \int_0^t s ds = \frac{1}{2}t^2.$$

The following properties of the Itô integral are easily verified for elementary functions and hence for all functions $f \in \mathcal{V}$:

(i) For any $0 \le c \le T$ we have $\int_0^T f dW = \int_0^c f dW + \int_c^T f dW$.

(ii) Itô integrals are linear: For any constant c and all $f, g \in \mathcal{V}$ we have $\int (cf + g)dW = c\int f dW + \int g dW$.

(iii) Expected values of Itô integrals are zero: $\mathbb{E}[\int f dW] = 0$.

(iv) Itô integrals are adapted: The random variable $\int_0^t f dW$ is $\mathcal{F}_t$-measurable.

Another important property of Itô integrals is the fact that they form martingales.

Theorem 2.2.7 *Let $f \in \mathcal{V}$. Then the stochastic process $\left(\int_0^{\cdot} f dW\right)_{t \geq 0}$ is an $(\mathcal{F}_t)$-martingale.*

Example 2.2.8 *Recall that*

$$2 \int_0^t W_s dW_s = W_t^2 - t = W_t^2 - < W >_t .$$

By the preceding theorem

$$M_t := 2 \int_0^t W_s dW_s$$

is a martingale. Hence $W_t^2 - < W >_t$ is a martingale; (we already knew this from Proposition 2.1.4). This is a special case of a more general result that states that if N is a martingale there exists a unique process $< N >$ (the compensator) that makes $N^2 - < N >$ a martingale.

It turns out that Itô integrals can be chosen to depend continuously on the upper integral boundary. The proof uses Doob's martingale inequality which allows for an estimate of the running maximum of a continuous martingale in terms of its p-th moment at the terminal time.

Theorem 2.2.9 *(Doob's martingale inequality) Let (M_t) is a martingale defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with almost surely continuous sample paths. For all $p \geq 1$, $T \geq 0$ and $\lambda > 0$ we have*

$$\mathbb{P}\left[\sup_{0 \leq t \leq T} |M_t| > \lambda\right] \leq \frac{1}{\lambda^p} \mathbb{E}[|M_T|^p].$$

We are now ready to show that Itô integrals depend continuously on the upper integral bound.

Theorem 2.2.10 *Let $f \in \mathcal{V}$. Then there exists a t-continuous version of the process*

$$\left(\int_0^t f dW_s\right)_{0 \leq t \leq T} .$$

2.2.2 Itô integral for a larger class of integrands

It will sometimes be necessary to consider Itô integrals for a more general class of integrands. For instance, let W be a Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with canonical filtration $(\mathcal{F}_t)$. Below we shall see that under certain conditions there exists an equivalent probability measure $\mathbb{P}^* \approx \mathbb{P}$ and a process W^* that is a Brownian motion w.r.t. $\mathbb{P}^*$ and $(\mathcal{F}_t)$ but whose canonical filtration $(\mathcal{F}_t^*)$ may be strictly small than $(\mathcal{F}_t)$. For such a process we will need to define integrals $\int f dW^*$ for $(\mathcal{F}_t)$-adapted integrands (i.e. for trading strategies that are measurable w.r.t. the original filtration, but not necessarily measurable w.r.t the filtration $(\mathcal{F}_t^*)$). In order to define such integrals we introduce the following class of functions.

Definition 2.2.11 *Let W be a Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\mathcal{G}_t)$ be an increasing family of σ-fields such that W_t is $\mathcal{G}_t$-adapted. We denote by $\mathcal{W}$ the class of all functions $f : [0, T] \times \Omega \to \mathbb{R}$ that satisfy the following conditions:*

(i) f *is* $\mathcal{B} \otimes \mathcal{F}$*-measurable*

(ii) $f(t, \cdot)$ *is* $\mathcal{G}_t$*-adapted*

(iii) $\mathbb{P}\left[\int_0^T f^2(s,\omega) < \infty\right] = 1$

If $f \in \mathcal{W}$, then one can show that there exist step functions f_n such that

$$\int_0^t |f_n - f|^2 ds \to 0$$

in probability. For such a sequence one has that $\int_0^t f_n dW$ converges in probability to a limit, denoted $\int f dW$. As before, there exists a t-continuous version. However, $(\int_0^\cdot f dW)$ will not be a martingale any more but only a local martingale in the sense of the following definition.

Definition 2.2.12 *An* $\mathcal{F}_t$*-adapted process* (Z_t) *is called a local martingale with respect to* $(\mathcal{F}_t)$ *if there exists a sequence of* $(\mathcal{F}_t)$*-stopping times* τ_n *such that*

$$\tau_n \to \infty \quad a.s. \text{ as } n \to \infty$$

and $(Z_{t \wedge \tau_n}) - Z_0$ *is an* $\mathcal{F}_t$*-martingale for every* n.

Note that no integrability assumption is imposed on Z_0. Because of this, it need not be possible to 'reduce' a local martingale to be a martingale.

Lemma 2.2.13 *A local martingale* (Z_t) *that is bounded from below and satisfies* $\mathbb{E}[Z_0] < \infty$ *is a super-martingale.*

The following lemma states conditions under which a local martingale is a (true) martingale.

Lemma 2.2.14 *Let* (Z_t) *be a local martingale with respect to* $(\mathcal{F}_t)$*. Then* (Z_t) *is a martingale if one of the following conditions holds:*

(i) The process is L^1 *bounded in the sense that*

$$\sup_{s \le t} |Z_s| \in L^1 \quad \text{for all } t > 0$$

(ii) The sequence (Z_t) *is uniformly integrable (e.g.* L^p *bounded for* $p > 1$*).*

2.2.3 The Generalized Itô Formula

In its simplest form the Itô formula allows us to evaluate integrals of the form

$$\int f(W_s) dW_s$$

when $f : \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth function. The preceding section introduced stochastic integrals

$$\int f(s,\omega) dW_s$$

for more general integrants. The definition of this integral can easily be extended to higher dimensions. To this end, let

$$W = (W^1, \ldots, W^n)$$

be a vector of n independent standard Brownian motions (the assumption of independence can be dropped). For a function $v : \mathbb{R} \times \Omega \to \mathbb{R}^{n \times n}$ we put

$$\int_0^t v dW = \int_0^t \begin{pmatrix} v_{1,1} & \cdots & v_{1,n} \\ \vdots & \vdots & \vdots \\ v_{n,1} & \cdots & v_{n,n} \end{pmatrix} \begin{pmatrix} dW_s^1 \\ \vdots \\ dW_s^n \end{pmatrix}$$

and denote again by $\mathcal{V}$ the class of all functions for which this definition makes sense. An *Itô process* is now defined as a stochastic process $X = (X_t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$X_t = X_0 + \int_0^t u(s,\omega) ds + \int_0^t v(s,\omega) dW_s(\omega) \tag{2.18}$$

where u is adapted, $v \in \mathcal{V}$ and the following integrability condition holds almost surely:

$$\int_0^t |u(s,\omega)| ds < \infty \quad \text{for all t.}$$

Example 2.2.15 *The simplest possible Itô process is a Brownian motion with drift and volatility:*

$$dX_t = \mu dt + \sigma dW_t$$

for some constants $\mu \in \mathbb{R}$ and $\sigma > 0$. Clearly,

$$X_t = \mu t + \sigma W_t.$$

The generalized Itô formula allows for an evaluation of smooth functions of Itô processes. Its proof follows from minor modifications of the arguments given in the proof of Theorem 2.1.18.

Theorem 2.2.16 *Let X be an Itô process as defined by (2.18) and let*

$$g(t,x) = (g_1(t,x), \ldots, g_p(t,x))$$

be continuously differentiable with respect to the time variable and twice continuously differentiable with respect to the space variable. Then

$$Y(t,\omega) := g(t, X_t(\omega))$$

defines an Itô process with

$$dY_k(t,\omega) = \frac{\partial}{\partial t} g(t, X_t) dt + \sum_i \frac{\partial}{\partial x_i} g_k(t, X_t) dX_t^i + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} g_k(t, X_t) dX_t^i dX_t^j$$

where $dW^i dW^j = \delta_{i,j} dt$ and $dW^i dt = dt dW_t^i = dt dt = 0$.

The following introduces a stochastic process that is the building block of the celebrated Black-Scholes option pricing model.

Example 2.2.17 *(Geometric Brownian motion) Given a BM W, an initial value S_0 and constants $\mu \in \mathbb{R}$ and $\sigma > 0$ the geometric Brownian motion process is defined by*

$$S_t = S_0 \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right).$$

In terms of the Itô processes $X_t = \sigma W_t$ and $Y_t = \left(\mu - \frac{1}{2}\sigma^2\right) t$ and the twice continuously differentiable function $G(x,y) = S_0 \exp(x+y)$ we have that

$$S_t = G(X_t, Y_t).$$

An application of the generalized Itô formula yields

$$\begin{aligned} S_t &= S_0 + \int_0^t G(X_s,Y_s)dX_s + \int_0^t G(X_s,Y_s)dY_s + \frac{1}{2}\int_0^t G(X_s,Y_s)\sigma^2 ds \\ &= S_0 + \int_0^t G(X_s,Y_s)\sigma dW_s + \int_0^t G(X_s,Y_s)(\mu - 1/2\sigma^2)ds + \frac{1}{2}\int_0^t G(X_s,Y_s)\sigma^2 ds \\ &= S_0 + \int_0^t \sigma S_s dW_s + \int_0^t \mu S_s ds. \end{aligned}$$

In other words, the geometric Brownian motion process solves the stochastic differential equation

$$dS_t = S_t\left(\sigma dW_t + \mu dt\right).$$

The next example illustrates the link between Itô processes and partial differential equations.

Example 2.2.18 *(Brownian motion and the reverse heat equation) Suppose that the functions F solves the heat equation*

$$F_t(t,x) + \frac{1}{2}F_{xx}(t,x) = 0 \tag{2.19}$$

on some (bounded or unbounded) domain $\mathcal{D} \subset \mathbb{R}$ with smooth boundary condition h. Let W be a one-dimensional standard Brownian motion, denote by

$$\tau := \inf\{t : W_t \notin \mathcal{D}\} \wedge T$$

the minimum of the first exit time of W from $\mathcal{D}$ and some terminal time T. The function F satisfies the boundary condition

$$F(\tau, x) = h(x) \quad on \quad \partial\mathcal{D}.$$

By Itô's formula we have

$$F(t,W_t) = F(0,0) + \int_0^t F_t(s,W_s)ds + \int_0^t F_x(s,W_s)dW_s + \frac{1}{2}\int_0^t F_{xx}(s,W_s)ds$$

up to the random time τ. Since F satisfies (2.19) we obtain

$$F(\tau, W_\tau) = F(0,0) + \int_0^\tau F_x(s,W_s)dW_s.$$

If F_x is sufficiently smooth, then the process $(\int_0^\cdot F_x(s,W_s)dW_s)$ is a martingale and

$$\mathbb{E}\int_0^\tau F_x(s,W_s)dW_s = 0.$$

Similarly, we can start the Brownian motion at time t in x. If we denote by $\mathbb{P}^{t,x}$ the resulting distribution of the Wiener process and by $\mathbb{E}^{t,x}$ the expected value with respect to $\mathbb{P}^{t,x}$ the boundary condition this yields

$$F(t,x) = \mathbb{E}^{t,x} h(\tau, W_\tau).$$

Hence we obtain a probabilistic representation of the solution to the heat equation with boundary condition h.

Example 2.2.19 *Prove that the process*

$$X_t = e^{\frac{1}{2}t} \sin(W_t)$$

is a martingale. Indeed, by Itô's formula we have

$$\begin{aligned} d(e^{\frac{1}{2}t}\sin(W_t)) &= \frac{1}{2}e^{\frac{1}{2}t}\sin(W_t) + e^{\frac{1}{2}t}\cos(W_t)dW_t + \frac{1}{2}e^{\frac{1}{2}t}(-\sin(W_t))dt \\ &= e^{\frac{1}{2}t}\cos(W_t)dW_t. \end{aligned}$$

Hence (X_t) can be represented as a stochastic integral with respect to Brownian motion. As such it is a martingale.

The generalized Itô formula also yields the following integration by parts formula; the proof is straightforward and left as an exercise.

Proposition 2.2.20 *(Integration-by-parts formula) Let X and Y be Itô processes on $\mathbb{R}$. Then the following integration-by-parts formula holds:*

$$\int X_s dY_s = X_t Y_t - X_0 Y_0 - \int Y_s dX_s + \int dX_s dY_s.$$

We close this section with a brief discussion of the quadratic variation of an Itô process

$$dX_t = v(t,\omega)dW_t$$

on the time interval $[0,T]$. By Theorem 2.2.10 the process X can be chosen to be continuous in t. The quadratic variation of X is defined as

$$< X(\omega) >_t \lim \sum_{i=1}^{n} |X_{t_i^n}(\omega) - X_{t_{i-1}^n}(\omega)|^2$$

where for each n the sequence $\{t_i^n\}_{i=1}^n$ is a partition of $[0,T]$. The limit is in probability, taken over all partitions with mesh sizes tending to zero as $n \to \infty$. For a simple integrant v it is easily verified that

$$< X(\omega) >_t = \int_0^t v^2(s,\omega)ds.$$

The standard approximation of general integrants by simple processes shows that this identity carries over to $v \in \mathcal{V}$. From this we immediately obtain the following result.

Theorem 2.2.21 *(Quadratic variation of Itô processes) Let X be an Itô process. The quadratic variation of X is given by*

$$< X(\omega) >_t = \int_0^t |v(s,\omega)|^2 ds.$$

In particular, the quadratic variation comes solely from the Itô integral $\int vdW$ and not from the ordinary integral $\int uds$.

2.2.4 The martingale representation theorem

By Theorem 2.2.7 Itô integrals are martingales with respect to the canonical filtration generated by the Wiener process W. In this section we prove a converse statement, namely that every square integrable martingale can be represented as a stochastic integral with respect to W. This martingale representation theorem turns out to be another pillar of the Black-Scholes option pricing model.
Throughout we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ that carries a standard Brownian motion W and assume that $(\mathcal{F}_t)$ is the Brownian filtration:

$$\mathcal{F}_t = \sigma(W_s : s \leq t).$$

With no loss of generality we also assume that $\mathcal{F} = \mathcal{F}_T$. In a first step we establish the *Itô representation*. It says that every square integrable $\mathcal{F}$-measurable random variable F can be written as a stochastic integral. For the proof we need the following lemma which we state without proof.

Lemma 2.2.22 *The linear span of random variables of the type*

$$\exp\left\{\int_0^T h dW - \frac{1}{2}\int_0^T h^2 dt\right\}; \quad h \in L^2([0,T]) \tag{2.20}$$

is dense in $L^2(\mathcal{F}_T, \mathbb{P})$.

We are now ready to state and prove a first version of Itô's representation theorem.

Theorem 2.2.23 *Let* $F \in L^2(\mathcal{F})$. *Then there exists a unique process* $f \in \mathcal{V}$ *such that*

$$F(\omega) = \mathbb{E}[F] + \int_0^T f(t,\omega) dW_t(\omega).$$

We are now ready to state and prove the martingale representation theorem.

Theorem 2.2.24 *Let* (M_t) *be an* $(\mathcal{F}_t)$*-martingale with* $M_t \in L^2$. *Then there exists a unique process* $g(t,\omega)$ *such that* $g \in \mathcal{V}$ *and*

$$M_t = \mathbb{E}[M_0] + \int_0^t g(s,\omega) dW_s \quad \mathbb{P}\text{-a.s. for all } t \geq 0.$$

The martingale representation theorem yields a fairly abstract result. A closed form expression for the integrant can only be obtained in special cases of which we discuss some below. We refer to Oksendal (2003), p.60 for more examples.

Example 2.2.25 *(i) Let* $F = W_1^2$ *and consider the martingale*

$$M_t := \mathbb{E}[F|\mathcal{F}_t] = W_t^2 + (1-t).$$

By Itô's formula

$$W_t^2 = t + 2\int_0^t W_s dW_s$$

so that we obtain the following representation of (M_t)*:*

$$M_t = 1 + 2\int_0^t W_s dW_s.$$

(ii) *Let* $F = \int_0^T W_s ds$. *The integration by parts formula yields*

$$\int_0^T W_s ds = TW_T - \int_0^T s dW_s - \int_0^T (T-t) dW_t.$$

(iii) *Let* $F = \exp(W_T)$. *Since*

$$d\left(\exp(W_T - \frac{1}{2}t)\right) = \exp(W_T - 1/2t) dW_t$$

we have

$$\exp(W_T - \frac{1}{2}T) = 1 + \int_0^T \exp(W_T - 1/2t) dW_t$$

and hence

$$F = \exp(\frac{1}{2}T) + \int_0^T \exp(W_T + \frac{1}{2}(T-t)) dW_t.$$

(iv) *Let* $F = \sin(W_T)$. *By Example 2.2.19 above*

$$d(e^{\frac{1}{2}t}\sin(W_t)) = e^{\frac{1}{2}t}\cos(W_t) dW_t.$$

or

$$e^{\frac{1}{2}t}\sin(W_T) = \int_0^T e^{\frac{1}{2}t}\cos(W_t) dW_t.$$

Hence

$$\sin(W_T) = \int_0^T e^{\frac{1}{2}(t-T)}\cos(W_t) dW_t.$$

2.2.5 Application: The Black Scholes Model

This section provides a first analysis of the Black-Scholes option pricing model. The assumption is that the dynamics of the stock price process (S_t) follows a geometric Brownian motion with drift μ and volatility σ. That is,

$$dS_t = S_t(\mu dt + \sigma dW_t).$$

We denote by S_t^0 the price of a riskless bond at time t that pays interest at a rate $r \geq 0$ so

$$dS_t^0 = S_t^0 r dt.$$

European Options

Our first goal is to price a European contingent claim

$$H = h(S_T)$$

on the risky asset with maturity T. To this end, we denote by

$$\phi(t, S_t) \quad \text{and} \quad \eta(t, S_t)$$

the number of stocks and bonds, respectively, an investor holds at time t. The value of her portfolio is hence given by

$$V(t, S_t) = S_t \cdot \phi(t, S_t) + S_t^0 \cdot \eta(t, S_t).$$

We assume that an agent can trade continuously so her gains from trading follow the stochastic process

$$G_t = \int_0^t \phi(t, S_t) dS_t + \int_0^t \eta(t, S_t) dS_t^0.$$

We also assume that the trading strategy (ϕ, η) is self-financing so that $V(t, S_t)$ equals the initial investment plus the gains from trading in the stock and bond market:

$$V(t, S_t) = V(0, S_0) + G_t.$$

The trading strategy (ϕ, η) replicates the claim H if by following (ϕ, η) the issuer can meet his payment obligations at maturity:

$$V(T, S_T) = H.$$

It turns out that the replicating strategy can be characterized as the solution to a linear partial differential equation.

Theorem 2.2.26 *(PDE for the replicating strategy) Let $V : [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ be a continuous solution to the terminal value problem*

$$V_t(t, S) + \frac{1}{2}\sigma^2 S^2 V_{SS}(t, S) + rSV_S(t, S) = rV(S, t) \quad \text{with} \quad V(T, \cdot) = h. \tag{2.21}$$

Then the trading strategy

$$\phi(t, S) = V_S(t, S) \quad \text{and} \quad \eta(t, S_t) = V(t, S_t) - \phi(t, S_t) S_t$$

along with the portfolio value $V(t, S)$ defines a self-financing trading strategy replicating H.

The preceding theorem states that by following the trading strategy (ϕ, η) the writer of an option can "eliminate" all the risk associated with issuing H. Implementing this strategy requires an initial investment $V(0, S_0)$, the claim's "fair" value. It turns out that for a European call option with strike K, i.e. for the claim

$$h(S_T) = (S_T - K)^+$$

this value can be given in closed form. The following lemma shows that pricing an option in the Black-Scholes model boils down to solving the (time-) reversed heat equation on $[0, T] \times \mathbb{R}$ with a terminal condition given by the payoff function at maturity.

Lemma 2.2.27 *Let $\tau(t) = \sigma^2(T - t)$. Define*

$$z(t, S) = \ln S - \left(r + \frac{1}{2}\sigma^2\right)(T - t)$$

and denote by u the solution to the heat equation

$$u_t = \frac{1}{2} u_{xx} \quad \text{with intial condition} \quad u(0, z) = (e^z - K)^+.$$

Then the function

$$C(t, S) = e^{-r(T-t)} u(\tau(t), z(t, S))$$

solves the terminal value problem (2.21) for a European call option.

From a previous example we know that the solution to the heat equation with initial condition $u(0,z) = u_0(z)$ equals

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(z + x\sqrt{t}) e^{-x^2/2} dx. \tag{2.22}$$

Hence a straightforward calculation yields the Black-Scholes price of a European call option:

$$C_{BS}(t,S,\sigma,r,K,T) = SN(d_1) - e^{-r(T-t)}KN(d_2) \tag{2.23}$$

where $N(\cdot)$ denotes the cumulative distribution function of the standard normal distribution and

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

The hedge portfolios in the stock and bond market are given in terms of the quantities $d_{1,2}$ as

$$\phi = \frac{\partial C_{BS}}{\partial S} = N(d_1) \quad \text{and} \quad \eta = e^{-r(T-t)}KN(d_2).$$

Remark 2.2.28 *One can show that u in (2.22) is continuously differentiable with respect to the time variable and twice continuously differentiable with respect to the space variable if u_0 is continuous. Differentiability conditions on u_0 are not needed (typically u_0 is not differentiable!). Continuity of the initial (terminal) condition is enough to guarantee that $u \in C^{1,2}$. This is due to the smoothing property of the normal density. This justifies - a posteriori - the application of Itô's formula when calculating the hedge portfolio.*

The derivative of the option value with respect to the current asset price is called the *option delta*. It specifies the number of shares the issuer needs to hold in her portfolio to eliminate the risk associated with her short position in the option. The second derivative with respect to S is called the *option gamma*. It yields a measure for the dependence of the hedge portfolio on asset prices. The derivatives with respect to interest rate, the volatility and the time to maturity are typically referred to the *vega*, *rho*, and *theta*, respectively.

Remark 2.2.29 *(i) Notice that the option value is independent of the drift μ. Notice furthermore that so far we did not consider any measure change. Our current approach is based solely on the idea of constructing a hedge portfolio that replicates the option's payoff at maturity. Below we consider a more advanced method to option pricing based on the concept of risk neutral valuation.*

(ii) Traders are concerned about the hedge portfolio - not the option value. As a result they are concerned about the option gamma. They typically try to hold a hedge portfolio with a small gamma as this limits the transactions (and hence the transaction costs) in the stock market necessary to hold a riskless (hedge) portfolio.

(iii) For a given market price of an option, the implied volatility *is defined as the volatility that makes the observed market price the Black-Scholes price. The Black-Scholes model assumes that the implied vola is independent of the strike and time to maturity. In reality, however, one typically observes a* volatility smile. *The implied vola is increasing in $|S-K|$, i.e., increasing in the option's "moneyness". It also depends on the strike generating what is known as an implied* volatility surface.

Barrier Options

So far we considered only European-style derivatives, i.e., derivatives whose payoff depends only on the price of the unerlying at maturity. By contrast, the payoff of a *barrier option* depends on whether the price of the underlying reaches a certain level before maturity. Most barrier options are either *knock-ins* or *knock-outs*. For instance, a *down-and-in put* with strike K and barrier B pays

$$C_{di}^{put} = \begin{cases} (K - S_T^i)^+ & \text{if } \min_{0\le t\le T} S_t^i \le B \\ 0 & \text{else} \end{cases}.$$

An *up-and-out call* with strike K and barrier B corresponds to

$$C_{uo}^{call} = \begin{cases} (S_T^i - K)^+ & \text{if } \max_{0\le t\le T} S_t^i < B \\ 0 & \text{else} \end{cases}.$$

Just like European options, within the Black-Scholes framework barrier options can be priced using PDE methods. The difference is that one obtains an additional boundary condition that captures the option's payoff when the asset price hits the threshold level B.

Proposition 2.2.30 *(PDE for up-and-out call options) Consider a up-and-out call option on S whose payoff at maturity is given by*

$$H = \begin{cases} (S_T - K)^+ & \textit{if } \max_{0\le t\le T} S_t < B \\ 0 & \textit{otherwise} \end{cases}.$$

The boundary value problem for option value reads

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S = rV \quad on \quad [0,T]\times(-\infty, B) \tag{2.24}$$

with terminal and boundary condition given by, respectively,

$$V(T,S) = (S-K)^+ \quad and \quad V(t,B) = 0. \tag{2.25}$$

This PDE is not as straightforward as the PDE for European options. The solution can nonetheless be obtained invoking probabilistic methods (the solution can be represented in terms of Brownian exit times). In order to illustrate the main idea, let us assume that $r = 0$, $\sigma = 1$, $S_0 = 1$ and the drift of the underlying is zero:

$$S_t = e^{W_t}.$$

The option payoff is

$$V_T = (e^{W_T} - K)\mathbf{1}_{\{W_T \ge k, M_T \le b\}}$$

where $M_T = \max_{0\le t\le T} W_t$ and

$$k = \log K, \quad b = \log B.$$

In the section on risk neutral pricing we show that within the current setting the fair value of the option is

$$V_t = \mathbb{E}[V_T \mid \mathcal{F}_t]. \tag{2.26}$$

This can be computed using the joint distribution of a Brownian motion and its running maximum derived above. In order to see the link to the above boundary value problem, let

$$\tau_B = \inf\{t \le T \ : \ S_t = B\} \quad (\tau_B := \infty \quad \text{if} \quad M_T < B)$$

On $\{0 \le t \le \tau_B\}$ the option has not been knocked out and $(V_{t\wedge\tau_B})_{0\le t\le T}$ is a martingale because τ_B is a stopping time. Moreover, from (2.26) we obtain the following result.

Lemma 2.2.31 *The process (V_t) is a Markov process up to τ_B, i.e.,*

$$V_t = v(t, S_t) \quad \textit{on} \quad 0 \le t \le \tau_B.$$

Due to the smoothing properties of the heat-kernel v satisfies the assumption of the generalized Itô formula ($v(t, S_t)$ is the option value under the assumption that the option has not been knocked out). We can therefore compute the differential and invoke the martingale property on $\{0 \le t \le \tau_B\}$ to deduce that

$$v_t(t,x) + \frac{1}{2}x^2 v_{x,x}(t,x) = 0 \quad \text{on} \quad [0,T]\times[0,B].$$

This holds because (t, S_t) can reach any point in $[0,T]\times[0,B]$ before the option knocks out. We refer to book of Björk, Chapter 18, for specific formulae for barrier options.

2.3 Topics in diffusion theory

So far, we used only PDE methods to analyze pricing and hedging problems. In order to study equivalent martingale measures and the risk-neutral approach to pricing and hedging in continuous time, we need to introduce the change of measure formula. This requires some knowledge of stochastic differential equations and stochastic exponentials.

2.3.1 Stochastic differential equations

Let W be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. An equation of the form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{2.27}$$

is called a *stochastic differential equation* (SDE) with coefficients $b(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$. We already solved the SDE

$$dX_t = X_t(\mu dt + \sigma dW_t)$$

using Itô's formula. X_t turned out to be a function of W_t and t, i.e., a strong solution in the sense of the following definition.

Definition 2.3.1 *The process X is called a strong solution to the SDE (2.27) if for all $t \ge 0$ the following holds:*

(i) X_t is a function of t and the Brownian path up to time t, i.e., X_t is of the form

$$X_t = F\left(t, (W_s)_{0\le s\le t}\right).$$

(ii) *The integrals* $\int_0^t b(s, X_s)ds$ *and* $\int_0^t \sigma(s, X_s)dW_s$ *exist.*

(iii) *The process satisfies the integral equation*

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s.$$

Example 2.3.2 *Consider the SDE* $dX_t = \alpha_t dW_t$ *for some deterministic function* $\alpha \in C^1$. *Then*

$$X_t = x + \int_0^t \alpha_s dW_s.$$

The integration by parts formula yields

$$X_t = x + \alpha_t W_t + \int_0^t W_s \alpha_s' ds$$

so X *is a strong solution.*

Example 2.3.3 *An Ornstein-Uhlenbeck process takes the form*

$$dX_t = -\alpha X_t dt + \sigma dW_t$$

for some positive constants α *and* σ. *Such a process fluctuates around zero (at least in the long run), due to the minus sign in front of the drift term. To solve this SDE we consider the process*

$$Y_t = X_t e^{\alpha t} \quad \text{so} \quad dY_t = e^{\alpha t} dX_t + \alpha e^{\alpha t} X_t dt.$$

Hence

$$dY_t = \sigma e^{\alpha t} dW_t.$$

In view of the previous example this SDE has a strong solution and

$$X_t = e^{-\alpha t}\left(x + \sigma \int_0^t e^{\alpha s} dW_s\right).$$

An Ornstein Uhlenbeck process is a Gauss process because X_t *follows a normal distribution. Specifically, if we define* $\gamma := \frac{\sigma}{\sqrt{2\alpha}}$ *then the martingale property of stochastic integrals implies*

$$X_t \sim N\left(x e^{-\alpha t}, \gamma^2 (1 - e^{-2\alpha t})\right).$$

Replacing the constant x *by a random variable* X_0 *that is independent of the Brownian motion and satisfies*

$$X_0 \sim N(0, \gamma^2) \quad \text{yields} \quad X_t \sim N(0, \gamma^2).$$

Thus, unlike Brownian motion that asymptotically fluctuates between infinity and negative infinity (remember the law of iterated logarithm), an Ornstein-Uhlenbeck process admits a stationary distribution. The covariance function of the stationary process is given by

$$\rho(s, t) = \gamma^2 e^{-|t-s|}.$$

A strong solution does not always exist. An example is the famous *Tanaka equation*

$$dX_t = \text{sign}(X_t)dW_t \quad \text{where} \quad \text{sign}(x) = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

To deal with such equation one works with a weaker solution concept, called *weak solutions* on which we shall not elaborate in this course. Instead we move on to the following existence and uniqueness results for SDEs. Its proof is similar to the standard Picard-Iteration in the theory of ordinary differential equation so we only give a give an outline.

Theorem 2.3.4 *Let $T > 0$ and let $b, \sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ be measurable functions that satisfy the linear growth condition*

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|) \quad (x \in \mathbb{R},\ t \in [0,T])$$

along with the Lipschitz condition

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x - y|.$$

Let Z be a square integrable random variable that is independent of W. Then the SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \quad \text{on} \quad [0,T]$$

with initial condition $X_0 = Z$ has a unique continuous strong solution.

It turns out that strong solutions to stochastic differential equations have the Markov property. We state this result without further proof.

Theorem 2.3.5 *Let the coefficients of the SDE*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$

satisfy the conditions of Theorem 2.3.4 and let $(\mathcal{F}_t)$ be the canonical filtration:

$$\mathcal{F}_t = \sigma(X_s : 0 \leq s \leq t).$$

It holds for all $t_1 \leq t_2 \leq T$ that

$$\mathbb{P}[X_{t_2} \leq y | \mathcal{F}_{t_1}] = \mathbb{P}[X_{t_2} \leq y | X_{t_1}] \quad \mathbb{P}\text{-}a.s.$$

2.3.2 Solutions to linear SDEs

As with ordinary differential equations, stochastic differential equations can rarely be solved in closed form. A closed form solution can be obtained for linear SDEs, i.e., for equations of the form

$$dX_t = (\alpha_t + \beta_t X_t)dt + (\gamma_t + \delta_t X_t)dW_t \tag{2.28}$$

where the coefficients are adapted stochastic processes and continuous in the time variable. The special case where $\alpha \equiv 0 \equiv \delta$ corresponds to a generalization of the Ornstein-Uhlenbeck process studied in Example 2.3.3.

Stochastic Exponential SDEs

When $\alpha_t = 0$ and $\gamma_t = 0$ the linear SDE (2.28) simplifies to

$$dU_t = \beta_t U_t dt + \delta_t U_t dW_t. \tag{2.29}$$

This SDE is of the form

$$dU_t = U_t dY_t$$

where the Itô process (Y_t) is defined by

$$dY_t = \beta_t dt + \delta_t dW_t \quad \text{with} \quad Y_0 = 1.$$

The SDE (2.29) is called the *stochastic exponential* of Y, denoted

$$U_t = \mathcal{E}(Y)_t.$$

It will key to the famous Girsanov theorem analyzed in the next section. By direct calculation we verify that

$$U_t = U_0 \exp\left(Y_t - Y_0 - \frac{1}{2} < Y >_t\right)$$

U_t is always positive if U_0 is. Since the bracket $< Y >$ of Y is given by $\frac{1}{2} \int \delta_s^2 ds$ we obtain that

$$U_t = U_0 \exp\left(\int_0^t (\beta_s - \frac{1}{2}\delta_s^2) ds + \int_0^t \delta_s dW_s\right). \tag{2.30}$$

It is checked directly by Itô's formula that this is indeed a solution to (2.29). Obviously, U is a strictly positive local martingale (as a stochastic integral w.r.t. Brownian motion). Conversely, every continuous strictly positive local martingale can be represented as a stochastic exponential.

Lemma 2.3.6 *Let U be a strictly positive local martingale. Then there exists a local martingale L such that $U = \mathcal{E}(L)$.*

General linear SDEs

The case of general linear SDEs can be reduced to stochastic exponentials by a variation of constants formula. More precisely, we look for solutions of the form

$$X_t = U_t V_t \quad \text{where} \quad dV_t = a_t dt + b_t dW_t$$

and (U_t) satisfies (2.29). If we put $U_0 = 1$ and $V_0 = X_0$ and take the differential of the product $U_t V_t$ we see that

$$b_t U_t = \gamma_t \quad \text{and} \quad a_t U_t = \alpha_t - \delta_t \gamma_t.$$

Using the representation (2.30) for U_t, this determines the functions a_t and b_t. Thus $V(t)$ is obtained and $X(t)$ is found to be

$$X_t = U_t \left(X_0 + \int_0^t \frac{\alpha_s - \delta_s \gamma_s}{U_s} ds + \int_0^t \frac{\gamma_s}{U_s} dW_s\right).$$

Example 2.3.7 *Let X satisfy the SDE*

$$dX_t = a_t X_t dt + dW_t.$$

To solve this equation using our general formula we first need to solve for the U process, i.e., solve

$$dU_t = \alpha_t U_t dt.$$

We obtain $U_t = \exp\left(\int_0^t a_s ds\right)$ so

$$X_t = e^{\int_0^t a_s ds}\left(X_0 + \int_0^t e^{-\int_0^s a_u du} dW_s\right).$$

Example 2.3.8 *(Brownian bridge) A linear SDE of the form*

$$dX_t = \frac{b - X_t}{T - t} dt + dW_t$$

starting in $X_0 = a$ is called a Brownian bridge and is of major importance in many financial market models of insider trading. With

$$\beta_t = -\frac{1}{T-t}, \quad \alpha_t = \frac{b}{T-t}, \quad \gamma_t = 1 \quad \text{and} \quad \delta_t = 0$$

the solution is given by

$$X_t = a\left(1 - \frac{t}{T}\right) + b\frac{t}{T} + (T-t)\int_0^t \frac{1}{T-s} dW_s.$$

We leave it as an exercise to prove that

$$\lim_{t\to T}(T-t)\int_0^t \frac{1}{T-s} dW_s = 0$$

almost surely so that $X_T = b$. Thus we do not only know the starting but also the end point of the process.

2.3.3 Change of variables and Girsanov's theorem

We have seen that changes of measures play a key role in pricing and hedging financial derivatives in discrete time models. In this section we discuss a change of measure formula (Girsanov's theorem) that is of fundamental importance in continuous time financial mathematics. Loosely speaking Girsanov's theorem states that if we change the drift coefficient of an Itô process the distribution of the process will not change dramatically. In fact, the law of the new process is absolutely continuous with respect to the law of the old process and we can explicitly calculate the density process. To make this more precise we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ carrying a standard Brownian motion W and recall that when the measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is absolutely continuous with respect to $\mathbb{P}$ on $\mathcal{F} = \mathcal{F}_T$, denoted $\mathbb{Q} \ll \mathbb{P}$, then there exists a random variable

$$Z := \frac{d\mathbb{Q}}{d\mathbb{P}}$$

called the *Radon-Nykodyn derivative* such that

$$\mathbb{Q}(A) = \int_A Z(\omega)\mathbb{P}(\omega) \qquad (A \in \mathcal{F}_T).$$

Furthermore, the restriction $\mathbb{Q}|_{\mathcal{F}_t}$ of $\mathbb{Q}$ to $\mathcal{F}_t$ is absolutely continuous with respect to the restriction $\mathbb{P}|_{\mathcal{F}_t}$ of $\mathbb{P}$ to $\mathcal{F}_t$ and the density process

$$Z_t := \frac{d(\mathbb{Q}|_{\mathcal{F}_t})}{d(\mathbb{P}|_{\mathcal{F}_t})}$$

is a uniformly integrable martingale. As always, we take càdlàg versions of Z. The first result shows how local martingales under $\mathbb{P}$ and $\mathbb{Q}$ are related.

Lemma 2.3.9 *Consider the stopping time* $R := \inf\{t \geq 0 \mid Z_t = 0\}$. *Then:*

1. $R = +\infty$ a.s. under $\mathbb{Q}$,
2. *For a non-negative adapted process* U *and* $s < t$ *we have*

$$\mathbb{E}_s^{\mathbb{Q}}[U_t] = 1_{\{Z_s \neq 0\}} \frac{1}{Z_s} \mathbb{E}_s^{\mathbb{P}}[U_t Z_t].$$

3. *3) For any* $\mathcal{F}_t$*-adapted process* Y*, if* YZ *is local martingale under* $\mathbb{P}$ *up to time* R*, then* Y *is a* $\mathbb{Q}$*-local martingale.*

Let us now state a first version of Girsanov's theorem. The proof uses Lévy's characterization of Brownian motion stated in Theorem 2.1.5.

Theorem 2.3.10 *(Girsanov Theorem I) Let* (α_t) *be an adapted process and* Y *be the solution to the SDE*

$$dY_t = \alpha_t dt + dW_t \qquad (t \in [0,T]).$$

Let the process M *be the stochastic exponential of* $-Y$*, i.e.,*

$$M_t = \exp\left(-\int_0^t \alpha_s dW_s - \frac{1}{2}\int_0^t \alpha_s^2 ds\right)$$

If M *is a martingale with respect to* $(\mathcal{F}_t)$ *and* $\mathbb{P}$*, then* Y *is a standard Brownian motion with respect to the measure* $\mathbb{Q}$ *defined by*

$$d\mathbb{Q} = M_T d\mathbb{P}.$$

We notice that stochastic exponentials are positive continuous local martingales and supermartingales but not martingales in general. A sufficient condition to guarantee that (M_t) is a martingale is the Novikov condition.

Theorem 2.3.11 (Novikov Condition) *The condition*

$$\mathbb{E}\left[\frac{1}{2}\int_0^T \alpha_s^2 ds\right] < \infty$$

implies that M *is a martingale. In particular,* M *is a martingale whenever the drift process of* Y *is bounded.*

The proof uses a clever application of the Hölder inequality. The following lemma is key.

Lemma 2.3.12 *Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. If*

$$\sup\left\{\mathbb{E}\left[\exp(\frac{\sqrt{p}}{2(\sqrt{p}-1)}M_\tau)\right] : \tau \text{ stopping time }, \tau \leq c\right\} < \infty \quad \text{for all} \quad c > 0,$$

then $\mathcal{E}(M)$ is an L^q-martingale.

We are now ready to prove the Novikov condition.

Example 2.3.13 *Suppose that (Y_t) is a Brownian motion on $[0, T]$ with drift $\mu \in \mathbb{R}$, i.e., that*

$$dY_t = \mu dt + dW_t \qquad (t \in [0, T]).$$

Then (Y_t) is a Browninan motion with respect to the measure $\mathbb{Q}$ where

$$d\mathbb{Q} = e^{-\mu W_T - \frac{1}{2}\mu^2 T} d\mathbb{P}.$$

Similar arguments as in the proof of Theorem 2.3.10 yield the following generalization.

Theorem 2.3.14 *(Girsanov Theorem II) Let $Y_t \in \mathbb{R}^n$ be an Itô process of the form*

$$dY_t = \beta_t dt + \theta_t dW_t$$

where (W_t) is an n-dimensional Brownian motion and $\beta_t \in \mathbb{R}^n$ and $\theta_t \in \mathbb{R}^{n,m}$ are adapted stochastic processes. Suppose there exist processes u_t and α_t such that

$$\theta_t(\omega)u_t(\omega) = \beta_t(\omega) - \alpha_t(\omega). \tag{2.31}$$

Put

$$M_t := \exp\left(-\int_0^t u_s dW_s - \frac{1}{2}\int_0^t u_s^2 ds\right) \qquad (0 \leq t \leq T)$$

and

$$dQ = M_T d\mathbb{P}.$$

Assume that M is a $\mathbb{P}$-martingale. Then $\mathbb{Q}$ is a probability measure and

$$\hat{W}_t = \int_0^t u_s ds + W_t \qquad (0 \leq t \leq T)$$

is a $\mathbb{Q}$-Brownian motion. In terms of $\hat{W}$ the process Y has the stochastic integral representation

$$dY_t = \alpha_t dt + \theta_t d\hat{W}_t.$$

We note gain that the Novikov condition

$$\mathbb{E}_\mathbb{P}\left[\frac{1}{2}\int_0^T u_s^2 ds\right] < \infty$$

is sufficient to guarantee that (M_t) is a martingale. We also point out that when $n = m$ and θ is invertible, then the process (u_t) satisfying (2.31) is uniquely given by

$$u_t(\omega) = \theta_t^{-1}(\omega)[\beta_t(\omega) - \alpha_t(\omega)].$$

In application in finance one usually chooses α to be equal to zero. In this case Y has the representation

$$dY_t = \theta_t d\hat{W}_t$$

under $\mathbb{Q}$. In particular, Y is a $\mathbb{Q}$-martingale and $\mathbb{Q}$ is called an equivalent martingale measure for Y.

Example 2.3.15 *Suppose that (Y_t) is a Brownian motion on $[0,T]$ with drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$, i.e., that*

$$dY_t = \mu dt + \sigma dW_t \qquad (t \in [0,T]).$$

Then (Y_t) is a Browninan motion with respect to the measure $\mathbb{Q}$ where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\frac{\mu}{\sigma}W_T - \frac{1}{2}(\frac{\mu}{\sigma})^2 T}.$$

Remark 2.3.16 *The assumption that θ is invertible has an economic interpretation. If Y is a vector of asset prices and θ is not invertible, then one of the assets can be represented as a combination of the others including the risk free bond. Hence that asset is redundant. As a result, we have more Brownian motions, i.e., sources of uncertainty than independent assets and the market is incomplete.*

Let is return to the Black-Scholes option pricing model where asset prices follow the geometric Brownian motion

$$dS_t = S_t(\mu dt + \sigma dW_t) \qquad (0 \le t \le T)$$

defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $r > 0$ be the risk-free interest rate, denote by $(\tilde{S}_t)$ the process of discounted stock prices,

$$\tilde{S}_t = e^{-rt} S_t,$$

and introduce a probability measure $\mathbb{P}^* \approx \mathbb{P}^*$ with density

$$M_T := \exp\left(- \int_0^T \frac{\mu - r}{\sigma} dW_s - \frac{1}{2}\int_0^T (\frac{\mu - r}{\sigma})^2 ds \right).$$

Under the equivalent martingale measure $\mathbb{P}^* \approx \mathbb{P}$ the process of discounted asset prices has the representation

$$d\tilde{S}_t = \sigma \tilde{S}_t d\hat{W}_t.$$

Thus, $(\tilde{S}_t)$ is a $\mathbb{Q}$-martingale. Hence the Itô representation stated in Theorem 2.2.23 asserts that any contingent claim $F \in L^2(\mathbb{Q}, \mathcal{F}_T)$ can be represented as a stochastic integral with respect to $(\tilde{S}_t)$:

$$F = \mathbb{E}_{\mathbb{Q}}[F] + \int_0^T \psi_u d\tilde{S}_u.$$

In the following section we will identify (ψ_s) as a trading strategy that replicates F and $\mathbb{E}_{\mathbb{Q}}[F]$ as the arbitrage-free price of F. We also generalize PDE characterization of option prices beyond the framework of the basic Black-Scholes model.

2.4 Risk neutral pricing

We are now ready to address to problem of pricing derivative securities in models with several assets whose price processes follow Itô processes. Throughout we work on stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ where $(\mathcal{F}_t)$ is the filtration generated by an n-dimensional Brownian motion W completed by the null-sets.

2.4.1 The market model

The financial market consists of $d \le n$ risky assets with price processes $S^1, ..., S^d$ and one risk-free with price process S^0. For simplicity we assume that the risk-free is $r = 0$ (this can be achieved by passing to discounted terms) so $S_t^0 \equiv 1$. The price processes of the risky assets follow the dynamics

$$dS_t^i = S_t^i dR_t^i \tag{2.32}$$

where the returns process $R = (R^i)_{i=1}^d$ is an Itô process given by

$$dR_t = \gamma_t dt + \sigma_t dW_t$$

with adapted processes γ and σ taking values in $\mathbb{R}^d$ and $\mathbb{R}^{d \times n}$, respectively. We assume that the model parameters satisfy the following condition.

Assumption 2.4.1 *There exists an adapted and (t, ω)-measurable process ξ that satisfies the Novikov condition such that*

$$\sigma_t(\omega)\xi_t(\omega) = \gamma_t(\omega) \quad \mathbb{P} \otimes \lambda\text{-}a.s. \tag{2.33}$$

Example 2.4.2 *Condition (2.33) is satisfied if the volatility matrix σ is of full rank, i.e.* $\det(\sigma_t \sigma_t^*) \neq 0$ *$dP \otimes dt$-a.s., where A^* denotes the transpose of the matrix A. In this case*

$$\xi := \sigma^* (\sigma\sigma^*)^{-1} \gamma. \tag{2.34}$$

We parameterize strategies by the amounts invested in each risky asset. Together with an initial capital this will define a value process by the self-financing requirement.

Definition 2.4.3 (Self-financing Trading Strategies) *A self-financing trading strategy φ is an adapted process $\varphi = (\varphi)_{i=1}^d$ on $[0, T]$ such that*

$$\int_0^T |\varphi_t^* \sigma_t|^2 dt < \infty \quad \text{a.s.}. \tag{2.35}$$

The wealth process $V = V^\varphi$ associated with a self-financing strategy is defined by

$$dV_t = \varphi_t^* dR_t$$

The model is said to allow arbitrage if there exists a a self-financing strategy φ such that the associated value process satisfies

$$V_0^\varphi = 0 \quad \text{and} \quad \mathbb{P}[V_T^\varphi \ge 0] = 1 \;\; \text{with} \;\; \mathbb{P}[V_T^\varphi > 0] > 0.$$

Without further assumptions on the trading strategies is will be impossible to exclude arbitrage. We therefore introduce the notion of admissible trading strategies.

Definition 2.4.4 *A self-financing trading strategy φ is called* admissible *if the associated value process V^φ satisfies at least one of the following two conditions:*

(i) The value process is bounded from below, that is $V^\varphi \ge -C$ for some $C \in \mathbb{R}$ ("bounded credit limit").

(ii) The process ξ is uniformly bounded (in t and ω) and φ satisfies the integrability condition

$$E\left[\int_0^T |\varphi_t^* \sigma_t|^2 dt\right] < \infty. \tag{2.36}$$

From now on, we consider only admissible strategies.

2.4.2 Equivalent martingale measures and absence of arbitrage

Just like in discrete time models there is a close link between the absence of arbitrage and the existence of (local) equivalent martingale measures.

Definition 2.4.5 *We say that $\mathbb{P}^*$ is an equivalent (local) martingale measure if $\mathbb{P}^*$ and $\mathbb{P}$ are equivalent and (discounted) asset prices are (local) martingales under $\mathbb{P}^*$.*

The link between the existence of an equivalent martingale measure and the absence of arbitrage (or, more generally, the condition of "no free lunch with vanishing risk") can be stated in great generality. We settle with the following result whose proof needs the Burkholder-Davis-Gundy inequality.

Lemma 2.4.6 (Burkholder-Davis-Gundy Inequality) *(See Revuz/Yor IV, Theorem 4.1) For a continuous local martingale M starting at 0, for any $p \in (0, \infty)$, there exist constants $c_p, C_p > 0$ such that*

$$c_p E\left[\langle M\rangle_\infty^{\frac{p}{2}}\right] \le E\left[\sup_{t\ge 0}|M_t|^p\right] \le C_p E\left[\langle M\rangle_\infty^{\frac{p}{2}}\right].$$

Theorem 2.4.7 *Under Assumption 2.4.1 the following holds:*

(i) There exists a local equivalent martingale measure.

(ii) The model is free of arbitrage

One can also prove a partial converse of the above result which we state without proof for completeness.

Proposition 2.4.8 *Suppose that the model is free of arbitrage. Then there exists an $\mathcal{F}_t$-adapted (t, ω)-measurable process ξ such that*

$$\sigma_t(\omega)\xi_t(\omega) = \gamma_t(\omega) \quad \mathbb{P} \otimes \lambda\text{-a.s.}$$

We say that the market model is complete if any contingent claim $H \in L^2(\mathbb{P})$ is attainable, i.e., if there exists an admissible trading strategy φ and a real number z such that

$$H = z + \int_0^T \varphi_u dS_u.$$

Proposition 2.4.9 *The market is complete if and only if the volatility matrix σ has a left-inverse σ^*, i.e.,*

$$\sigma_t^* \sigma^t = I \quad a.s.$$

In this case, there exists a unique replicating strategy for any claim $H \in L^2$.

Characterization of equivalent martingale measures

In general, the above market model is incomplete and so there exist may EMMs. It turns out, though, that the EMMs can be conveniently characterized. To this end, it will be useful to re-parameterize strategies. The value process satisfies

$$dV_t = \phi_t^* \sigma_t d\hat{W}_t = (\sigma_t^* \phi_t)^* d\hat{W}_t$$

so the range

$$C_t = \mathrm{Im}\ \sigma_t^*$$

of the matrix σ_t^* is important for the set of attainable payoffs. We have that

$$\mathbb{R}^n = \mathrm{Im}\ \sigma_t^* \oplus \mathrm{Ker}\ \sigma_t \equiv C_t \oplus C_t^{\perp},$$

since $(\mathrm{Im}\ \sigma_t^*)^{\perp} = \mathrm{Ker}\ \sigma_t$. Hence, any $z \in \mathbb{R}^n$ can be decomposed accordingly into its orthogonal projections as

$$z = \Pi_{C_t}(z) \oplus \Pi_{C_t^{\perp}}(z) = \sigma_t^*(\sigma_t \sigma_t^*)^{-1} \sigma_t z \oplus (Id - \sigma_t^*(\sigma_t \sigma_t^*)^{-1} \sigma_t) z.$$

Letting $\phi_t := \sigma_t^* \varphi_t$ ($\phi := \sigma^* \varphi$) be the "image strategy", we can rewrite the value process associated to φ via

$$dV_t = \phi_t^*(\xi_t dt + dW_t) = \phi_t^* d\hat{W}_t.$$

We have a bijection between $\{\varphi\}$ and $\{\phi\}$ by $\phi = \sigma^* \varphi$ and $\varphi = (\sigma^*)^+ \phi$ for the "pseudo-inverse"

$$(\sigma^*)^+ := (\sigma \sigma^*)^{-1} \sigma.$$

Remark 2.4.10 *Another way of phrasing the preceding result is to say that there exist n correlated Brownian motions $\bar{W}^1, ...\bar{W}^n$ such that the investor can fully hedge the risk coming from the first n Browninan motions, but not the risk coming the remaining $d - n$ Brownian motions.*

Remark 2.4.11 *To construct an equivalent (local) martingale measure ("EMM") one can use the Girsanov transformation. This yields that for the probability measure $\hat{Q}$ defined via*

$$\frac{d\hat{Q}}{dP} := \mathcal{E}\Big(-\int_0^{\cdot} \xi_t dW_t\Big)_T \quad (\xi\, bounded)$$

$\hat{W}$ is a $\hat{Q}$-Brownian motion. Hence $dR = \sigma d\hat{W}$ is a $\hat{Q}$-martingale if and only if S is a $\hat{Q}$)-martingale since S^i, $\frac{1}{S^i}$ are locally bounded and $dS^i = S^i dR^i$ and $dR^i = \frac{1}{S^i} dS^i$. The measure $\hat{Q}$ is called the "minimal martingale measure".

To construct other martingale measures, let $Q \sim P$, with density process $Z = \mathcal{E}(\int_0^{\cdot} \lambda_t dW_t)$ using Itô representation theorem (Theorem 2.2.24) for $\int_0^{\cdot} \frac{1}{Z_t} dZ_t = \int_0^{\cdot} \lambda_t dW_t$ with λ predictable, $\int_0^T |\lambda_t|^2 dt < \infty$, and $W^Q = W - \int_0^{\cdot} \lambda_t dt$ a Q-Brownian motion. Now, since

$$\sigma_t d\hat{W}_t = \sigma_t(\xi_t dt + dW_t) = \sigma_t\left((\xi_t + \lambda_t)dt + dW_t^Q\right)$$

holds, then Q is an EMM if and only if $\sigma(\xi + \lambda) = 0$ ($dP \otimes dt$ a.e.); that is

$$\lambda_t^Q = \lambda_t = -\xi_t + \eta_t,$$

with

$$\eta_t = \eta_t^Q \in \text{Ker } \sigma_t \equiv C_t^{\perp}.$$

Furthermore, since $\xi_t \in C_t$, then $\eta \cdot \xi = \eta^* \xi = 0$ ($dP \otimes dt$ a.e.). This implies that any EMM Q must have density process of the form

$$Z_t = \left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \mathcal{E}\left(\int_0^{\cdot} \lambda_s^Q dW_s\right)_t = \mathcal{E}\left(-\int_0^{\cdot} \xi_s dW_s\right)_t \mathcal{E}\left(\int_0^{\cdot} \eta_s^Q dW_s\right)_t, \tag{2.37}$$

since $\mathcal{E}\left(\langle -\int_0^{\cdot} \xi_s dW_s, \int_0^{\cdot} \eta_s^Q dW_s \rangle\right) \equiv 1$. Thus, we have shown the following result.

Proposition 2.4.12 (Characterization of the EMM) *a) Any EMM Q has density process of the form (2.37) with predictable $\lambda = -\xi + \eta$, with $-\xi_t = \Pi_{C_t}(\lambda_t)$, $\eta_t = \Pi_{C_t^{\perp}}(\lambda_t)$ satisfying*

$$\int_0^T |\lambda_t|^2 dt = \int_0^T |\xi_t|^2 dt + \int_0^T |\eta_t|^2 dt < \infty.$$

In particular, $\eta \cdot \xi \equiv 0$, and λ, η and ξ are unique $dP \otimes dt$ a.e.

b) In turn, any predictable λ with $\Pi_{C_t}(\lambda_t) = -\xi_t$ ($dP \otimes dt$ a.e.) and $\int_0^T |\lambda_t|^2 dt < \infty$ and such that $Z := \mathcal{E}\left(\int_0^{\cdot} \lambda_t dW_t\right)$ is a martingale, defines an equivalent local martingale measure Q via (2.37).

The range of option prices

We know from discrete time finance that the set of arbitrage free option prices is essentially identical to the set of expected option payoffs under the equivalent martingale measures. A similar result holds within the Black-Scholes framework. In order to see this, let us consider a European option on a contingent claim H. The maximal price the buyer is willing to pay is

$$p(H) := \sup\left\{y : \text{ there exists an admissible } \varphi \text{ with } V_0^{\varphi} = 0 \text{ s.t. } -y + V_T^{\varphi} \geq -H \text{ a.s.}\right\}$$

On the other hand, the minimal price the seller is willing to accept is

$$q(H) := \inf\left\{z : \text{ there exists an admissible } \varphi \text{ with } V_0^{\varphi} = 0 \text{ s.t. } z + V_T^{\varphi} \geq H \text{ a.s.}\right\}$$

If $p(H) = q(H)$, then this common value is the option price. In general, we have the following result.

Proposition 2.4.13 *Let Q be an equivalent martingale measure such that*

$$\frac{dQ}{d\mathbb{P}} = \exp\left(-\int_0^T \xi_s dW_s - \frac{1}{2}\xi_s^2 ds\right)$$

and ξ satisfies the Novikov condition and $\sigma^* \xi = \gamma$. Then

$$\textit{ess inf} H \leq p(H) \leq \mathbb{E}_Q[H] \leq q(H) \leq \infty.$$

Pricing European optionas in the classical BS model

It turns out that in the Black-Scholes model every option $h \in L^2(\mathbb{P}^*)$ is replicable. This is essentially due to the fact that in this case, the equivalent martingale measure $\mathbb{P}^*$ is unique. The value of any replicating portfolio is given by its discounted expected payoff under $\mathbb{P}^*$.

Theorem 2.4.14 *In the Black-Scholes model very option $h \in L^2(\mathbb{P}^*)$ is replicable and the value of any replicating portfolio is given by*

$$V_t = \mathbb{E}^* \left[e^{-r(T-t)} h | \mathcal{F}_t \right]. \tag{2.38}$$

The explicit formula (2.38) along with some tedious but straightforward algebra yields the following formula for the for the price C_{BS} of a European call option with strike K and maturity K: From this we obtain the Black-Scholes price of a European call option:

$$C_{BS}(t, S, \sigma, r, K, T) = SN(d_1) - e^{-r(T-t)} KN(d_2) \tag{2.39}$$

where $N(\cdot)$ denotes the cumulative distribution function of the standard normal distribution and

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

We derived the same formula in a previous section using PDE methods. One of the features of the Black-Scholes model (and its success) is the fact that the pricing formula depends on only one non-observable parameter, the volatility $\sigma > 0$. In practice there two methods to evaluate σ. The first is the empirical variance of the logarithmic stock price. Since, in the framework of the Black-Scholes model, logarithmic asset prices have stationary and independent increments, σ can easily be estimated from historic data. The second technique is methods the implied volatility. From observed option prices one can deduce the volatility that makes the option prices the Black-Scholes-price.

Hedging European options in the classical BS model

The proof of Theorem 2.4.14 is based on the martingale representation theorem, a rather technical result that is not satisfactory in practice. However, it turns out that for a derivative of the form $h = f(S_T)$ the hedging strategy is given by the option delta, the derivative of its Black-Scholes price with respect to the current spot. To see this, notice first that the geometric Brownian motion process satisfies

$$S_t = S_0 \exp\left(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t\right) = S_u \exp\left(\sigma(W_t - W_u) + (\mu - \frac{1}{2}\sigma^2)(t-u)\right) \quad (u \le t)$$

and because the Brownian increment $W_t = W_u$ is independent of the stock price S_u. We deduce that the value $\tilde{V}_t$ must be of the form

$$\tilde{V}_t = e^{-rt} F(t, S_t)$$

for some function F. The function is given by

$$F(t,x) \quad = \quad \mathbb{E}^* \left[e^{-r(T-t)} f(xe^{r(T-t)} e^{\sigma(W_T - W_t) - \frac{\sigma^2}{2}(T-t)}) \right].$$

For a large class of functions f including calls and puts F is of class C^∞ on $[0,T[\times\mathbb{R}$. If we set $\tilde{F}(t,x) = e^{-rt} F(t, xe^{rt})$ we have $\tilde{V}_t = \tilde{F}(t, \tilde{S}_t)$ and for $t < T$ the Itô formula yields

$$\tilde{F}(t, \tilde{S}_t) = \tilde{F}(0, \tilde{S}_0) + \int_0^t \tilde{F}_x(u, \tilde{S}_u) d\tilde{S}_u + \int_0^t \tilde{F}_t(u, \tilde{S}_u) du + \frac{1}{2} \int_0^t \tilde{F}_{x,x}(u, \tilde{S}_u) d < \tilde{S} >_u .$$

From $d\tilde{S}_t = \sigma\tilde{S}_t dt$ we deduce $d < \tilde{S} >_u = \sigma^2 \tilde{S}_u^2 du$ so $\tilde{F}$ can be written

$$\tilde{F}(t, \tilde{S}_t) = \tilde{F}(0, \tilde{S}_0) + \int_0^t \tilde{F}_x(u, \tilde{S}_u) d\tilde{S}_u + \int_0^t K_u du.$$

Under the risk neutral measure the discounted values of a replicating portfolio form a martingale so $K_u = 0$ almost surely. From this we see that the replicating portfolio is given by:

$$\phi_t = \tilde{F}_x(t, \tilde{S}_t) \quad \text{and} \quad \eta_t = \tilde{F}(t, \tilde{S}_t) - \phi_t \tilde{S}_t.$$

2.4.3 Option Pricing and PDE

We have seen that the value of a European option in the geometric Brownian motion model can be calculated using partial differential equations. In this section we extend this approach to more general diffusion models. Specifically, we assume that asset prices follow the diffusion

$$dS_t^x = b(t, S_t^x)dt + \sigma(t, S_t^x)dW_t \quad \text{with} \quad S_0^x = x. \tag{2.40}$$

where W is a d-dimensional Brownian motion and the drift and diffusion coefficients satisfy the assumptions of Theorem 2.3.4. We assume that the model is complete and denote by $\mathbb{P}^*$ the unique equivalent martingale measure.

The *infinitesimal generator* at time t of (S_t) is the differential operator A_t that acts on all sufficiently function $u : \mathbb{R} \to \mathbb{R}$ according to

$$(A_t u)(x) = \frac{1}{2}\sigma^2(t, x)\frac{\partial^2 u}{\partial x^2} + b(t, x)\frac{\partial u}{\partial x}.$$

For the time-homogenous case $\sigma(t, x) \equiv \sigma(x)$ and $b(t, x) \equiv b(x)$ this reduces to

$$(Au)(x) = \frac{d}{dt}\mathbb{E}[u(S_t^x)]\,|_{t=0}\,.$$

The operator $\frac{\partial}{\partial t} + A_t$ is called the *Dynkin operator*. It arises naturally when solving stochastic differential equations. In fact, we have the following result whose proof is an immediate consequence of Itô's formula.

Theorem 2.4.15 *Let $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable with respect to the state variable and continuously differentiable with respect to the time variable. If u has bounded derivatives with respect to x, then the process (M_t) defined by*

$$M_t = u(t, S_t^x) - \int_0^t \left(\frac{\partial u}{\partial t} + A_t u\right)(v, S_v^x)dv$$

is a martingale.

In order to deal with discounted quantities we also state a slightly more general result in the following proposition.

Proposition 2.4.16 *Under the assumptions of Theorem 2.4.15 and if $r(t, x)$ is a bounded continuous function the following process is a martingale:*

$$M_t = e^{-\int_0^t r(u, S_u^x)du}u(t, S_t^x) - \int_0^t e^{-\int_0^v r(s, S_s^x)ds}\left(\frac{\partial u}{\partial t} + A_t u - ru\right)(v, S_v^x)dv.$$

As in the benchmark geometric Brownian motion model the option value is given by

$$V_t = \mathbb{E}^* \left(e^{-\int_0^t r(u,S_u^x)du} f(S_T^x) | \mathcal{F}_t \right)$$

and we can prove that

$$V_t = F(t, S_t^x)$$

where the function F is defined by

$$F(t,x) = \mathbb{E}^* \left[e^{-\int_0^T r(u,S_u^x)du} f(S_T^{t,x}) \right]$$

when we denote by $(S_u^{t,x})$ the unique strong solution of (2.40) that starts from x at time t. The following result characterizes the function F as a solution of a partial differential equation.

Theorem 2.4.17 *Let u be as in Theorem 2.4.15. If u satisfies the boundary value problem*

$$\frac{\partial u}{\partial t} + A_t u = 0 \quad \text{for all} \quad (t,x) \in [0,T] \times \mathbb{R} \tag{2.41}$$

and

$$u(T,x) = f(x) \quad \text{for all} \quad x \in \mathbb{R} \tag{2.42}$$

then

$$u(t,x) = F(t,x).$$

The previous result reduces the problem of pricing a European contingent claim to solving of terminal value problem. For this terminal value problem to have a unique classical solution one needs to impose some regularity assumptions on the volatility coefficients and the operator A_t often needs to be elliptic.

2.5 The fundamental theorem of asset pricing in continuous time

In section 1.2.2, the fundamental theorem of asset pricing in one periode established that the existence of an equivalent martingale measure coincides with the arbitrage free property of a financial market model. The objective of the following paragraphs is to sketch the proof of an analogous result in the setting of continuous time due to Delbaen and Schachermayer, which states that

Theorem 2.5.1 *A financial market model S given by a continuous locally bounded semi-martingale admits an equivalent local martingale measure if and only if S satisfies NFLVR*[1]*.*

The difficulty of the proof lies in demonstrating the inverse sense, that is that the condition NFLVR implies the existence of an equivalent local maringale measure. This part will exploit the Kreps-Yan theorem established in the second subsection to this chapter, which in turn crucially relies on a separation theorem due to the Hahn-Banach theorem. Yet, in order to be able to use this theorem, one must show in an additional step that a certain set $C \subset L^\infty$ introduced in the following subsection is weak-*-closed. This step will turn out to be the longest and was therefore moved into a fourth

[1] The new appearing notions of semi-martingales and NFLVR will be defined in the following subparagraph

subsection.

Throughout the chapter, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the usual assumptions. The material covered is based on Delbean & Schachermayer (2006).

2.5.1 Definitions

Definition 2.5.2 *Let $M = (M_t)_{t\in[0,T]}$ be a continuous local $(\mathcal{F}_t)_t$-martingale and $A = (A_t)_{t\in[0,T]}$ an adapted continuous process of finite variation starting in zero, i.e. $A_0 = 0$. Then the process $S = (S_t)_{t\in[0,T]}$ definied by $S_t = S_0 + M_t + A_t$ is called a continuous semi-martingale.*

Theorem 2.5.3 *The space of semi-martingales equipped with the translation invariant metric*

$$d(S,0) := \sup\left\{\mathbb{E}[\min\left\{|\int_0^T H_s dS_s|, 1\right\}] \mid H \text{ predictable}, \ |H| \le 1\right\}$$

is a complete metric space. The induced topology is called topology of semi-martingale convergence.

In the following, we assume that the continuous semi-martingale S is one dimensioal and models our financial market (or more percisly it's risky asset). For the following, one may assume S to be an It-process. The fundamental theorem of asset pricing is however true in much greater generality (actually for all locally bounded, not even necessarily continuous semimartingales).

Definition 2.5.4 *A stochastic process $H = (H_t)_{t\in[0,T]}$ is called an admissible strategy given the financial market S if H is predictable, H is S-integrable, that is the integral $\int_0^T H_s dS_s$ is well defined and there exists an $M > 0$ such that*

$$\int_0^T H_s dS_s \ge -M.$$

Intuitivly, when trading in continuous time, one can only use the information "just before time t" when investing in the market S. Our "infinitesimal profit dV_t" is given by $dV_t = H_t dS_t$, translating the discrete expression $V_{t+\Delta t} - V_t = H_t(S_{t+\Delta t} - S_t)$.

Definition 2.5.5 *A process H is called predictable if it is measureable with respect to the σ-algebra on $[0,T] \times \Omega$ generated by the adapted left-continuous processes.*

Remark 2.5.6 *Every predictable process is progressivly measurable and hence adapted. Sometimes H is simply assumed to be adapted. In this case there exists $\tilde{H}$ predictable such that $H_t(\omega) = \tilde{H}_t(\omega)$ $\mathbb{P} \otimes \lambda$-almost surely. Moreover if S is an It process, one has equality of the associated stochastic integrals, i.e.*

$$\int_0^t H_s dS_s = \int_0^t \tilde{H}_s dS_s.$$

Definition 2.5.7 *Given a financial market S a free lunch with vanishing risk (FLVR) is a sequence $(H^n)_{n\in\mathbb{N}}$ of admissible strategies such that for all $n \in \mathbb{N}$*

$$V_T^n := \int_0^T H_t^n dS_t \ge -\varepsilon_n$$

where $(\varepsilon_n)_n$ is a sequence of real numbers converging to zero and such that $V_T^n \to g$ almost surely with $\mathbb{P}(g > 0) > 0$ and $g \geq 0$ almost surely. We say a financial market S satisfies the no free lunch with vanishing risk (NFLVR) condition if there exists no free lunch with vanishing risk in S.

Remark 2.5.8 *Every arbitrage possibility is a FLVR, hence a market admitting no free lunch with vanishing risk (NFLVR) has also no arbitrage opportunity (NA).*

We introduce the following sets to give a characterization of NFLVR and NA, with which we will be working in later demonstrations.

- Set of final payoffs of admissible strategies: $K_0 := \{\int_0^T H_s dS_s;\ H \,\text{admissible}\}$
- Cone of dominated payoffs: $C_0 := K_0 - L_+^0$

We further restrict ourselves to bounded payoffs, i.e. take

$$K := K_0 \cap L^\infty; \qquad C := C_0 \cap L^\infty = \{f \in L^\infty |\ \exists g \in K:\ f \leq g\}$$

These sets now allow us to give the following characterization:

- There holds NFLVR if and only if $\overline{C} \cap L_+^\infty = \{0\}$
- There holds NA if and only if $C \cap L_+^\infty = \{0\}$

where the above closure is being taken with respect to the (strong) L^∞ norm.

In order to introduce the Kreps-Yan theorem, which will provide us with one of the crucial steps in the demonstration of the fundamental theorem of asset pricing, we also introduce the above notions in the setting of simple admissible strategies:

Definition 2.5.9 *Let $S = (S_t)_{t\in[0,T]}$ be a locally bounded real valued stochastic process, providing a financial market model. A real valued process $H = (H_t)_{t\in[0,T]}$ is called a simple trading strategy if H is of the form*

$$H = \sum_{i=1}^n h_i \chi_{(\tau_{i-1},\tau_i]}$$

where $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n < \infty$ are finite stopping times and h_i are $\mathcal{F}_{\tau_{i-1}}$ measurable real valued functions. A simple process H is called admissible if S^{τ_n} and the functions $h_1, \dots, h_n$ are uniformly bounded.

Starting from this notion, we again introduce the associated spaces:

- Set of final payoffs of simlpe admissible strategies:

$$K^{simple} := \left\{\int_0^T H_s dS_s;\ H \,\text{simple, admissible}\right\}$$

- Cone of dominated payoffs:

$$C^{simple} := K^{simple} - L_+^\infty = \{f - k \mid f \in K^{simple}, f \in L^\infty, k \geq 0\}$$

Note that in this setting, we do not need to restrict ourselves to bounded functions, as by definition of simple admissibility we already have this property.

We further define no arbitrage according to our previous characterization in the non-simple case:

Definition 2.5.10 *The process S satisfies the condition no arbitrage with respect to simple integrands (NA^{simple} if $K^{simple} \cap L^\infty_+ = \{0\}$ or equivalently $C^{simple} \cap L^\infty_+ = \{0\}$.*

We finally also introduce the new notion of no free lunch (NFL), of which the following Kreps-Yan theorem will make use.

Definition 2.5.11 (No free lunch) *A financial market model S satisfies the condition no free lunch (NFL) if one has*

$$\overline{C^{simple}}^* \cap L^\infty_+ = \{0\}$$

where $\overline{C^{simple}}^*$ denotes the closure of the set C^{simple} with respect to the weak--topology of L^∞.*

2.5.2 The Kreps-Yan theorem

The objective of this subsection is to prove the following theorem, which will be key to proving the fundamental theorem. It is based on the following separation theorem due to the Hahn-Banach theorem.

Theorem 2.5.12 *Let A, B be non-empty, disjoint convex subsets of a locally convex space such that A is closed and B is compact. There exists a closed real hyperplane in E strictly separating A and B.*

Theorem 2.5.13 *(Kreps-Yan) A bounded stochastic process S satisfies the condition NFL if and only if there exists an equivalent martingale measure for S.*

2.5.3 The proof of the fundamental theorem of asset pricing

The proof will make use of the following theorem, whose proof will be postponed to the following subsection:

Theorem 2.5.14 *Let S be a bounded semi-martingale satisfying NFLVR, then $C \subset L^\infty$ is weak-*-closed.*

Theorem 2.5.15 *(Fundamental theorem of asset pricing) Let S be a bounded semi-martingale. Then S admits an equivalent martingale measure if and only if S satisfies NFLVR.*

We are also able to formulate the fundamental theorem in stetting where our continuous semi-martingale is only locally bounded. In this case, we replace the equivalent martingale measure by an equivalent local martingale measure.

Corollary 2.5.16 *Let S be a locally bounded semi-martingale. Then S admits an equivalent local martingale measure if and only if S satisfies NFLVR.*

2.5.4 On the weak-*-closedness of C

The following subsection is entirly dedicated to proving the weak-*-closedness of C in L^∞. To this end, we first introduce the notion of Fatou-closedness.

Definition 2.5.17 *A subset $D \subset L^0$ is Fatou-closed if for every sequence $(f_n)_{n\geq 1}$ uniformly bounded from below with $f_n \to f$ almost surely, we have $f \in D$.*

Remark 2.5.18 *Suppose D is a cone. Then we can choose the lower uniform bound to be -1, i.e. D is Fatou-closed if for all $(f_n)_{n\geq 1}$ such that $f_n \geq -1$ for all $n \in \mathbb{N}$ and such that $f_n \to f$ almost surely, we have $f \in D$.*

The notion of Fatou-closedness will help us to deduce the desired weak-*-closedness of the set $C = C_0 \cap L^\infty$. To this end, we will use the Krein-Smulian theorem:

Theorem 2.5.19 *(Krein-Smulian) A convex cone $C \subset L^\infty$ is weak-*-closed if and only if for each sequence $(f_n)_{n\geq 1} \subset C$ converging in probability to a function f and such that $\|f_n\| \leq 1$ uniformly in n, we have $f \in C$.*

Lemma 2.5.20 *Suppose the convex cone C_0 is Fatou-closed. Then the set $C = C_0 \cap L^\infty$ is weak-*-closed in L^∞*

We therefore need to show that C_0 is Fatou closed, i.e. that for all sequences $(h_n)_{n\geq 1} \subset C_0$ such that $h_n \geq -1$ and $h_n \to h$ almost surely we have $h \in C_0$, which by definition means that there exists an $f \in K_0$ depending on our chosen sequence such that $h \leq f$. We will use one final reformulation of this task to be proven using the following notion.

Definition 2.5.21 *An element $f_0 = \int_0^T H_s dS_s$ is called maximal if there is no other admissible strategy $\tilde{H}$ such that $\int_0^T \tilde{H}_s dS_s \geq f_0$ almost surely and $\mathbb{P}[\int_0^T \tilde{H}_s dS_s > f_0] > 0$.*

Next introduce the following set

$$\mathcal{D}_h := \left\{ f \in L^0 \mid \exists (K^n)_{n\geq 1}, \int_0^T K_s^n dS_s \geq -1, \quad \int_0^t K_s^n dS_s \to f \ a.s., \quad h \leq f \right\}$$

Lemma 2.5.22 *The set $\mathcal{D}_h$ is non-empty and contains a maximal element f_0.*

The proof makes use of transfinite induction and can be found in the book by Delbaen and Schachermeyer.

Lemma 2.5.23 *Let $(h_n)_{n\geq 1} \subset C_0$ be a sequence such that $h_n \geq -1$ and $h_n \to h$ almost surely. Let f_0 be the maximal element in $\mathcal{D}_h$. If we have $f_0 \in K_0$, then $h \in C_0$. In particular C_0 is Fatou-closed.*

Remark 2.5.24 *The above lemma and it's proof show that we can restrict ourselves to a sequence converging to the maximal element f_0 instead of considering $(h_n)_{n\geq 1}$ converging to h. Hence, our task is to show the following:*

Let $f_0 \in \mathcal{D}_h$ be maximal. By definition, there exists a sequence $(H^n)_{n\geq 1}$ of admissible strategies such that $f_n := \int_0^T H_s^n dS_s \geq -1$ and $f_n \to f_0$ almost surely. We must show that $f_0 \in K_0$, i.e. that there exists an admissible strategy H such that $f_0 = \int_0^T H_s dS_s$. By maximality of f_0, it would suffices to show $f_0 \leq \int_0^T H_s dS_s$.

Upon succeeding to show the above, we can conclude by lemma 2.5.23 the Fatou-closedness of C_0, which in turn would give us that C is weak-*-closed by lemma 2.5.20.

The remainder of this subsection will now be dedicated to the proof to the above claim in six steps, whose outline we want to sketch here. It is at this point that finally the semi-martingale property of S will come crucially into play. The main challange will be to show that for a sequence $(L^k)_{k\geq 1}$ of certain convex combinations of integrands $L^k \in \operatorname{conv}\{H^k, H^{k+1}, \dots\}$ (whose integrals still converge to f_0 almost surely) we have convergence of the integrals $\int_0^T L_s^k dS_s$ in the semi-martingale topology.

At this point, Memin's theorem allows us to conclude the existence of an admissible process L such that $f_0 = \int_0^T L_s dS_s$, i.e. percisly our claim (step 6). In order to establish convergence in the semi-martingale topology we consider the integrals associated to the local martingale part M and the finite variation part A and establish convergence in the semi-martingale topology separatly (steps 4 and 5). In order to do so, we need some auxilliary lemmas (step 3) as well as a change of the underlying probability measure (steps 1 and 2) in order to be able to work in an L^2 setting that is natural when working with stochastic integrals.

Lemma 2.5.25 *(Step 1) Given the above notation, the random variable*

$$F_{n,m} := \sup_{t\in[0,T]} \Big| \int_0^t H_s^n dS_s - \int_0^t H_s^m dS_s \Big|$$

converges in probability to zero.

A priori, we do not know if the stochastic integrals are in $L^2(\mathbb{P})$. Yet, in order to show convergences of semi-martingales, an L^2 setting is much desirable. This first step will therefore enable us to define a new measure $\mathbb{Q}$ (not to be confused with an equivalent martingale measure from the fundamental theorem), under which we are in the $L^2(\mathbb{Q})$-setting. The proof relies on the maximality of f_0.

Lemma 2.5.26 *(Step 2: Change of measure) The random variable*

$$q := \sup_n \sup_{t\in[0,T]} \Big| \int_0^t H_s^n dS_s \Big|$$

is strictly positive and finite almost surely. Hence we can define a measure $\mathbb{Q}$ absolutly continuous with respect to our departure measure $\mathbb{P}$ and with Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{e^{-q}}{\mathbb{E}^{\mathbb{P}}[e^{-q}]}$$

Moreover under the new measure $\mathbb{Q}$ one has

$$\lim_{n,m\to\infty} \| \sup_{t\in[0,T]} | \int_0^t H_s^n dS_s - \int_0^t H_s^m dS_s| \|_{L^2(\mathbb{Q})} = 0$$

The following third step comprises two lemmas which essentially prepare the proof of the following step four. We refer again to the book by Delbean and Schachermayer for the proofs.

Lemma 2.5.27 *(Step 3.1) For any $c > 0$, set $\tau_c^n := \inf\{t \mid |\int_0^t H_s^n DM_s| \geq c\} \wedge T$. Stopping the local martingale $\int_0^\cdot H_s^n dM_s$ at τ_c^n causes an error $\int_0^T K_s^{n,c} dM_s$ compared with the integral takien over the complete time, where $K_t^{n,c} = \chi_{(\tau_c^n,T]}(t) H_t^n$. Given this setting, for all $\varepsilon > 0$ there is a $c_0 > 0$ such that for arbitrary n, for all convex weights $(\lambda_1, \ldots, \lambda_n)$ and all $c \geq c_0$ we have*

$$\mathbb{Q}\left[\sup_{t\in[0,T]} |\left(\sum_{i=1}^n \lambda_i \int_0^t K_s^{i,c} dM_s\right)| > \varepsilon\right] < \varepsilon$$

Lemma 2.5.28 *(Step 3.2) Given the setting of lemma 2.5.27, for all $\varepsilon > 0$ there exists a $c_0 > 0$ such that for all h predictable such that $|h| \leq 1$, all convex weights $(\lambda_1, \ldots, \lambda_n)$ and all $c \geq c_0$ we have*

$$\mathbb{Q}\left[\sup_{t\in[0,T]} |\left(\sum_{i=1}^n \lambda_i \int_0^t h_s K_s^{i,c} dM_s\right)| > \varepsilon\right] < \varepsilon$$

Lemma 2.5.29 *(Step 4: Convergence of the martingale part) There exists a sequence $(L^n)_{n\geq 1}$ of convex combinations $L^n \in \operatorname{conv}\{H^k, k \geq n\}$ such that the process $\int_0^\cdot L_s^n dM_s$ converges in the semi-martingale topology.*

Lemma 2.5.30 *(Step 5: Convergence of the bounded variation part) The sequence $(L^n)_{n\geq 1}$ from the previous lemma 2.5.29 is such that the process $\int_0^\cdot L_s^n dA_s$ converges in semi-martingale topology as well.*

Using these preparations, we can finish the proof using Memin's theorem:

Theorem 2.5.31 *(Memin, 1980) Let $X = (X_t)_{t\in[0,T]}$ be a semi-martingale. Let $\mathbb{L}(X)$ be the space of equivalence classes of X-integrable processes produced by the equivalence relation*

$$Y \sim Z \Leftrightarrow \int_0^T Y_s dX_s = \int_0^T Z_s dX_s$$

Then for $Y \in \mathbb{L}(X)$,

$$\|Y\|_{\mathbb{L}(X)} := d(\int_0^T Y_s dX_s, 0)$$

defines a quasi norm on $\mathbb{L}(X)$, where $d(\cdot,\cdot)$ denotes the semi-martingale metric. The induced topological space is a complete topological vectorspace.

Lemma 2.5.32 *(Step 6: Finishing the proof) For the sequence $(L^n)_{n\geq 1}$ from lemma 2.5.29 the process $\int_0^\cdot L_s^n dS_s$ converges in semi-martingale topology. There exists a predictable process L such that $\int_0^T L_s dS_s \geq -1$ and such that the limit of the semi-martingale convergence can be given by*

$$\int_0^\cdot L_s^n dS_s \to_d \int_0^\cdot L_s dS_s.$$

Moreover we have the almost sure convergence

$$\int_0^T L_s^n dS_s \to \int_0^T L_s dS_s = f_0$$

and therefore in particular $f_0 \in K_0$.

Chapter 3

Stochastic Optimal Control

3.1 Introduction to basic concepts

In a complete financial market model each contingent claim under some conditions can be replicated by continuously trading in the stock and bond market. In an incomplete market, a perfect hedge may not be possible and a hedging error may occur. A hedging error also occurs in a complete market if the writer of an option invests less than the fair value in hedging. The problem of *quantile hedging* consists in finding the optimal strategy that maximizes the probability of a perfect hedge subject to capital constraints. This is a typical example of a *stochastic optimal control problem.*

Example ("Quantile Hedging") Consider a complete financial market model with a single risky asset and unique equivalent martingale measure $\mathbb{P}^*$. The fair value of a contingent claim $H \in L^2$ is given by $\mathbb{E}^*[H]$. If the writer of the claim faces a capital constraint in the sense that he cannot (or does not want to) invest more than $v < \mathbb{E}^*[H]$ Euros for hedging, then a perfect hedge is not possible. For a self-financing trading strategy ξ, let us thus denote by (V_t^ξ) the associated value process with initial investment V_0^ξ and introduce the set

$$\mathcal{S} := \{\xi : \xi \text{ self-financing, square intergrable with } V_0^\xi \leq v\}$$

of trading strategies that require an initial investment of no more than v. The problem of quantile hedging consists of

$$\max_{\xi \in \mathcal{S}} \mathbb{P}[V_T^\xi \geq H].$$

This section provides an introduction into the theory of stochastic optimal control. We closely follow the nice book "Continuous-time Stochastic Control and Optimization with Financial Applications" by H. Pham.

3.1.1 Examples of stochastic optimization problems

In the sequel we present several examples of stochastic optimization problems arising in economics and finance.

Portfolio allocation

We consider a financial market consisting of a riskless asset with strictly positive price process S^0 representing the savings account, and n risky assets of price process S representing stocks. An agent may invest in this market at any time t, with a number of shares α_t in the n risky assets. By denoting by X_t its wealth at time t, the number of shares invested in the savings account at time t is $(X_t - \alpha_t \cdot S_t)/S_t^0$. The self-financed wealth process evolves according to

$$dX_t = (X_t - \alpha_t \cdot S_t)\frac{dS_t^0}{S_t^0} + \alpha_t \cdot dS_t$$

The control is the process α valued in A, subset of $\mathbb{R}^n$. The portfolio allocation problem is to choose the best investment in these assets. A classical approach for describing the behavior and preferences of agents and investors is the expected utility criterion. It relies on the theory of choice under uncertainty: the agent compares random incomes for which he knows the probability distributions. Under some conditions on the preferences, Von Neumann and Morgenstern show that they can be represented through the expectation of some function U, called utility. The random income X is preferred to a random income X' if $\mathbb{E}[U(X)] \geq \mathbb{E}[U(X')]$. The utility function U is nondecreasing and concave, this last feature formulating the risk aversion of the agent.
In this portfolio allocation context, the criterion consists of maximizing the expected utility from terminal wealth on a finite horizon $T < \infty$:

$$\sup_{\alpha} \mathbb{E}[U(X_T)]$$

A class of possible utility functions would be

$$U(x) = \begin{cases} \frac{x^p - 1}{p},\ x \geq 0 \\ -\infty,\ x < 0 \end{cases}.$$

with $0 < p < 1$, the limiting case $p = 0$ corresponding to a logarithmic utility function: $U(x) = \ln x$, $x > 0$. These popular utility functions are called CRRA (Constant Relative Risk Aversion) since the relative risk aversion defined by $\eta = -xU''(x)/U'(x)$, is constant in this case and equal to $1 - p$.

Production-consumption model

We consider the following model for a production unit. The capital valued K_t at time t evolves according to the investment rate I_t in capital and the price S_t per unit of capital by

$$dK_t = K_t \frac{dS_t}{S_t} + I_t dt$$

The debt L_t of this production unit evolves in terms of the interest rate r, the consumption rate C_t and the productivity rate P_t of capital:

$$dL_t = r dL_t - \frac{K_t}{L_t} dP_t + (I_t + C_t)dt$$

We choose a dynamics model for (S_t, P_t):

$$dS_t = \mu S_t dt + \sigma_1 S_t dW_t^1,$$

$$dP_t = bdt + \sigma_2 dW_t^2$$

where (W^1, W^2) is a two-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t), \mathbb{P})$ and μ, b, σ_1, σ_2 are constants, $\sigma_1, \sigma_2 > 0$. The net value of this production unit is

$$X_t = K_t - L_t$$

We impose the constraints

$$K_t \geq 0,\, C_t \geq 0,\, X_t > 0, \quad t \geq 0$$

We denote by $k_t = K_t / X_t$ and $c_t = C_t / X_t$ the control variables for investment and consumption. The dynamics of the controlled system is then governed by:

$$dX_t = X_t \left[k_t \left(\mu - r + \frac{b}{S_t} \right) + (r - c_t) \right] dt + k_t X_t \sigma_1 dW_t^1 + k_t \frac{X_t}{S_t} \sigma_2 dW_t^2$$

$$dS_t = \mu S_t dt + \sigma_1 S_t dW_t^1,$$

Given a discount factor $\beta > 0$ and a utility function U, the objective is to determine the optimal investment and consumption for the production unit:

$$\sup_{(k,c)} \mathbb{E} \left[\int_0^\infty e^{-\beta t} U(c_t X_t) dt \right].$$

Irreversible investment model

We consider a firm with production goods (electricity, oil, etc.) The firm may increase its production capacity by transferring capital form an activity sector to another one. The controlled dynamics of its production capacity then evolves according to

$$dX_t = X_t(-\delta dt + \sigma dW_t) + \alpha_t dt.$$

$\delta \geq 0$ is the depreciation rate of the production, $\sigma > 0$ is the volatility, $\alpha_t dt$ is the capital-unit number obtained by the firm for a cost $\lambda \alpha_t dt$. $\lambda > 0$ is interpreted as a conversion factor from an activity sector to another one. The control α is valued in $\mathbb{R}_+$. This is an irreversible model for the capital expansion of the firm. The profit function of the company is an increasing, concave function Π from $\mathbb{R}_+$ into $\mathbb{R}$, and the optimization problem for the firm is

$$\sup_{\alpha} \mathbb{E} \left[\int_0^\infty e^{-\beta t} (\Pi(X_t) - \lambda \alpha_t) dt \right]$$

3.1.2 Controlled diffusion processes

We consider a control model where the state of the system is governed by a stochastic differential equation (SDE) valued in $\mathbb{R}^n$:

$$dX_s = b(X_s, \alpha_s) ds + \sigma(X_s, \alpha_s) dW_s, \tag{3.1}$$

where W is a d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions.

Assumption 3.1.1 *We assume that the following conditions are satisfied.*

(i) The control $\alpha = (\alpha_s)$ is a progressively measurable process taking values in some set $A \subset \mathbb{R}^m$.

(ii) The measurable functions $b : \mathbb{R}^n \times A \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times A \to \mathbb{R}^{n\times d}$ satisfy a uniform Lipschitz condition:

$$|b(x,a) - b(y,a)| + |\sigma(x,a) - \sigma(y,a)| \le K|x-y| \quad \textit{for all} \quad x, y \in \mathbb{R}^n, \ a \in A. \tag{3.2}$$

In the sequel, for $0 \le t \le T < \infty$, we denote by $\mathcal{T}_{t,T}$ the set of stopping times valued in $[t,T]$ and by $\mathcal{A}$ the set of control processes α such that

$$\mathbb{E}\left[\int_0^T |b(0,\alpha_t)|^2 + |\sigma(0,\alpha_t)|^2 dt\right] < \infty. \tag{3.3}$$

The conditions (3.2) and (3.3) ensure for all $\alpha \in \mathcal{A}$ and for any initial condition $(t,x) \in [0,T] \times \mathbb{R}^n$, the existence and uniqueness of a strong solution to the SDE (with random coefficients) (3.1) starting from x at $s = t$. We denote by $\{X_s^{t,x},\ s \in [t,T]\}$ this solution with a.s. continuous paths. Under the above conditions on b, σ and α we also have:

$$\mathbb{E}[\sup_{s\in[t,T]} |X_s^{t,x}|^2] < \infty \tag{3.4}$$

$$\lim_{h\downarrow 0^+} \mathbb{E}[\sup_{s\in[t,t+h]} |X_s^{t,x} - x|^2] = 0 \tag{3.5}$$

Now, let $f : [0,T] \times \mathbb{R}^n \times A \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be two measurable functions describing running and terminal revenues, respectively We suppose that

$$\begin{aligned} &(i)\, g \text{ is lower-bounded} \\ \textbf{or } &(ii)\, g \text{ satisfies a quadratic growth condition: } |g(x)| \le C(1+|x|^2),\ \forall x \in \mathbb{R}^n, \\ &\quad \text{for some constant } C \text{ independent of } x. \end{aligned} \tag{3.6}$$

For $(t,x) \in [0,T] \times \mathbb{R}^n$, we denote by $\mathcal{A}(t,x)$ the subset of controls α of $\mathcal{A}$ such that

$$\mathbb{E}\left[\int_t^T |f(s, X_s^{t,x}, \alpha_s)| ds\right] < \infty, \tag{3.7}$$

and we assume that $\mathcal{A}(t,x)$ is not empty for all $(t,x) \in [0,T] \times \mathbb{R}^n$. We can then define under (3.6) the gain function:

$$J(t,x,\alpha) := \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, \alpha_s) ds + g(X_T^{t,x})\right],$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$ and $\alpha \in \mathcal{A}(t,x)$. The objective is to maximize over control processes the gain function J. To this end, we introduce the associated *value function*:

$$v(t,x) := \sup_{\alpha\in\mathcal{A}(t,x)} J(t,x,\alpha).$$

Definition 3.1.2 *(i) We say that $\hat{\alpha} \in \mathcal{A}(t,x)$ is an optimal control if $J(t,x,\hat{\alpha}) = v(t,x)$.*

(ii) A control α is called a Markovian control if it has the form $\alpha_s = a(s, X_s^{t,x})$ for some measurable function a from $[0,T] \times \mathbb{R}^n$ into A.

In the sequel, we shall implicitly assume that the value function v is measurable in its arguments. This point is not trivial a priori, but can be proven using the so called measurable selection theorem.

3.1.3 Dynamic programming principle

The dynamic programming principle (DPP) is a fundamental principle in the theory of stochastic control.

Theorem 3.1.3 *Let $(t, x) \in [0, T] \times \mathbb{R}^n$. Then*

$$\begin{aligned} v(t,x) &= \sup_{\alpha \in \mathcal{A}(t,x)} \sup_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s) ds + v(\theta, X_\theta^{t,x})\right] \\ &= \sup_{\alpha \in \mathcal{A}(t,x)} \inf_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s) ds + v(\theta, X_\theta^{t,x})\right] \end{aligned} \tag{3.8}$$

The interpretation of the DPP is that the optimization problem can be split in two parts: an optimal control on the whole time interval $[t, T]$ may be obtained by first searching for an optimal control from time θ given the state value $X_\theta^{t,x}$, i.e. compute $v(\theta, X_\theta^{t,T})$, and then maximizing over controls on $[t, \theta]$ the quantity

$$\mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s) ds + v(\theta, X_\theta^{t,x})\right].$$

The proof of the preceding theorem requires cumbersome measurable selection arguments. The following weak DPP can be established without measurable selection. To this end, we assume that the value function is locally bounded and denote by

$$\begin{aligned} v^*(t,x) &:= \limsup_{(x',t') \to (x,t)} v(t', x') \\ v_*(t,x) &:= \liminf_{(x',t') \to (x,t)} v(t', x') \end{aligned}$$

its upper- and lower-semicontinous envelope, respecyively. If we know that the value function is continuous (we state a sufficient condition below), then $v = v_* = v^*$ and the weak DPP stated in the following theorem reduces to the standard DPP.

Theorem 3.1.4 *Assume that v is locally bounded. Let $t \in [0, T]$ be fixed and let $\{\theta^\alpha : \theta^\alpha \in \mathcal{T}_{t,T}, \alpha \in \mathcal{A}(t, \cdot)\}$ be a family of stopping times independent of $\mathcal{F}_t$. Then*

$$v(t,x) \le \sup_{\alpha \in \mathcal{A}(t,x)} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s) ds + v^*(\theta, X_\theta^{t,x})\right].$$

If, furthermore, g is lower semi-continuous and $X_\theta^{t,x} \mathbf{1}_{[t,\theta^\alpha}$ is L^∞-bounded, then

$$v(t,x) \ge \sup_{\alpha \in \mathcal{A}(t,x)} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s) ds + v_*(\theta, X_\theta^{t,x})\right].$$

The following proposition states a sufficient condition that guarantees the continuity of the value function. We state and prove the result for $f \equiv 0$. An extension to general running cost function is straightforward.

Proposition 3.1.5 *Suppose that $f \equiv 0$ and that g is Lipschitz continuous. Then v is Lipschitz continuous in x, uniformly in $t \in [0, T]$. If, furthermore, the set of admissible controls A is bounded, then v is $\frac{1}{2}$-Hölder continuous in t and*

$$|v(t,x) - v(t',x)| \le C(1 + |x|)\sqrt{|t - t'|}$$

In this case, v is continuous.

3.1.4 Hamilton-Jacobi-Bellman equation

The Hamilton-Jacobi-Bellman equation (HJB) is the infinitesimal version of the dynamic programming principle: it describes the local behavior of the value function when we send the stopping time $\theta \in \mathcal{T}_{t,T}$ to t.

Formal derivation of HJB

Let us consider the time $\theta = t + h$ with $h > 0$ and a control $\alpha \in \mathcal{A}(t,x)$. According to DPP:

$$v(t,x) \geq \mathbb{E}\left[\int_t^{t+h} f(s, X_s^{t,x}, \alpha_s)ds + v(t+h, X_{t+h}^{t,x})\right] \tag{3.9}$$

By assuming that v is smooth enough, we may apply Itô's formula:

$$v(t+h, X_{t+h}^{t,x}) = v(t,x) + \int_t^{t+h}\left(\frac{\partial v}{\partial t} + \mathcal{L}^{\alpha_s}v\right)(s, X_s^{t,x})ds + \text{(local) martingale}$$

where the operator $\mathcal{L}^a$ (for $a \in A$) is defined by

$$\mathcal{L}^a v := b(x,a)\cdot D_x v + \frac{1}{2}tr(\sigma(x,a)\sigma'(x,a)D_x^2 v).$$

Substituting into (3.9) we get

$$\mathbb{E}\left[\int_t^{t+h}\left(\frac{\partial v}{\partial t} + \mathcal{L}^{\alpha_s}v\right)(s, X_s^{t,x}) + f(s, X_s^{t,x}, \alpha_s)ds\right] \leq 0$$

Dividing by h and letting $h \to 0$ we obtain:

$$\frac{\partial v}{\partial t}(t,x) + \mathcal{L}^a v(t,x) + f(t,x,a) \leq 0$$

for all $a \in A$. Choosing an optimal control $\alpha^* \in \mathcal{A}(t,x)$ s.t.

$$v(t,x) = \mathbb{E}\left[\int_t^{t+h} f(s, X_s^{t,x}, \alpha_s^*)ds + v(t+h, X_{t+h}^{t,x})\right]$$

we would have obtained

$$\frac{\partial v}{\partial t}(t,x) + \mathcal{L}^{\alpha_t^*}v(t,x) + f(t,x,\alpha_t^*) = 0$$

after $h \to 0$. This suggests that v should satisfy

$$\frac{\partial v}{\partial t}(t,x) + \sup_{a\in A}[\mathcal{L}^a v(t,x) + f(t,x,a)] = 0, \qquad \forall (t,x) \in [0,T)\times\mathbb{R}^n$$

We often rewrite this PDE in the form

$$-\frac{\partial v}{\partial t}(t,x) - H(t,x,D_x v(t,x), D_x^2 v(t,x)) = 0, \qquad \forall (t,x) \in [0,T)\times\mathbb{R}^n \tag{3.10}$$

where for $(t,x,p,M) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}_n$:

$$H(t,x,p,M) = \sup_{a\in A}[b(x,a)\cdot p + \frac{1}{2}tr(\sigma(x,a)\sigma'(x,a)M) + f(t,x,a)].$$

This function is called the Hamiltonian of the associated control problem. The equation (3.10) is called the dynamic programming equation or the Hamilton-Jacobi-Bellman equation. The terminal condition associated to this PDE is

$$v(T,x) := g(x), \quad \forall x \in \mathbb{R}^n,$$

which results from the very definition of the value function v considered at the horizon date T.

Verification theorem

The crucial step in the classical approach to dynamic programming consists in proving that, given a smooth solution to the HJB equation, this candidate coincides with the value function. This result is called the verification theorem, and allows us to exhibit as byproduct an optimal Markovian control. The proof relies essentially on Itô's formula.

Theorem 3.1.6 *Let w be a function in $C^{1,2}([0,T)\times\mathbb{R}^n)\cap C^0([0,T]\times\mathbb{R}^n)$, and satisfying a quadratic growth condition, i.e. there exists a constant C such that*

$$|w(t,x)| \le C(1+|x|^2), \quad \forall (t,x)\in[0,T]\times\mathbb{R}^n.$$

(i) Suppose that

$$-\frac{\partial w}{\partial t}(t,x) - \sup_{a\in A}[\mathcal{L}^a w(t,x) + f(t,x,a)] \ge 0, \qquad \forall (t,x)\in[0,T)\times\mathbb{R}^n \tag{3.11}$$

$$w(T,x) \ge g(x), \qquad \forall x\in\mathbb{R}^n \tag{3.12}$$

Then $w \ge v$ on $[0,T]\times\mathbb{R}^n$.

(ii) Suppose in addition that $w(T,\cdot) = g$, and there exists a measurable function $\hat{\alpha}(t,x)$, $(t,x)\in[0,T)\times\mathbb{R}^n$, valued in A such that

$$-\frac{\partial w}{\partial t}(t,x) - \mathcal{L}^{\hat{\alpha}(t,x)}w(t,x) - f(t,x,\hat{\alpha}(t,x)) = 0, \quad \forall (t,x)\in[0,T)\times\mathbb{R}^n, \tag{3.13}$$

s.t. the SDE

$$dX_s = b(X_s,\hat{\alpha}(s,X_s))ds + \sigma(X_s,\hat{\alpha}(s,X_s))dW_s$$

admits a unique solution, denoted by $\hat{X}_s^{t,x}$, given an initial condition $X_t = x$, and the process $\{\hat{\alpha}(s,\hat{X}_s^{t,x}),\ s\in[t,T]\}$ lies in $\mathcal{A}(t,x)$.
Then $w = v$ on $[0,T]\times\mathbb{R}^n$ and $\hat{\alpha}$ is an optimal Markovian control.

3.1.5 Application: Portfolio optimization for power utilities

We consider again the example described in Section 3.1.1 in the framework of the Black-Scholes-Merton model over a finite horizon T. An agent invests at any time t a *portion* α_t of his wealth in a stock of price S (governed by a geometric Brownian motion) and $1-\alpha_t$ in a bond of price S^0 with interest rate r. The investor faces the portfolio constraint that at any time t, α_t is valued in A compact convex subset of $\mathbb{R}$. His wealth process evolves according to the SDE

$$\begin{aligned} dX_t &= \frac{X_t\alpha_t}{S_t}dS_t + \frac{X_t(1-\alpha_t)}{S_t^0}dS_t^0 = \\ &= X_t(\alpha_t\mu + (1-\alpha_t)r)dt + X_t\alpha_t\sigma dW_t. \end{aligned}$$

Given a portfolio strategy $\alpha\in\mathcal{A}$ we denote by $X^{t,x}$ the corresponding wealth process starting from an initial capital $X_t = x$ at time $t>0$. The agent wants to maximize the expected utility from terminal wealth at horizon T. The value function of the utility maximization problem is then defined by

$$v(t,x) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}[U(X_T^{t,x})], \qquad (t,x)\in[0,T]\times\mathbb{R}_+. \tag{3.14}$$

The utility function U is increasing and concave on $\mathbb{R}_+$.
The HJB equation for the stochastic control problem (3.14) is

$$-\frac{\partial w}{\partial t} - \sup_{a\in A}[\mathcal{L}^a w(t,x)] = 0, \tag{3.15}$$

together with the terminal condition

$$w(T,x) = U(x), \qquad x \in \mathbb{R}_+. \tag{3.16}$$

Here, $\mathcal{L}^a w(t,x) = x(a\mu + (1-a)r)\frac{\partial w}{\partial x} + \frac{1}{2}x^2a^2\sigma^2\frac{\partial^2 w}{\partial x^2}$. It turns out that for the particular case of power utility functions of CRRA type, as considered originally by Merton

$$U(x) = \frac{x^p}{p}, \qquad x \geq 0,\ 0 < p < 1$$

one can find explicitly a smooth solution to the above problem. We are looking for a candidate solution in the form

$$w(t,x) = \phi(t)U(x)$$

for some positive function ϕ. By substituting into (3.15)-(3.16), we derive that ϕ should satisfy the ordinary differential equation

$$\phi'(t) + \rho\phi(t) = 0, \qquad \phi(T) = 1$$

where

$$\rho = p\sup_{q\in A}[a(\mu - r) + r - \frac{1}{2}a^2(1-p)\sigma^2].$$

We obtain $\phi(t) = \exp(\rho(T-t))$. Hence, the function

$$w(t,x) = \exp(\rho(T-t))U(x), \qquad (t,x) \in [0,T] \times \mathbb{R}_+$$

is a smooth solution to (3.15)-(3.16). Furthermore, the function $A \ni a \longmapsto a(\mu - r) + r - \frac{1}{2}a^2(1-p)\sigma^2$ is strictly concave on the closed convex set A, and thus attains its maximum at some constant $\hat{a}$. By construction, $\hat{a}$ attains the supremum of $\sup_{a\in A}[\mathcal{L}^a w(t,x)]$. Moreover, the wealth process associated to the constant control $\hat{a}$

$$dX_t = X_t(\hat{a}\mu + (1-\hat{a})r)dt + X_t\hat{a}\sigma dW_t$$

admits a unique solution given an initial condition. From the verification Theorem 3.1.6, this proves that the value function to the utility maximization problem (3.14) is equal to w, and the optimal proportion of wealth to invest in stock is constant given by $\hat{a}$. Finally notice that when A is large enough, the values $\hat{a}$ and ρ are explicitly given by

$$\hat{a} = \frac{\mu - r}{\sigma^2(1-p)}$$

and

$$\rho = \frac{(\mu - r)^2}{2\sigma^2}\frac{p}{1-p} + rp.$$

3.2 Viscosity solutions

In order to solve a given stochastic optimal control problem, that is to calculate the value function

$$v(t,x) = \sup_{\alpha \in \mathcal{A}(t,x)} J(t,x,\alpha) \tag{3.17}$$

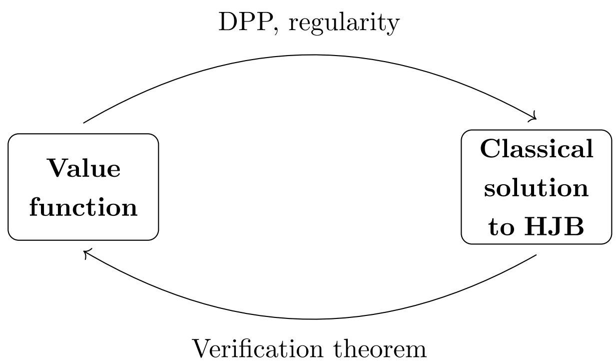
and if existent, an optimal control $\hat{\alpha} \in \mathcal{A}(t,x)$, the general philosophy in the preceding sections was as follows:
Given one knew that the value function had some regularity properties, one was able to deduce from 3.17 a DPP and subsequently the corresponding HJB equation

$$-\frac{\partial v}{\partial t}(t,x) - H(t,x,D_x v(t,x), D_x^2 v(t,x)) = 0, \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^n \tag{HJB}$$

where for $(t,x,p,M) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}_n$:

$$H(t,x,p,M) = \sup_{a \in A}[b(x,a) \cdot p + \frac{1}{2} tr(\sigma(x,a)\sigma'(x,a)M) + f(t,x,a)].$$

Under even more conditions (among others a quadratic growth condition) these deductions are invertible using the verification theorem.



Hence, instead of solving the infinite dimensional optimization problem in 3.17 directly, one tries to solve the corresponding HJB equation, subsequently applying the verification theorem to conclude that the solution to the HJB equation coincides with the value function.
The problem to this approach is that the verification theorem requires the HJB equation to have a *classical* solution, that is $v \in C^{1,2}([0,T) \times \mathbb{R}^d) \cap C([0,T] \times \mathbb{R}^d)$, which is only the case in some very narrow problem settings.
In order to still apply the strategy described (that is computing the value function by solving a corresponding PDE and applying some kind of verification theorem) to more general settings, one thus needs to weaken the definition of what it means for a function to be solution to a given differential equation.
There are several possibilities as to how one can do this. The most common approach constitutes the notion of a weak solution. It will turn out that for our purposes, a natural weaking of the solution concept is given by so called viscosity solutions. Yet, we must "pay" for this weakening by potentially losing uniqueness of our new solution concept. In order to still guarantee uniqueness, which is crucial in

going from the HJB equation back to the value function, one needs to make sure the PDE considered admits a so called comparison principle, which will thus in a certain sense replace the verification theorem in the classical solution setting.

3.2.1 Framework, Motivation and Definition

Throughout this section, we will consider nonlinear PDEs of the form

$$F\big(x, u(x), Du(x), D^2u(x)\big) = 0 \qquad \text{on } \mathcal{O} \subseteq \mathbb{R}^n \text{ open} \tag{PDE}$$

where $F : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ is continuous and $\mathcal{S}^n$ denotes the set of symmetric $n \times n$ matrices. Moreover, we will demand F to be elliptic, that means that

$$F(x, r, p, A) \le F(x, r, p, B)$$

for all $x \in \mathcal{O}, r \in \mathbb{R}, p \in \mathbb{R}^n$ and $A \ge B$, meaning $A - B$ is positive semi-definite[1]. In case we are dealing with time dependent problems, we consider the following generalization to PDE

$$F\big(t, x, u(t,x), \partial_t u(t,x), D_x u(t,x), D_x^2 u(t,x)\big) = 0 \qquad \text{on } [0,T) \times \mathcal{O} \subseteq \mathbb{R} \times \mathbb{R}^n \tag{PDEt}$$

where one imposes in addition a parabolicity condition, that is for all $t \in [0,T), x \in \mathcal{O}, r \in \mathbb{R}, q \le \tilde{q} \in \mathbb{R}, p \in \mathbb{R}^n, M \in \mathcal{S}^n$ one demands

$$F(t, x, r, q, p, M) \ge F(t, x, r, \tilde{q}, p, M).$$

Note that these conditions (ellipticity and parabolicity) are in particular satisfied by the HJB equation where

$$F(t, x, r, q, p, M) = -q - \sup_{a \in A} \Big[b(x,a) \cdot p + \frac{1}{2} tr(\sigma(x,a)\sigma'(x,a)M) + f(t,x,a) \Big],$$

as the matrix $\sigma(x,a)\sigma'(x,a)$ is symmetric positive semi-definite.

In order to introduce a weakend solution concept for PDEs, one must find a characterization of solutions that in the classical case of sufficiently regular functions coincides with the classical solution concept while at the same time allowing more general cases to be considered. Such a characerization is given by the following proposition:

Proposition 3.2.1 *A function $u \in C^{1,2}([0,T) \times \mathcal{O})$ is a classical solution to PDEt if and only if it is a classical supersolution i.e.*

$$F\big(t, x, u(t,x), \partial_t u(t,x), D_x u(t,x), D_x^2 u(t,x)\big) \ge 0 \qquad \text{on} [0,T) \times \mathcal{O} \subseteq \mathbb{R} \times \mathbb{R}^n$$

as well as a classical subsolution, i.e.

$$F\big(t, x, u(t,x), \partial_t u(t,x), D_x u(t,x), D_x^2 u(t,x)\big) \le 0 \qquad \text{on} [0,T) \times \mathcal{O} \subseteq \mathbb{R} \times \mathbb{R}^n.$$

[1] Note that as A, B are symmetric matrices over $\mathbb{R}$ all eigenvalues of $A - B$ are real.

It is a classical supersolution if and only if for all $(t_0, x_0) \in [0,T) \times \mathcal{O}$ and $\varphi \in \times C^{1,2}([0,T) \times \mathcal{O})$ such that (t_0, x_0) is a minimizer of

$$(t,x) \mapsto u(t,x) - \varphi(t,x),$$

we have

$$F\big(t, x_0, u(t,x_0), \partial_t\varphi(t,x_0), D_x\varphi(t,x_0), D_x^2\varphi(t,x_0)\big) \geq 0.$$

It is a classical subsolution if and only if for all $(t_0, x_0) \in [0,T) \times \mathcal{O}$ and $\varphi \in \times C^{1,2}([0,T) \times \mathcal{O})$ such that (t_0, x_0) is a maximizer of

$$(t,x) \mapsto u(t,x) - \varphi(t,x),$$

we have

$$F\big(t, x_0, u(t,x_0), \partial_t\varphi(t,x_0), D_x\varphi(t,x_0), D_x^2\varphi(t,x_0)\big) \leq 0.$$

Remark 3.2.2 *Note that the given characterizations of super- and subsolutions to PDEt theirselves do not require u to be differentiable but only the test function φ which is smooth by default. This permits us to introduce the concept of viscosity solutions that is independet of the harsh regularity requirements of a classical solution. As one last step before the formal definition, we need to introduce upper-and lower-semicontinous envelopes. Their utility will become clear in the upcoming remark.*

Definition 3.2.3 (Envelopes) *Let $u : [0,T) \times \mathcal{O} \to \mathbb{R}$ be locally bounded. Then the limits*

$$u^*(t,x) := \limsup_{(x',t') \to (x,t)} u(t',x')$$
$$u_*(t,x) := \liminf_{(x',t') \to (x,t)} u(t',x')$$

exist. u^* is called upper-semicontinous envelope and u_* is called lower-semicontinous envelope of u.

Definition 3.2.4 *A locally bounded function $u : [0,T) \times \mathcal{O} \to \mathbb{R}$ is called*

*(i) a viscosity supersolution to (*PDEt*) if*

$$F\big(t_0, x_0, u_*(t_0,x_0), \partial_t\varphi(t_0,x_0), D_x\varphi(t_0,x_0), D_x^2\varphi(t_0,x_0)\big) \geq 0$$

for all $(t_0, x_0) \in [0,T) \times \mathcal{O}$ and $\varphi \in \times C^{1,2}([0,T) \times \mathcal{O})$ such that (t_0, x_0) is a minimizer of $u_* - \varphi$ on $[0,T) \times \mathcal{O}$.

*(ii) a viscosity subsolution to (*PDEt*) if*

$$F\big(t_0, x_0, u^*(t_0,x_0), \partial_t\varphi(t_0,x_0), D_x\varphi(t_0,x_0), D_x^2\varphi(t_0,x_0)\big) \leq 0$$

for all $(t_0, x_0) \in [0,T) \times \mathcal{O}$ and $\varphi \in \times C^{1,2}([0,T) \times \mathcal{O})$ such that (t_0, x_0) is a maximizer of $u^* - \varphi$ on $[0,T) \times \mathcal{O}$.

(iii) a viscosity solution if it is a viscosity sub- and a supersolution.

Remark 3.2.5

a) By Proposition 3.2.1 classical solutions are viscosity solutions.

b) A viscosity solution does not need to be smooth, in particular not even continuous.

c) By density arguments we may assume that $\varphi \in C^\infty([0,T) \times \mathcal{O})$ *instead of just* $\varphi \in C^{1,2}([0,T) \times \mathcal{O})$.

d) We can assume without loss of generality that (t_0, x_0) *is the unique minimizer or maximizer, that means we can assume for a supersolution*

$$u_*(t,x) > \varphi(t,x) \quad \forall (t,x) \neq (t_0,x_0), \qquad u_*(t_0,x_0) = \varphi(t_0,x_0)$$

and for a subsolution

$$u^*(t,x) < \varphi(t,x) \quad \forall (t,x) \neq (t_0,x_0), \qquad u^*(t_0,x_0) = \varphi(t_0,x_0).$$

Note that in order to make this point (which will be crucial in upcoming proves) we need the envelope notion: If u was not lower-semicontinuous, the minimum might not be attained, which is why we need to consider the lower-semicontinuous envelope of u instead of u itself. Analogous considerations hold for the viscosity subsolution case.

Example 3.2.6 *Consider the ordinary differential equation*

$$|u'(x)| - 1 = 0 \qquad \text{on } \mathbb{R}.$$

The function $f(x) = |x|$ *is not a viscosity supersolution. Indeed take* $x_0 = 0$ *and* $\varphi \equiv 0$. *Then* $\varphi(x_0) = f(x_0)$ *and* $f > \varphi$ *on* $\mathbb{R}\backslash\{0\}$. *But*

$$|D\varphi(x)| - 1 = -1 \ngeq 0.$$

The function $g(x) = -|x|$ *is a viscosity solution. Indeed we only need to test in* $x_0 = 0$. *For the subsolution property take* $\varphi \in C^2(\mathbb{R})$ *with* $\varphi(0) = 0$ *and* $\varphi > g$ *on* $\mathbb{R}\backslash\{0\}$. *For* $\varphi(0) = 0$ *and* $\varphi \geq g$ *we get* $|\varphi'(0)| < 1$, *so*

$$F\big(D\varphi(0)\big) = |D\varphi(0)| - 1 < 0.$$

The supersolution property is fulfilled because there is no smooth function $\varphi \leq g$ *with* $\varphi(0) = g(0)$.

3.2.2 From the value function to a viscosity solution of HJB via DPP

In the previous sections, we saw that under a lot of regularity and growth conditions on the value function, one was able to deduce that it solves HJB in the classical sense. As we now introduced viscosity solutions to PDEs of this form, we can considerably loosen the requirements on the value function for it to be a viscosity solution to HJB. By definition, we need to show the viscosity sub- and supersolution properties.

Proposition 3.2.7 *Suppose the value function v is locally bounded on* $[0,T) \times \mathbb{R}$, *that*

$$|f(t,x,a)| \leq C(1 + |x|^2) + \kappa(a)$$

for some $\kappa(a) \geq 0$ *and*

$$(t,x) \mapsto f(t,x,a) \quad \text{for } a \in A$$

is continuous. Then v is a viscosity supersolution to

$$-\frac{\partial v}{\partial t} - H(t,x,D_x v, D_{xx} v) = 0 \qquad \text{on } [0,T) \times \mathbb{R}^n$$

.

Remark 3.2.8 *Note that the proof did not rely on properties of the Hamiltonian $H(\cdot)$. This will not be the case in the viscosity subsolution case. In attempting to prove this property, one needs to account for potential singulari ties of the Hamiltionian, which will be done exploiting the following assumption:*

Define

$$\mathrm{dom}(H) = \{(t,x,p,M) \mid H(t,x,p,M) < \infty\}.$$

We denote by (A) the assumptions, that H is continuous on

$$\mathrm{int}\big(\mathrm{dom}(H)\big)$$

and that there exists a continuous $\mathcal{G}$ such that

$$H(t,x,p,M) < \infty \qquad \Longleftrightarrow \qquad \mathcal{G}(t,x,p,M) \geq 0.$$

Proposition 3.2.9 *Assume that v is locally bounded and (A) holds. Then v is a viscosity subsolution to the HJB variational inequality*

$$\min\Big(-\frac{\partial v}{\partial t} - H(t,x,D_x v, D_{xx} v), \mathcal{G}(t,x,D_x v, D_{xx} v)\Big) = 0 \qquad \text{on } (0,T)\times\mathbb{R}.$$

One can combine the two previous propositions to obtain:

Theorem 3.2.10 *Under the assumptions of Proposition 6.14 and 6.15 the value function is a viscosity solution to*

$$\min\Big(-\frac{\partial v}{\partial t} - H(t,x,D_x v, D_{xx} v), \mathcal{G}(t,x,D_x v, D_{xx} v)\Big) = 0 \qquad \text{on } (0,T)\times\mathbb{R}.$$

Remark 3.2.11 *Note that if H is bounded, i.e. $\mathrm{dom}(H) = [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}^n$ and continuous, then condition (A) is satisfied using any strictly positive continuous function $\mathcal{G} : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}^n \to \mathbb{R}$. Taking one such function, and given the assumptions of the previous theorem, one can conclude that in fact the value function is not just a viscosity solution to the HJB variational inequality, but in fact a solution to the HJB equation*

$$-\frac{\partial v}{\partial t} - H(t,x,D_x v, D_{xx} v) = 0$$

itself.

Remark 3.2.12 *Note that the only conditions imposed on the value function in the theorem is local boundedness, where before we needed $v \in C^{1,2}([0,T)\times\mathbb{R}^n)$ just for it to be considered a solution candidate for the HJB equation.*

Up to this point, we neglected entirly the terminal condition since the definition of viscosity solutions does not depend on it. As we intend to establish a link from the value function to the HJB equation (which can only be complete together its terminal condition) one must ask the question in what sense the value function at terminal time satisfies this terminal condition, i.e. in what sense do we have

$$v(T,x) = g(x) \qquad \forall x \in \mathbb{R}^n? \tag{3.18}$$

This question is answered by the following theorem.

Theorem 3.2.13 *Assume f and g are lower bounded or of quadratic growth in x and (A) holds.*

(i) If g is lower semi continuous, then $v_*(T,\cdot)$ is a viscosity supersolution to

$$\min\Big(v_*(T,x) - g, \mathcal{G}(T,x,D_x v(T,\cdot), D_{xx} v(T,\cdot)\Big) = 0 \qquad \text{on } \mathbb{R}^n.$$

(ii) If g is upper semi continuous, then $v^*(T,\cdot)$ is a viscosity subsolution to

$$\min\Big(v^*(T,x) - g, \mathcal{G}(T,x,D_x v(T,\cdot), D_{xx} v(T,\cdot)\Big) = 0 \qquad \text{on } \mathbb{R}^n.$$

3.2.3 From the viscosity solution to the value function via a comparison principle

The previous subsection, more precisely theorem 3.2.10, established under which conditions one can infer that the value function is a viscosity solution to the HJB equation. As in the classical solution case, one intends to invert this implication, that is, one tries to find conditions under which the viscosity solution to the HJB equation coincides with the value function. The main obstacle to this pursuit is that we do not know a priori whether or not viscosity solutions are unique for a given PDE. Even if we were to know that such a PDE had a unique classical solution, our weakening of this concept might well imply there to be several viscosity solutions. The following subsection is thus committed to the establishment of uniqueness results for the HJB equation. The key and main task to this establishment will be the demonstration of a so called comparison principle for the HJB equation.

Definition 3.2.14

(i) We say that a strong comparison principle holds for PDE if the following implication holds: Let v be an upper semi continuous viscosity subsolution to PDE on $\mathcal{O}$ and w be a lower semicontinuous viscosity supersolution to PDE on $\mathcal{O}$ such that $v \le w$ on $\partial\mathcal{O}$. Then one also has $v \le w$ on $\mathcal{O}$.

(ii) We say that a strong comparison principle holds for PDEt if the following implication holds: Let v be an upper semi continuous viscosity subsolution to PDEt on $[0,T)\times\mathcal{O}$ and w be a lower semicontinuous viscosity supersolution to PDEt on $[0,T)\times\mathcal{O}$ such that $v(T,\cdot) \le w(T,\cdot)$ on $\mathcal{O}$. Then one also has $v \le w$ on $[0,T)\times\mathcal{O}$.

Lemma 3.2.15 *Assume v,w are viscosity solution to PDE with the same continuous Dirichlet-boundary condition g, that is $v|_{\partial\mathcal{O}} = w|_{\partial\mathcal{O}} = g$. If PDE admits a strong comparison principle, then $u = v$ and moreover v is continuous on $\mathcal{O}$.*

In exactly the same fashion, one can proof:

Lemma 3.2.16 *Assume v,w are viscosity solution to PDEt with the same continuous terminal condition g, that is $v(T,\cdot) = w(T,\cdot) = g$. If PDEt admits a strong comparison principle, then $u = v$ and moreover v is continuous on $[0,T)\times\mathcal{O}$.*

In other words, we have shown uniqueness and continuity of a viscosity solution to a PDE, provided it admits a comparison principle and some continuity on the boundary or initial value. Combining this with the result of theorem 3.2.10 we can infer that the unique viscosity solution to the HJB equation coincides with the value function, concluding our way back.

The remainder of this subsection will therefore be dedicated to proving that a comparison principle holds for the HJB equation given some additional constraints. Recall in particular that the verification theorem demanded the classical solution to HJB to be of quadratic growth. It will turn out that while we disposed of regularity requirements, we will still need to impose a polynomial growth condition in order to receive a comparison principle, illustrating the close analogy of the roles played by the verification theorem and the comparison principle.

3.2.4 Comparison principles

The proof of a comparison principle for the full second order Hamiltonian will be gradually approached in three steps. A first step will prove a comparison principle for classical solutions, followed by a second step proving a comparison principle for first order Hamiltonians (no dependence on the Hessian) and a final third step, discussing the fully general Hamiltonian.

For reasons that will become clear in the upcoming proofs, we will consider the slightly generalized HJB equation

$$\begin{cases} -\frac{\partial v}{\partial t}(t,x) + \beta v - H(t,x,D_x v, D_x^2 v) & = 0, \quad \text{on } [0,T) \times \mathbb{R}^n, \\ v(T,\cdot) & = g(\cdot), \quad \text{on } \mathbb{R}^n. \end{cases} \tag{3.19}$$

with Hamiltonian

$$H(t,x,p,M) = \sup_{a \in A} \left(b(x,a).p + \frac{1}{2} tr(\sigma(x)\sigma'(x)M) + f(t,x,a) \right)$$

for $(t,x,p,M) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n$, $\beta \in \mathbb{R}$. A standing assumptions will be that b, σ satisfy linear growth conditions in x, uniformly in $a \in A$.

Comparison principle for classical solutions

Theorem 3.2.17 *Let $u, v \in C^{1,2}([0,T) \times \mathbb{R}) \cap C^0([0,T] \times \mathbb{R})$ and u a subsolution and v a supersolution with polynomial growth in x and with*

$$u(T,\cdot) \leq v(T,\cdot) \qquad \text{on } \mathbb{R}.$$

Then there holds

$$u \leq v \qquad \text{on } [0,T] \times \mathbb{R}$$

i.e. a strong comparison principle for classical solutions.

Comparison principle for first order Hamiltonians

The preceding proof of the comparison principle for classical solutions made use of certain conditions of optimality that relied on the fact that classical solutions are smooth. Trying to proof a comparison principle for viscosity solutions will therefore need to require another tool, which will be provided by the "doubling of variables" technique, which will be able to cover HJB equations with first order Hamiltonians that is

$$-\frac{\partial w}{\partial t} - \beta w - H\big(t, x, D_x w(t,x)\big) = 0 \qquad \text{on } [0,T) \times \mathbb{R} \tag{HJB-1}$$

with

$$H(t,x,p) = \sup_a \big(b(x,a)p + f(t,x,a)\big).$$

We assume that

(i) $|b(x,a) - b(y,a)| \leq L|x-y|$ for all $a \in A$,

(ii) $f(t,x,a)$ is uniformly continuous in (t,x) uniformly in $a \in A$.

Notice that due to these assumptions we have

$$\begin{aligned} |H(t,x,p) - H(s,y,p)| &\leq L|p||x-y| + \sup_a |f(t,x,a) - f(s,y,a)|\big) \\ &=: \mu\big(|t-s| + (1+|p|)(x-y)\big) \end{aligned}$$

for some $\mu(\cdot)$ such that $\mu(z) \to 0$ if $z \to 0$. We call this the key inequality.

Proposition 3.2.18 *Let u be an upper semi continuous viscosity subsolution and v be a lower semi continuous viscosity supersolution to (*HJB-1*) with polynomial growth, that is*

$$|u(t,x)| + |v(t,x)| \leq C(1+|x|^p)$$

for some $p \geq 1$. If

$$u(T,\cdot) \leq v(T,\cdot) \qquad \text{on } \mathbb{R}^n,$$

then

$$u \leq v \qquad \text{on} [0,T) \times \mathbb{R}^n$$

that is there holds a comparison principle.

Comparison principle for second order Hamiltonians

The preceding proof of a comparison principle for first order Hamiltonians relied crucially on the key inequality

$$|H(t,x,p) - H(s,y,p)| \leq \mu\big(|t-s| + (1+|p|)(x-y)\big)$$

as well as

$$|t_\varepsilon - s_\varepsilon| = o(\varepsilon) \qquad |x_\varepsilon - y_\varepsilon| = o(\varepsilon) \qquad |p_\varepsilon| = \frac{2}{\varepsilon}(x_\varepsilon - y_\varepsilon) = O(1)$$

which permitted us to complete our proof by contradiction. In the second order setting, we encounter the difficulty to find a bound on the expression

$$tr\Big((\sigma(x_\varepsilon,a)\sigma'(x_\varepsilon,a)(\frac{2}{\varepsilon}\,\mathrm{Id})\Big) - tr\Big((\sigma(y_\varepsilon,a)\sigma'(y_\varepsilon,a)(-\frac{2}{\varepsilon}\,\mathrm{Id})\Big).$$

Such a bound will be provided by Ishii's Lemma, for whose application we will need to introduce the notion of second order super- and subjets. This notion will permit us to characterize viscosity solutions, i.e. we will have a one-to-one identification of the two notions.

Definition 3.2.19 *The second-order superjet of an upper semi continuous function u at $(\bar{t},\overline{x})$ is given by*

$$\mathcal{P}^{2,+}u(\bar{t},\overline{x}) = \Big\{(\overline{q},\overline{p},\overline{M}) \in \mathbb{R}\times\mathbb{R}^n\times\mathcal{S}^n \mid u(t,x) \le u(\bar{t},\overline{x}) + \overline{q}(t-\bar{t}) + \overline{p}(x-\overline{x}) + \frac{1}{2}\overline{M}(x-\overline{x})^2 + o(|t-\bar{t}| + |x-\overline{x}|^2)\Big\}.$$

The second-order subjet of a lower semi continuous function v at $(\bar{t},\overline{x})$ is given by

$$\mathcal{P}^{2,-}v(\bar{t},\overline{x}) = \Big\{(\overline{q},\overline{p},\overline{M}) \in \mathbb{R}\times\mathbb{R}^n\times\mathcal{S}^n \mid v(t,x) \ge v(\bar{t},\overline{x}) + \overline{q}(t-\bar{t}) + \overline{p}(x-\overline{x}) + \frac{1}{2}\overline{M}(x-\overline{x})^2 + o(|t-\bar{t}| + |x-\overline{x}|^2)\Big\}.$$

Lemma 3.2.20 *An upper semi continuous function u is a viscosity subsolution to (3.10) if and only if for all $(t,x) \in [0,T)\times\mathbb{R}$ and for all $(q,p,M) \in \overline{\mathcal{P}}^{2,+}u(t,x)$*

$$-q + \beta w(t,x) - H(t,x,p,M) \le 0.$$

Likewise a lower semi continuous function v is a viscosity supersolution to (3.10) if and only if for all $(t,x) \in [0,T)\times\mathbb{R}$ and for all $(q,p,M) \in \overline{\mathcal{P}}^{2,-}v(t,x)$

$$-q + \beta w(t,x) - H(t,x,p,M) \ge 0.$$

With the notion of second-order super- and subjets at hand, we are ready to state Ishii's Lemma.

Lemma 3.2.21 *(Ishii)*
Let u be an upper semi continuous function and v be a lower semi continuous function on $[0,T)\times\mathbb{R}^n$ and let $\Phi \in C^{1,1,2,2}$ and $(\bar{t},\overline{s},\overline{x},\overline{y})$ be a local maximum of

$$u(t,x) - v(s,y) - \Phi(t,s,x,y).$$

Then for $\eta > 0$ there exists $M, N \in \mathcal{S}^n$ such that

$$\Big(\frac{\partial\Phi}{\partial t}(\bar{t},\overline{s},\overline{x},\overline{y}), D_x\Phi(\bar{t},\overline{s},\overline{x},\overline{y}), M\Big) \in \overline{\mathcal{P}}^{2,+}u(\bar{t},\overline{x})$$

and

$$\Big(\frac{\partial\Phi}{\partial s}(\bar{t},\overline{s},\overline{x},\overline{y}), D_y\Phi(\bar{t},\overline{s},\overline{x},\overline{y}), M\Big) \in \overline{\mathcal{P}}^{2,-}v(\bar{t},\overline{x})$$

and further

$$\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \le D^2_{xy}\Phi(\bar{t},\overline{s},\overline{x},\overline{y}) + \eta\big(D^2_{xy}\Phi(\bar{t},\overline{s},\overline{x},\overline{y})\big)^2.$$

The following corollary will be the key component to expand our proof from the comparison principle for first order Hamlitonian so second order Hamlitonians, circumventing the difficulty presented in the introduction to this subsection.

Corollary 3.2.22 *Let $u, v : [0,T) \times \mathbb{R}^n \to \mathbb{R}$ be upper semi continuous respectivly lower semicontinuous. Take $\phi_\varepsilon(t,s,x,y) := \frac{1}{2\varepsilon}\left((t-s)^2 + (x-y)^2\right) \in C^{1,1,2,2}$ and $(\bar{t},\bar{s},\bar{x},\bar{y})$ a local minimum of the function*

$$u(t,x) - v(s,y) - \phi_\varepsilon(t,s,x,y).$$

Then there exist $M, N \in \mathcal{S}^n$ satisfying

$$\left(\frac{1}{\varepsilon}(\bar{t}-\bar{s}), \frac{1}{\varepsilon}(\bar{x}-\bar{y}), M\right) \in \overline{\mathcal{P}}^{2,+}u(\bar{t},\bar{x})$$

and

$$\left(\left(\frac{1}{\varepsilon}(\bar{t}-\bar{s}), \frac{1}{\varepsilon}(\bar{x}-\bar{y}), N\right) \in \overline{\mathcal{P}}^{2,-}v(\bar{t},\bar{x})$$

and further for all matrices $C, D \in \mathbb{R}^{n\times d}$ there holds

$$tr(CC'M - DD'N) \leq \frac{3}{\varepsilon}|C-D|^2.$$

Theorem 3.2.23 *Comparison principle*
Let u be an upper semi continuous viscosity subsolution and v be a lower semi continuous viscosity supersolution to the second order generalized HJB equation

$$\begin{cases} -\frac{\partial v}{\partial t}(t,x) + \beta v - H(t,x,D_x v, D_x^2 v) & = 0, \quad on\ [0,T) \times \mathbb{R}^n, \\ v(T,\cdot) & = g(\cdot), \quad on\ \mathbb{R}^n. \end{cases} \tag{3.20}$$

with Hamiltonian

$$H(t,x,p,M) = \sup_{a\in A}\left(b(x,a).p + \frac{1}{2}tr(\sigma(x)\sigma'(x)M) + f(t,x,a)\right)$$

Assume both u and v are of polynomial growth in x and such that $u(T,\cdot) \leq v(T,\cdot)$. Then $u \leq v$ on $[0,T) \times \mathbb{R}^n$, i.e. a comparison principle holds.

3.3 Backward Stochastic Equations and Optimal Control

This chapter is an introduction to the theory of BSDEs and its applications. It became now very popular, and is an important field of research due to its connections with stochastic control, mathematical finance and partial differential equations. BSDEs provide a probabilistic representation of nonlinear PDEs. As a consequence, BSDEs can also be used for designing numerical algorithms to nonlinear PDEs.

3.3.1 Existence and uniqueness results

Let $W = (W_t)_{t\in[0,T]}$ be a standard d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the natural filtration on W, and T is a fixed finite horizon.

We denote by $\mathbb{S}^2(0,T)$ the set of real-valued progressively measurable processes Y such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^2\right] < \infty$$

and $\mathbb{H}^2(0,T)^d$ the set of $\mathbb{R}^d$-valued progressively measurable processes Z such that

$$\mathbb{E}\left[\int_0^T |Y_t|^2 dt\right] < \infty.$$

We are given a pair (ξ, f), called the terminal condition and generator (or driver), satisfying:

- (A) $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$
- (B) $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ s.t.
  - $f(\cdot, t, y, z)$ written for simplicity $f(t,y,z)$, is progressively measurable for all y, z
  - $f(t,0,0) \in \mathbb{H}^2(0,T)$
  - f satisfies a uniform Lipschitz condition in (y,z), i.e. there exists a constant C_f such that

$$|f(t,y_1,z_1) - f(t,y_2,z_2)| \le C_f(|y_1 - y_2| + |z_1 - z_2|), \quad \forall y_1, y_2, \forall z_1, z_2, \quad dt \otimes \mathbb{P} \text{ a.e.}$$

We consider the (unidimensional) backward stochastic differential equation (BSDE):

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t \cdot dW_t, \quad Y_T = \xi \tag{3.21}$$

The second equality is called the terminal condition.

Definition 3.3.1 *A solution to the BSDE (3.21) is a pair* $(Y,Z) \in \mathbb{S}^2(0,T) \times \mathbb{H}^2(0,T)^d$ *satisfying*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0,T].$$

We prove an existence and uniqueness result for the above BSDE.

Theorem 3.3.2 *Given a pair* (ξ, f) *satisfying (A) and (B), there exists a unique solution* (Y,Z) *to the BSDE (3.21).*

Linear BSDE

We consider the particular case where the generator f is linear in y and z. The linear BSDE is written in the form

$$-dY_t = (A_t Y_t + Z_t \cdot B_t + C_t)dt - Z_t \cdot dW_t, \quad Y_T = \xi, \tag{3.22}$$

where A and B are bounded progressively measurable processes valued in $\mathbb{R}$ and $\mathbb{R}^d$, and C is a process in $\mathbb{H}^2(0,T)^d$. We can solve this BSDE explicitly.

Proposition 3.3.3 *The unique solution* (Y,Z) *to the linear BSDE (3.22) is given by*

$$\Gamma_t Y_t = \mathbb{E}\left[\Gamma_t \xi + \int_t^T \Gamma_s C_s ds \middle| \mathcal{F}_t\right], \tag{3.23}$$

where Γ *is the adjoint (or dual) process, solution to the linear SDE*

$$d\Gamma_t = \Gamma_t(A_t dt + B_t \cdot dW_t), \quad \Gamma_0 = 1$$

Comparison principle

We state a very useful comparison principle for BSDEs.

Theorem 3.3.4 *Let (ξ^1, f^1) and (ξ^2, f^2) be two pairs of terminal conditions and generators satisfying conditions (A) and (B), let (Y^1, Z^1), (Y^2, Z^2) be the solutions to their corresponding BSDEs. Suppose that:*

- $\xi^1 \leq \xi^2$ *a.s.*
- $f^1(t, Y_t^1, Z_t^1) \leq f^2(t, Y_t^1, Z_t^1)$ $dt \otimes \mathbb{P}$ *a.e.*
- $f^2(t, Y_t^1, Z_t^1) \in \mathbb{H}^2(0, T)$.

Then $Y_t^1 \leq Y_t^2$ for all $t \in [0, T]$, a.s.
Furthermore, if, in addition, $Y_0^1 = Y_0^2$, then $Y_t^1 = Y_t^2$, $t \in [0, T]$ a.s. and thus $Z^1 = Z^2$ a.e. In particular, if $\mathbb{P}(\xi^1 < \xi^2) > 0$ or $f^1(t, Y_t^1, Z_t^1) < f^2(t, Y_t^1, Z_t^1)$ on a set of strictly positive measure $dt \otimes \mathbb{P}$, then $Y_0^1 < Y_0^2$.

BSDE, PDE and nonlinear Feynman-Kac formula

According to the Feynman-Kac formula, the solution to the linear parabolic PDE

$$-\frac{\partial v}{\partial t} - \mathcal{L}v - f(t, x) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^n, \tag{3.24}$$

$$v(T, x) = g(x), \quad x \in \mathbb{R}^n, \tag{3.25}$$

has the representation

$$v(t, x) = \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}) ds + g(X_T^{t,x})\right],$$

where $\mathcal{L}$ is the second-order operator

$$\mathcal{L}v = b(x) \cdot D_x v + \frac{1}{2}\mathrm{tr}(\sigma(x)\sigma'(x) D_{xx}^2 v),$$

and $\{X_s^{t,x}, s \in [t, T]\}$ is the solution to the SDE

$$dX_s = b(X_s)ds + \sigma(X_s)dW_s, \quad s \in [t, T], \quad X_t = x$$

In this section we study an extension of Feynman-Kac formula for semilinear PDE in the form

$$-\frac{\partial v}{\partial t} - \mathcal{L}v - f(t, x, v, \sigma' D_x v) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^n, \tag{3.26}$$

$$v(T, x) = g(x), \quad x \in \mathbb{R}^n, \tag{3.27}$$

with $\mathbb{R}^n$-valued b and $\mathbb{R}^{n \times d}$-valued σ both satisfying a Lipschitz condition on $\mathbb{R}^n$. f is a continuous function on $[0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$ satisfying a linear growth condition in (x, y, z) and a Lipschitz condition in (y, z), uniformly in (t, x). The continuous function g satisfies a linear growth condition. We shall represent the solution to this PDE by means of the BSDE

$$-dY_s = f(s, X_s, Y_s, Z_s)ds - Z_s \cdot dW_s, \quad s \in [0, T], \quad Y_T = g(X_T), \tag{3.28}$$

and the forward SDE valued in $\mathbb{R}^n$:

$$dX_s = b(X_s)ds + \sigma(X_s)dW_s. \tag{3.29}$$

By a standard estimate on the second moment of X, we see that the terminal condition and the generator of the BSDE (3.28) satisfy the conditions (A) and (B) stated in section 3.3.1. Thus there exists a unique solution $(Y,Z) \in \mathbb{S}^2(0,T) \times \mathbb{H}^2(0,T)^d$ satisfying (3.28). By the Markov property of the diffusion X we notice that $Y_t = v(t, X_t)$, where

$$v(t,x) := Y_t^{t,x}$$

is a deterministic function of (t,x) in $[0,T] \times \mathbb{R}^n$, $\{X_s^{t,x}, s \in [t,T]\}$ is the solution to the SDE (3.29) starting from x at t, and $\{(Y_s^{t,x}, Z_s^{t,x}), s \in [t,T]\}$ is the solution to the BSDE (3.28) with $X_s = X_s^{t,x}$, $s \in [0,T]$.[2] We call this framework a Markovian case for the BSDE.

The next result is analogous to the verification theorem for Hamilton-Jacobi-Bellman equations, and shows that a classical solution to the semilinear PDE provides a solution to the BSDE.

Theorem 3.3.5 *Let $v \in C^{1,2}([0,T) \times \mathbb{R}^n) \cap C^0([0,T] \times \mathbb{R}^n)$ be a classical solution to (3.26)-(3.27), satisfying a linear growth condition and such that for some positive constants C, q: $|D_x v(t,x)| \leq C(1+|x|^q)$ for all $x \in \mathbb{R}^n$. Then, the pair (Y,Z) defined by*

$$Y_t = v(t,X_t), \quad Z_t = \sigma'(X_t)D_x v(t,X_t), \quad t \in [0,T],$$

is the solution to the BSDE (3.28).

3.3.2 Control and BSDE

Theorem 3.3.6 *Let (Y,Z), (Y^α, Z^α) be solutions to the BSDEs given by (ξ, f), (ξ^α, f^α), $\alpha \in \mathcal{A}$ subset of progressively measurable processes. Let (ξ,f), (ξ^α, f^α), $\alpha \in \mathcal{A}$ satisfy (A) and (B) in section 3.3.1. Suppose that there exists some $\hat{\alpha} \in \mathcal{A}$ such that*

$$f(t,Y_t,Z_t) = \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}} f^\alpha(t,Y_t,Z_t) = f^{\hat{\alpha}}(t,Y_t,Z_t), \quad dt \otimes \mathbb{P} \ a.e.$$

$$\xi = \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}} \xi^\alpha = \xi^{\hat{\alpha}}.$$

Then $Y_t = \operatorname{ess\,inf}_{\alpha \in \mathcal{A}} Y_t^\alpha = Y_t^{\hat{\alpha}}$, $t \in [0,T]$ a.s.

In Chapter 3.1 we studied how to solve a stochastic control problem by the dynamic programming method. We present here an alternative approach, called Pontryagin maximum principle, and based on optimality conditions for controls.

We consider the framework of a stochastic control problem as defined in Chapter 3.1: let X be a controlled diffusion on $\mathbb{R}^n$ governed by

$$dX_s = b(X_s, \alpha_s) + \sigma(X_s, \alpha_s)dW_s, \tag{3.30}$$

[2] Note that $X_s^{t,x}$ and also $g(X_T^{t,x})$ are independent of $\mathcal{F}_t$, therefore $Y_s^{t,x}, Z_s^{t,x}$ must be independent of $\mathcal{F}_t$ as well, according to the fixed point construction in section 3.3.1. Hence $Y_t^{t,x}$ is deterministic.

where W is a d-dimensional standard Brownian motion, and $\alpha \in \mathcal{A}$, the control process, is a progressively measurable process valued in A. The gain functional to maximize is

$$J(\alpha) = \mathbb{E}\left[\int_0^T f(t, X_t, \alpha_t)dt + g(X_T)\right],$$

where $f : [0,T] \times \mathbb{R}^n \times A \to \mathbb{R}$ is continuous in (t,x) for al a in A, $g : \mathbb{R}^n \to \mathbb{R}$ is a concave C^1 function, and f, g satisfy a quadratic growth condition in x.
We define the generalized Hamiltonian $\mathcal{H} : [0,T] \times \mathbb{R}^n \times A \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ by

$$\mathcal{H}(t,x,a,y,z) = b(x,a)\cdot y + \mathrm{tr}(\sigma'(x,a)z) + f(t,x,a),$$

and we assume that $\mathcal{H}$ is differentiable in x with derivative denoted by $D_x\mathcal{H}$. We consider for each $\alpha \in \mathcal{A}$, the BSDE, called the adjoint equation:

$$-dY_t = D_x\mathcal{H}(t, X_t, \alpha_t, Y_t, Z_t)dt - Z_t dW_t, \qquad Y_T = D_x g(X_T). \tag{3.31}$$

Theorem 3.3.7 *Let $\hat{\alpha} \in \mathcal{A}$ and $\hat{X}$ the associated controlled diffusion. Suppose that there exists a solution $(\hat{Y}, \hat{Z})$ to the associated BSDE (3.31) such that*

$$\mathcal{H}(t, \hat{X}_t, \hat{\alpha}_t, \hat{Y}_t, \hat{Z}_t) = \max_{a\in A} \mathcal{H}(t, \hat{X}_t, a, \hat{Y}_t, \hat{Z}_t), \quad t \in [0,T] \ a.s. \tag{3.32}$$

and

$$(x,a) \longmapsto \mathcal{H}(t, x, a, \hat{Y}_t, \hat{Z}_t) \text{ is a concave function,} \tag{3.33}$$

for all $t \in [0,T]$. Then $\hat{\alpha}$ is an optimal control, i.e.

$$J(\hat{\alpha}) = \sup_{\alpha\in\mathcal{A}} J(\alpha).$$

We conclude this section by providing the connection between maximum principle and dynamic programming. The value function of the stochastic control problem considered above is defined by

$$v(t,x) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, \alpha_s)ds + g(X_T^{t,x})\right] \tag{3.34}$$

where $\{X_s^{t,x},\ s \in [t,T]\}$ is the solution to (3.30) starting from x at t. Recall that the associated Hamilton-Jacobi-Bellman equation is

$$-\frac{\partial v}{\partial t} - \sup_{a\in A}[\mathcal{G}(t,x,a,D_x v, D_x^2 v)] = 0, \tag{3.35}$$

where for $(t,x,a,p,M) \in [0,T] \times \mathbb{R}^n \times A \times \mathbb{R}^n \times \mathcal{S}_n$,

$$\mathcal{G}(t,x,a,p,M) = b(x,a)\cdot p + \frac{1}{2}\mathrm{tr}(\sigma\sigma'(x,a)M) + f(t,x,a).$$

Theorem 3.3.8 *Suppose that $v \in C^{1,3}([0,T) \times \mathbb{R}^n) \cap C^0([0,T] \times \mathbb{R}^n)$, and there exists an optimal control $\hat{\alpha} \in \mathcal{A}$ to (3.34) with associated controlled diffusion $\hat{X}$. Then*

$$\mathcal{G}(t, \hat{X}_t, \hat{\alpha}_t, D_x v(t,\hat{X}_t), D_x^2 v(t,\hat{X}_t)) = \max_{a\in A} \mathcal{G}(t, \hat{X}_t, a, D_x v(t,\hat{X}_t), D_x^2 v(t,\hat{X}_t)), \tag{3.36}$$

and the pair

$$(\hat{Y}_t, \hat{Z}_t) = (D_x v(t,\hat{X}_t), D_x^2 v(t,\hat{X}_t)\sigma(\hat{X}_t, \hat{\alpha}_t)), \tag{3.37}$$

is the solution to the adjoint BSDE (3.31).

3.3.3 Application: Portfolio optimization

Exponential utility maximization with option payoff

We consider a financial market with one riskless asset of price $S^0 = 1$ and one risky asset of price process

$$dS_t = S_t(b_t dt + \sigma_t dW_t),$$

where W is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t), \mathbb{P})$ equipped with the natural filtration $\mathbb{F}$ of W, b and σ are two bounded progressively measurable processes, $\sigma_t \geq \varepsilon$, for all t, a.s. with $\varepsilon \geq 0$. An agent, starting from a capital x, invests an amount α_t at any time t in the risky asset. His wealth process, controlled by α, is given by

$$X_t^{x,\alpha} = x + \int_0^t \alpha_u \frac{dS_u}{S_u} = x + \int_0^t \alpha_u (b_u du + \sigma_u dW_u).$$

We denote by $\mathcal{A}$ the set of progressively measurable processes α valued in $\mathbb{R}$, such that $\int_0^T |\alpha_t|^2 dt < \infty$ a.s. and $X^{x,\alpha}$ is lower-bounded. The agent must provide at maturity T an option payoff represented by a bounded random variable ξ $\mathcal{F}_T$-measurable. Given his risk aversion characterized by an exponential utility

$$U(x) = -\exp(-\eta x), \quad x \in \mathbb{R}, \quad \eta > 0,$$

the objective of the agent is to solve the maximization problem:

$$v(x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}[U(X_t^{x,\alpha} - \xi)]. \tag{3.38}$$

The approach adopted here for determining the value function v and the optimal control $\hat{\alpha}$ is quite general, and is based on the following argument. We construct a family of processes $(J_t^\alpha)_{t \in [0,T]}$, $\alpha \in \mathcal{A}$, satisfying the properties:

(i) $J_T^\alpha = U(X_T^{x,\alpha} - \xi)$ for all $\alpha \in \mathcal{A}$

(ii) J_0^α is a constant independent of $\alpha \in \mathcal{A}$

(iii) J^α is a supermartingale for all $\alpha \in \mathcal{A}$, and there exists an $\hat{\alpha} \in \mathcal{A}$ such that $J^{\hat{\alpha}}$ is a martingale

Indeed, in this case, for such $\hat{\alpha}$, we have for any $\alpha \in \mathcal{A}$,

$$\mathbb{E}[U(X_t^{x,\alpha} - \xi)] = \mathbb{E}[J_T^\alpha] \leq J_0^\alpha = J_0^{\hat{\alpha}} = \mathbb{E}[J_T^{\hat{\alpha}}] = \mathbb{E}[U(X_T^{x,\hat{\alpha}} - \xi)]$$

which proves that $\hat{\alpha}$ is an optimal control and $v(x) = J_0^{\hat{\alpha}}$.
We construct such a family (J^α) in the form

$$J_t^\alpha = U(X_t^{x,\alpha} - Y_t), \quad t \in [0,T], \quad \alpha \in \mathcal{A},$$

with (Y, Z) solution to the BSDE

$$Y_t = \xi + \int_t^T f(s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0,T],$$

where f is a generator to be determined. The conditions (i) and (ii) are clearly satisfied. In order to satisfy condition (iii), we shall exploit the particular structure of the exponential utility function U. Indeed, by substituting the definitions of $X^{x,\alpha}$ and Y into $U(X_t^{x,\alpha} - Y_t)$ we obtain:

$$J_t^\alpha = -\exp(-\eta(X_t^{x,\alpha} - Y_t)) = M_t^\alpha C_t^\alpha,$$

where M^α is the local martingale given by

$$M_t^\alpha = \exp(-\eta(x - Y_0)) \exp\left(-\int_0^t \eta(\alpha_u\sigma_u - Z_u)dW_u - \frac{1}{2}\int_0^t |\eta(\alpha_u\sigma_u - Z_u)|^2 du)\right),$$

and

$$C_t^\alpha = -\exp\left(\int_0^t \rho(u, \alpha_u, Z_u)du\right),$$

with

$$\rho(t, a, z) = \eta\left(\frac{\eta}{2}|a\sigma_t - z|^2 - ab_t - f(t, z)\right).$$

We are then looking for a generator f such that the process (C_t^α) is nonincreasing for all $\alpha \in \mathcal{A}$, and constant for some $\hat{\alpha} \in \mathcal{A}$. In other words, the problem is reduced to finding f such that

$$\rho(t, \alpha_t, Z_t) \geq 0, \quad t \in [0, T]$$

for all $\alpha \in \mathcal{A}$ and

$$\rho(t, \hat{\alpha}_t, Z_t) = 0, \quad t \in [0, T].$$

These two conditions are satisfied for

$$f(t, z) := -z\frac{b_t}{\sigma_t} - \frac{1}{2\eta}\left|\frac{b_t}{\sigma_t}\right|^2, \tag{3.39}$$

and

$$\hat{\alpha}_t := \frac{1}{\sigma_t}\left(Z_t + \frac{1}{\eta}\frac{b_t}{\sigma_t}\right), \quad t \in [0, T], \tag{3.40}$$

which can be seen by rewriting ρ in the form

$$\frac{1}{\eta}\rho(t, a, z) = \frac{\eta}{2}\left|a\sigma_t - z - \frac{1}{\eta}\frac{b_t}{\sigma_t}\right|^2 - z\frac{b_t}{\sigma_t} - \frac{1}{2\eta}\left|\frac{b_t}{\sigma_t}\right|^2 - f(t, z).$$

Theorem 3.3.9 *The value function to problem (3.38) is equal to*

$$v(x) = U(x - Y_0) = -\exp(-\eta(x - Y_0)),$$

where (Y, Z) is the solution to the BSDE

$$Y_t = \xi + \int_t^T f(s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T],$$

with a generator f given by (3.39). Furthermore, an optimal control $\hat{\alpha}$ is given by (3.40).

Mean-variance criterion for portfolio selection

We consider a Black-Scholes financial model. There is one riskless asset of price process

$$dS_t^0 = rS_t^0 dt,$$

and one stock of price process

$$dS_t = S_t(bdt + \sigma dW_t),$$

with constants $b > r$ and $\sigma > 0$. An agent invests at any time t an amount α_t in the stock, and his wealth process is governed by

$$dX_t = \alpha_t \frac{dS_t}{S_t} + (X_t - \alpha_t)\frac{dS_t^0}{S_t^0} = [rX_t + \alpha_t(b-r)]dt + \sigma\alpha_t dW_t, \quad X_0 = x.$$

We denote by $\mathcal{A}$ the set of progressively measurable processes α valued in $\mathbb{R}$, such that

$$\mathbb{E}\left[\int_0^T |\alpha_t|^2 dt\right] < \infty.$$

The mean-variance criterion for portfolio selection consists in minimizing the variance of the wealth under the constraint that its expectation is equal to a given constant:

$$V(m) = \inf_{\alpha\in\mathcal{A}} \{\mathrm{Var}(X_T) : \mathbb{E}[X_T] = m\}, \quad m \in \mathbb{R}.$$

We shall see later, by the Lagrangian method, that this problem is reduced to the resolution of an auxiliary control problem

$$\tilde{V}(\lambda) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}[(X_T - \lambda)^2], \quad \lambda \in \mathbb{R}.$$

We shall solve this auxiliary problem by the stochastic maximum principle described in section 3.3.2. In this case, the Hamiltonian $\mathcal{H}$ takes the form:

$$\mathcal{H}(x, a, y, z) = [rx + a(b-r)]y + \sigma a z.$$

The adjoint BSDE (3.31) is written for any $\alpha \in \mathcal{A}$ as

$$-dY_t = rY_t dt - Z_t dW_t, \quad Y_T = 2(X_T - \lambda).$$

Let $\hat{\alpha} \in \mathcal{A}$ a candidate for the optimal control, and $\hat{X}$, $(\hat{Y}, \hat{Z})$ the corresponding processes. Then,

$$\mathcal{H}(x, a, \hat{Y}_t, \hat{Z}_t) = rx\hat{Y}_t + a[(b-r)\hat{Y}_t + \sigma\hat{Z}_t].$$

Since this expression is linear in a, we see that conditions (3.32) and (3.33) will be satisfied iff

$$(b-r)\hat{Y}_t + \sigma\hat{Z}_t = 0, \quad t \in [0,T] \text{ a.s.}$$

We are looking for the $(\hat{Y}, \hat{Z})$ solution to the adjoint BSDE in the form

$$\hat{Y}_t = \varphi(t)\hat{X}_t + \psi(t),$$

for some deterministic C^1 functions φ and ψ. By substituting into the adjoint BSDE and using the definition of X we observe through comparison of the finite variation and martingale parts:

$$\varphi'(t)\hat{X}_t + \varphi(t)(r\hat{X}_t + \hat{\alpha}_t(b-r)) + \psi'(t) = -r(\varphi(t)\hat{X}_t + \psi(t)), \tag{3.41}$$

$$\varphi(t)\sigma\hat{\alpha}_t = \hat{Z}_t,$$

We also have terminal conditions

$$\varphi(T) = 2, \quad \psi(T) = -2\lambda.$$

By using $(b-r)\hat{Y}_t + \sigma\hat{Z}_t = 0$ and $\varphi(t)\sigma\hat{\alpha}_t = \hat{Z}_t$ we have

$$\hat{\alpha}_t = \frac{(r-b)\hat{Y}_t}{\sigma^2\varphi(t)} = \frac{(r-b)(\varphi(t)\hat{X}_t + \psi(t))}{\sigma^2\varphi(t)}.$$

On the other hand, using (3.41), we also have

$$\hat{\alpha}_t = \frac{(\varphi'(t) + 2r\varphi(t))\hat{X}_t + \psi'(t) + r\psi(t)}{(r-b)\varphi(t)}.$$

By comparing the two expressions for $\hat{\alpha}$ we obtain ODEs

$$\varphi'(t) + \left(2r - \frac{(b-r)^2}{\sigma^2}\right)\varphi(t) = 0, \quad \varphi(T) = 2$$

$$\psi'(t) + \left(r - \frac{(b-r)^2}{\sigma^2}\right)\psi(t) = 0, \quad \psi(T) = -2\lambda,$$

whose explicit solutions are

$$\varphi(t) = 2\exp\left(\left(2r - \frac{(b-r)^2}{\sigma^2}\right)(T-t)\right),$$

$$\psi(t) = -2\lambda\exp\left(\left(r - \frac{(b-r)^2}{\sigma^2}\right)(T-t)\right).$$

With this choice of φ and ψ the corresponding processes $(\hat{Y}, \hat{Z})$ solve the adjoint BSDE and the conditions for the maximum principle in Theorem 3.3.7 are satisfied. We therefore have an optimal control given by

$$\hat{\alpha}_\lambda(t,x) = \frac{(r-b)(\varphi(t)x + \psi(t))}{\sigma^2\varphi(t)},$$

written in the Markovian form.

The value function $\tilde{V}(\lambda) = \mathbb{E}[(X_T - \lambda)^2]$ with X given by control $\hat{\alpha}_\lambda$ can be calculated explicitely using Itô's formula and the definition of $\hat{\alpha}_\lambda$:

$$\tilde{V}(\lambda) = \exp\left(-\frac{(b-r)^2}{\sigma^2}T\right)(\lambda - e^{rT}x)^2, \quad \lambda \in \mathbb{R}.$$

Finally we show how the two optimization problems associated with V and $\tilde{V}$ are related.

Proposition 3.3.10 *We have the two conjugate relations*

$$\tilde{V}(\lambda) = \inf_{m \in \mathbb{R}} [V(m) + (m - \lambda)^2], \quad \lambda \in \mathbb{R}, \tag{3.42}$$

$$V(m) = \sup_{\lambda \in \mathbb{R}} [\tilde{V}(\lambda) - (m - \lambda)^2], \quad m \in \mathbb{R}. \tag{3.43}$$

For any $m \in \mathbb{R}$, the optimal contol of $V(m)$ is equal to $\hat{\alpha}_{\lambda_m}$ where λ attains the maximum in (3.43), i.e.

$$\lambda_m = \frac{m - x \exp\left(\left(r - \frac{(b-r)^2}{\sigma^2}\right) T\right)}{1 - \exp(-\frac{(b-r)^2}{\sigma^2} T)}. \tag{3.44}$$

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