Optimal Order Display in Limit Order Markets with Liquidity Competition[*]

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Abstract

Order display is associated with benefits and costs. Benefits arise from increased execution-priority, while costs are due to adverse market impact. We analyze a structural model of optimal order placement that captures trade-off between costs and benefits of order display. For a benchmark model of pure liquidity competition we give closed-form solution for optimal display sizes. We show that liquidity competition incentivizes the use of hidden orders to prevent losses due to over-bidding. On the other hand, we predict that the use of hidden orders is more prevalent in stocks with low depth. Our theoretical considerations are accompanied by an empirical analysis using high-frequency order-message data from NASDAQ.

JEL Classification Codes: C51, C60, D01, D4, G1

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1 Introduction

The use of hidden liquidity among the major stock exchanges has considerably increased in the recent years. Nowadays, hidden orders, Iceberg orders or so called reserve orders have become prevalent features of modern electronic markets.[2] Exchanges still require openly displayed quotes to effectively organize trade, though. By giving displayed orders higher execution priority than hidden orders most exchanges encourage market participants to openly display their orders. At the same time, order exposure is associated with risk; adverse selection, quote-matching and front-running can lead to increased transaction costs when traders expose their trade intentions. Market participants, therefore, need to balance the costs and benefits of order exposure when making trading decisions.

In this paper we analyze a structural model of optimal order display in a limit order book which lends itself to both, a theoretical and empirical analysis. The model captures the trade-off between benefits and costs of order display and provides implications for optimal display strategies. We show that the intensity of liquidity competition, market depths relative to market order arrivals, and order sizes are key determinants of optimal display ratios. Our analysis suggests that hidden orders are more beneficial in liquid markets, due to increased competition in liquidity provision.

Specifically, we consider a broker trade problem of executing a limit order at minimal cost. Brokers typically face a time constraint within which to execute a client's position and execution performance is benchmarked against a pre-agreed reference price. We consider the case of trade execution of a single limit order when the broker/trader has the option to shield any fraction of the order from public view. Order execution is governed by arrivals of market and limit orders as well as cancellations of standing volume. Incoming order flows determine the execution volume at terminal time. Due to a liquidation constraint, un-executed orders are cancelled and traded against the best prevailing opposite price.

Biais et al. (1995), Ranaldo (2004) Griffiths et al. (2000), among others, find that the visible order book and thus order display affects trading dynamics as market participants observe changes in the order book and adapt their trading strategies accordingly. Our key assumption is, therefore, that the trader's display decision affects both, the dynamics of order flow at the same side of the market and opposite-side price dynamics. For a benchmark model of pure liquidity competition where order display only affects the supply side of liquidity we provide an explicit characterization of optimal display strategies. The assumption that order display primarily affects liquidity supply

[2] A growing body of empirical studies indicate the wide-spread use of hidden orders. For instance, Pascual Gasco and Veredas (2008) report that 26% of all trades on the Spanish Stock Exchanges involve hidden volume. Frey and Sandas (2009) report that 9.3% of *submitted* and 15.9% of *executed* shares contain Iceberg orders on the German Xetra Stock Exchange.[3] Further studies confirm that hidden liquidity is particularly prevalent among large investors: D'Hondt et al. (2004) report that 81% of orders with total sizes in the largest quartile are Icebergs or (partly) hidden orders. Supplementing this findings, Frey and Sandas (2009) find that Iceberg orders are on average 12-20 times larger than limit orders.

is motivated by the fact that liquidity supply accounts for most of the trading activities in limit order markets. Hasbrouck and Saar (2009) attribute more than 90% of trading activity to liquidity provision and only a marginal portion to liquidity demand. Our benchmark model suggests that traders should increasingly hide their orders when the intensity of liquidity competition is high. We show that this is particularly relevant for liquid stocks that are traded at high prices and have a low (relative) tick size.

Using high-frequency order-message ITCH data from NASDAQ we estimate model parameters and derive optimal display strategies for a range of stocks from the S&P 500 for the period of January to April 2011 for different market and trade settings. Complementing previous literature on hidden orders including Harris (1997) and Bessembinder et al. (2009) we show that partially hiding orders can lead to significant cost savings over full-display strategies. Cost savings are most significant when order book depths at the submission price level are low. Partial display may lead to significant performance enhancements even for small orders. This contrasts earlier empirical findings that suggests that hidden orders are more prevalent among large orders.

This paper contributes to the optimal liquidation literature and the literature on market impact and hidden liquidity. The literature on optimal liquidation, pioneered by the work of Bertsimas and Lo (1998) and Almgren and Chriss (1999), follows a qualitative approach to modeling trading and liquidation problems in illiquid markets. The analysis is typically confined to liquidation strategies within a stylized order book model using active (market) orders only. This restriction is mostly owned to the fact that the market impact of active orders is comparably easy to model. When passive (limit) orders are considered, the analysis is usually confined to dark pool orders which induce little or no market impact.[4] An exception is Esser and Mönch (2007) who allow for market impact of passive orders on prices, but not liquidity competition. Complementing previous work on the market impact of limit orders of Hautsch and Huang (2012) we show that limit order placements primarily affect the supply side of liquidity through an increase of liquidity competition.

The theoretical literature on hidden orders is quite sparse. Buti and Rindi (2013) and Moinas (2010) study equilibrium models of optimal order placement. Our approach distinguishes in several ways. Most importantly, we do not assume that information is asymmetric; following the optimal liquidation literature we consider purely liquidity driven trades. In fact, Madhavan et al. (1997) and Huang and Stoll (1997) argue that trading frictions do not derive from informational asymmetries alone. Second, our structural approach to modeling the impact of visible orders is general and flexible enough to account for a wide range of market impact scenarios.[5] Finally, we link display

[4]There is by now a significant literature on optimal portfolio liquidation using market orders including Almgren and Chriss (1999); Almgren (2003, 2001); Obizhaeva and Wang (2013); Alfonsi, Fruth, and Schied (2010); Horst and Naujokat (2014). Models with market and dark pools orders are analyzed in, e.g. Kratz and Schöneborn (2014); Horst and Naujokat (2014); Graewe, Horst, and Qiu (2014).

[5]For instance, Buti and Rindi (2013) do not account for effects on the demand side of liquidity and Moinas (2010) does not capture effects on the supply side of liquidity. Esser and Mönch (2007) assume market impact only on prices, but not on order flows.

decision to liquidity characteristics. This is particularly important for understanding the origination of hidden liquidity in limit order markets. Our findings show that non-informational mechanisms suffice to rationalize the presence of hidden orders. In that respect, our approach is similar to that proposed in Hollifield et al. (2006). However, while their focus is the trade-off between price and execution risk our focus is on the trade-off between displaying and hiding trade intentions.

The remainder of this paper is structured as follows. In Section 2, we introduce the model, including the order flow and price dynamics, and derive an explicit representation of optimal display ratios. In Section 3, we estimate the model parameters and provide estimates for the optimal exposure size for various stocks and market settings. Section 4 concludes.

2 The Model

Following the optimal liquidation literature we consider a trader ("she") who trades for liquidity reasons. The trader aims to buy a fixed (large) position of N shares over a (short) trading period $[0, T]$. Her reference price is the prevailing best bid price (B_0) at which she submits her order. The trader can choose to openly display any number $\Delta \in [0, N]$ of shares in the order book. The remaining $N - \Delta$ shares are shielded from public view and remain hidden until execution or cancellation. A random number Z^Δ of shares is executed before the end of the trading period. In order to enforce full liquidation the un-executed part $N - Z^\Delta$ of the order is cancelled at the terminal time and executed against standing sell limit orders at the then prevailing best ask price A_T^Δ. For simplicity, we assume that market orders incur no transaction costs. The dependence of the execution volume Z^Δ and best ask price A_T^Δ on the display size Δ accounts for the possible *impact* visible orders have on the dynamics of the order book.

The *absolute* transaction costs are then given by

$$\widetilde{P}^\Delta := Z^\Delta B_0 + \left(N - Z^\Delta\right) A_T^\Delta.$$

We define the relative execution price P^Δ as the relative ratio between the absolute transaction price and the best available bid price at submission time. That is,

$$P^\Delta := \frac{\widetilde{P}^\Delta - NB_0}{NB_0} = \left(1 - \frac{Z^\Delta}{N}\right) S_T^\Delta \tag{2.1}$$

where $S_T^\Delta := (A_T^\Delta - B_0)/B_0$ denotes the *effective spread*. The term $(1 - \frac{Z^\Delta}{N})$ represents the unexecuted proportion of the order while S_T^Δ represents the additional costs of trading relative to the initial best bid B_0.

For simplicity we assume that the two sides of the market are conditionally given Δ. More precisely, we shall assume that order flow dynamics and hence market impact depend only/materializes only on/through imbalances in standing volumes at the top of the book. We believe that this a reasonable assumption on short time scales which considerably simplifies our analysis. For a fixed display size Δ, the expected relative execution price then reads

$$W(\Delta) := \mathbb{E}[P^\Delta] = \Big(1 - \frac{\mathbb{E}[Z^\Delta]}{N}\Big) \cdot \mu(\Delta), \tag{2.2}$$

where $\mu(\Delta) := \mathbb{E}[S_T^\Delta]$ denotes the conditional expected effective spread at the terminal time, given Δ. We denote the expected execution volume by $V := \mathbb{E}[Z^\Delta]$. The trader's problem is to find the display size Δ^* that minimises the expected relative execution costs, i.e., the implementation shortfall (cf. Perold (1988)).

Definition 1 (Optimal Display)**.** *The optimal display size Δ^* is defined as*

$$\Delta^* := \arg\min_{0 \le \Delta \le N} \{W(\Delta)\}. \tag{2.3}$$

2.1 Order Arrival and Trade Dynamics

In our model, order execution is determined by incoming order flow. Sell market orders execute against standing buy limit orders and improve the chance of execution; incoming buy limit orders add liquidity to the same side of the book and hence impede the chance of execution. Modeling the full dynamics of individual order arrivals and cancellations would render the analysis of our model too complex. To enhance tractability, we use a reduced-form model of aggregate order flow consolidating order flows into single submissions. This reduces our model to a 4-stage model: first, the trader submits her order, then aggregate limit orders arrive (or cancel), followed by aggregate market order arrivals. Finally, the trader cancels all unexecuted orders and resubmits them as market orders to guarantee full execution.

We assume that orders at the same side of the market arrive independently of the stock price[6]. Aggregate market order volume is denoted $x \ge 0$, aggregate limit order volume at the submission price level is denoted $y \ge 0$ and aggregate limit order volume at more competitive price levels is denoted $\hat{y} \ge 0$.

Execution of standing limit orders is settled according to a set of priority rules. We follow the standard rule of order-driven markets where orders submitted at more competitive prices have priority over orders submitted at less competitive prices, displayed orders have priority over hidden orders at the same price level and orders at the same price level and with the same display status are served according to the time of arrival.

At the time of order submission our trader faces a depth of $D = D_{bid}$ of visible shares at the submission price level. These orders have higher time-priority than the trader's submission. Assuming a cancellation proportion $c \in [0, 1]$ before market order arrival, the volume that has higher execution-priority than the trader's *displayed* order is

$$Q^d := D(1 - c) + \hat{y}. \tag{2.4}$$

The quantity $\hat{y}$ reflects aggressive limit orders that improved the trader's (submission-)

[6] Again, our main assumption will be that the market dynamics depend on order imbalances only.

price level; they also have higher execution priority than the trader's displayed order. The volume that has higher priority than the trader's *hidden* order is

$$Q^h := Q^d + \Delta + y$$

as the visible volume $\Delta + y$ has display priority. The trader's total execution volume Z^Δ can then be represented in terms of the observable quantities D, Δ, N, the cancellation ratio c and the random (unobservable) quantities y, $\hat{y}$, and x as:

$$Z^\Delta = \begin{cases} 0 & x \le Q^d, \\ x - Q^d & Q^d < x \le \Delta + Q^d, \\ \Delta & \Delta + Q^d < x \le Q^h, \\ \Delta + x - Q^h & Q^h < x \le Q^h + N - \Delta, \\ \Delta + (N - \Delta) & Q^h + N - \Delta < x. \end{cases} \tag{2.5}$$

On high-frequency time scales, (aggregate) order flow volumes are known to have a positive probabilistic mass at zero, i.e., there is a significant non-zero probability that no orders arrive over short horizons (see e.g., Hautsch et al. (2013)). This is particularly relevant for less actively traded and less liquid stocks. To account for the possibility of non-arrivals and while keeping the model simple we propose a zero-augmented exponential distribution for the flow variables x, y and $\hat{y}$. The respective densities read:

$$f_y(s) = (1-q) \cdot \mathbf{1}_{\{s=0\}} + \tfrac{q}{\beta} \cdot e^{-\frac{s}{\beta}} \cdot \mathbf{1}_{\{s>0\}} \ , \tag{2.6}$$

$$f_{\hat{y}}(t) = (1-\hat{q}) \cdot \mathbf{1}_{\{t=0\}} + \tfrac{\hat{q}}{\hat{\beta}} \cdot e^{-\frac{t}{\hat{\beta}}} \cdot \mathbf{1}_{\{t>0\}} \ , \tag{2.7}$$

$$f_x(u) = (1-p) \cdot \mathbf{1}_{\{u=0\}} + \tfrac{p}{\alpha} \cdot e^{-\frac{u}{\alpha}} \cdot \mathbf{1}_{\{u>0\}} \ , \tag{2.8}$$

where $\mathbf{1}$ denotes the indicator function. The parameters $\alpha, \beta, \hat{\beta}$ are assumed non-negative throughout and $p, q, \hat{q} \in [0,1]$. Assuming that y, $\hat{y}$, and x are conditionally independent given Δ, the expected execution volume reads

$$V := \mathbb{E}[Z^\Delta] = \int_0^\infty \int_0^\infty \int_0^\infty Z^\Delta f_y(s) f_{\hat{y}}(t) f_x(u) ds dt du. \tag{2.9}$$

With our choice of arrival dynamics, the expected transaction volume can be given in closed form. The proof is standard, yet tedious, and hence omitted.

Proposition 1 (Expected Execution Volume)**.** *Let* $p \cdot \alpha > 0$ *and* $N > 0$*. Then*

$$V = \alpha p (1 - \hat{\beta}_r) e^{-\frac{D_{bid}(1-c)}{\alpha}} \left\{ (1 - \beta_r) \left(e^{-\frac{\Delta}{\alpha}} - e^{-\frac{N}{\alpha}} \right) + \left(1 - e^{-\frac{\Delta}{\alpha}} \right) \right\}, \tag{2.10}$$

with

$$\hat{\beta}_r := \hat{q} \frac{\hat{\beta}}{\alpha + \hat{\beta}}, \qquad \beta_r := q \frac{\beta}{\alpha + \beta}. \tag{2.11}$$

The first term in the curly brackets in (2.10) corresponds to the execution of the hidden part of the order. It depends on the parameters characterizing submission-level liquidity

(β) relative to the market order volume (α), the total order size (N), and the display ratio relative to the expected market order volume. The term $(1-\beta_r)$ reflect the loss in time-priority due to incoming visible orders at the submission price level, respectively. The second term $(1 - e^{-\frac{\Delta}{\alpha}})$ corresponds to the execution of the visible part; it only depends on Δ.

2.2 Market Impact and Optimal Display Strategies

It is known that visible changes in order-imbalances at the top of the book affect market dynamics, especially incoming order flows and prices movements.[7] To accommodate this feature, we assume that the parameters $\beta, \hat{\beta}, p, \alpha$ and μ governing order flow and price movements are functions of the order imbalance

$$I := I(\Delta) = D_{bid} - D_{ask} + \Delta, \qquad \Delta \in [0, N] \tag{2.12}$$

where D_{bid} and D_{ask} denote the visible standing volume at the best bid and ask respectively by the time our trader submits her order. Positive (negative) values of I represent bid-side (sell-side) excess-liquidity. Due to the dependence of flow and price parameters on order imbalances, transaction costs depend on the trader's display strategy both directly through losses in time priority and indirectly through its impact on order flow and price dynamics. To capture both dependencies we express W as:

$$W(\Delta, I(\Delta)) := W\Big(\Delta, p(I(\Delta)), \alpha\big(I(\Delta)\big), \beta\big(I(\Delta)\big), \hat{\beta}\big(I(\Delta)\big), \mu\big(I(\Delta)\big).\Big), \tag{2.13}$$

The following provides sensitivity analysis of W with respect to the various model parameters. The proof is standard and hence omitted.

Lemma 1. *The partial derivatives satisfy:*

$$\frac{\partial W}{\partial \Delta} < 0, \quad \frac{\partial W}{\partial \hat{\beta}} > 0, \quad \frac{\partial W}{\partial \beta} > 0, \quad \frac{\partial W}{\partial \alpha} < 0, \quad \frac{\partial W}{\partial p} < 0.$$

2.2.1 Optimal Order Display: Priority-gain vs Market Impact

Taking the total derivative of (2.13) shows that the impact of infinitesimal changes in the display size can be decomposed into two "market microstructure terms":

$$\begin{aligned}\frac{d}{d\Delta} W &= \underbrace{\left(\frac{\partial p}{\partial I}\frac{\partial}{\partial p} + \frac{\partial \alpha}{\partial I}\frac{\partial}{\partial \alpha} + \frac{\partial \beta}{\partial I}\frac{\partial}{\partial \beta} + \frac{\partial \hat{\beta}}{\partial I}\frac{\partial}{\partial \hat{\beta}} + \frac{\partial \mu}{\partial I}\frac{\partial}{\partial \mu}\right) W}_{=: M_{Impact}} - \underbrace{\left(-\frac{\partial W}{\partial \Delta}\right)}_{=: M_{Priority} > 0} \\ &= M_{Impact} - M_{Priority},\end{aligned}$$

where we used the fact that $I'(\Delta) = 1$. The term $M_{Priority}$ captures the differential gain in execution priority due to differential increases in the trader's display size Δ. It is strictly negative (see Lemma 1) as displayed orders gain time-priority over incoming

[7] See for instance, Ranaldo (2004); Cao et al. (2009) and Esser and Mönch (2007)

orders at the same price level. The term M_{Impact} captures the impact on trading cost through changes in other traders' submissions as a response to changes in the order display Δ.

Whether market impact reduces or increases transaction costs depends on how the market variables react to changes in volume imbalances. For instance, in view of (2.2.1) and Lemma 1, differential increases in liquidity competition (i.e., $\frac{\partial \hat{\beta}}{\partial \Delta} > 0$ and $\frac{\partial \beta}{\partial \Delta} > 0$), together with decreases in liquidity demand (i.e., $\frac{\partial \mu}{\partial \Delta} < 0$) and adverse movements of prices (i.e., $\frac{\partial \mu}{\partial \Delta} > 0$), materialize in increased transaction costs when traders increase the display size. In fact, the empirical and theoretic literature has provided substantial evidence that markets react in exactly this way: Harris (1997) and Buti and Rindi (2013) suggest that order display causes greater liquidity competition, i.e., $\frac{\partial \hat{\beta}}{\partial \Delta} > 0$ and $\frac{\partial \beta}{\partial \Delta} > 0$, Moinas (2010) predicts that order display reduces liquidity demand, i.e., $\frac{\partial \alpha}{\partial \Delta} < 0$. Hautsch and Huang (2012) find evidence that limit order submissions move prices away from the submission price level, i.e., $\frac{\partial \mu}{\partial \Delta} > 0$.

The optimal display size marks a trade-off between the cost of market impact and the gain in execution priority. More formally, if the mapping $\Delta \mapsto W(\Delta)$ is strictly (quasi-) convex, then the unique optimal display size $\Delta^* \in (0, N)$ is characterised by the first order condition $\frac{d}{d\Delta} W = 0$, or equivalently

$$M_{\text{Market}}(\Delta^*) = M_{\text{Priority}}(\Delta^*). \tag{2.14}$$

2.2.2 A Reduced Model of Pure Liquidity Competition

In the absence of market impact ($M_{Impact} = 0$), fully display is clearly optimal. Absence of market impact is an unrealistic assumption, though. Due to the highly non-linear nature of (2.2.1), explicit solutions for the optimal display size under market impact will not be available in general. In this section, we analyze a reduced model of pure liquidity competition within wich a closed form solution can indeed be obtained. Our key assumption is to allow only limit order competition at more aggressive price levels to depend on order imbalances, while leaving all other parameters constant. In this case the differential (2.2.1) simplifies to

$$\frac{d}{d\Delta} W = \frac{\partial \hat{\beta}}{\partial \Delta} \frac{\partial}{\partial \hat{\beta}} W - \left(- \frac{\partial W}{\partial \Delta} \right). \tag{2.15}$$

There are several economic reasons to single out limit order competition. First, it is known that limit order submission activity captures most of the dynamics in a limit book market.[8] Second, the microstructure literature has shown that limit order competition often takes place through "quote-matching" and "front-running" strategies which aggressively overbid openly displayed orders; see Harris (1997) for details. We

[8]For instance, Biais et al. (1995) show that only 4% of order activity is associated with market orders, while more than 90% are associated with limit orders.

further assume that liquidity competition reacts linearly to order display, i.e.,

$$\hat{\beta}(\Delta) = \hat{\beta}_0 + \hat{\beta}_1 \Delta, \tag{2.16}$$

where $\hat{\beta}_1$ measures the aggressiveness with which liquidity competitors overbid displayed orders. Harris (1997) suggests that $\hat{\beta}_1$ is positive.

Definition 2 (The Intensity of Liquidity Competition)**.** *Assume $\alpha > 0$. We define the intensity of liquidity competition ξ as*

$$\xi := \frac{\alpha \hat{\beta}_1}{\alpha + \hat{\beta}_0}. \tag{2.17}$$

Proposition 2, whose proof is given in the appendix, derives a closed form expression of optimal display sizes as a function of ξ. Optimal display sizes will be characterized in terms of the *Lambert function* (c.f. Corless et al. (1996)). A Lambert function Φ is any function that solves the equation:

$$w = \Phi(w) e^{\Phi(w)}, w \in \mathbb{C}. \tag{2.18}$$

We will use of the lower branch Φ_{-1} of the Lambert function. It is defined for the interval $[-1/e, 0]$, is strictly negative, bounded from above, unbounded from below, monotonically decreasing and obeys $\Phi_{-1} \leq \Phi_{-1}(-1/e) = -1$.

Proposition 2 (Optimal Display in Markets with Liquidity Competition)**.** *Assume the trader wants to buy (sell) N shares and that all model parameters except $\hat{\beta}$ are independent of the display size and that $\hat{\beta}$ satisfies (2.16). Assume moreover that $\hat{q} = 1$, $\hat{\beta}_0 \geq 0$ and $\hat{\beta}_1 > 0$. Then,*

$$\frac{\Delta^*}{N} = \begin{cases} 1 & \text{if} \quad \xi \leq \xi_- \\ -\frac{\alpha}{N}(1 + \xi^{-1} + \Phi_{-1}(\overline{w})) & \text{if} \quad \xi_- < \xi < \xi_+ \\ 0 & \text{if} \quad \xi \geq \xi_+ \end{cases}, \tag{2.19}$$

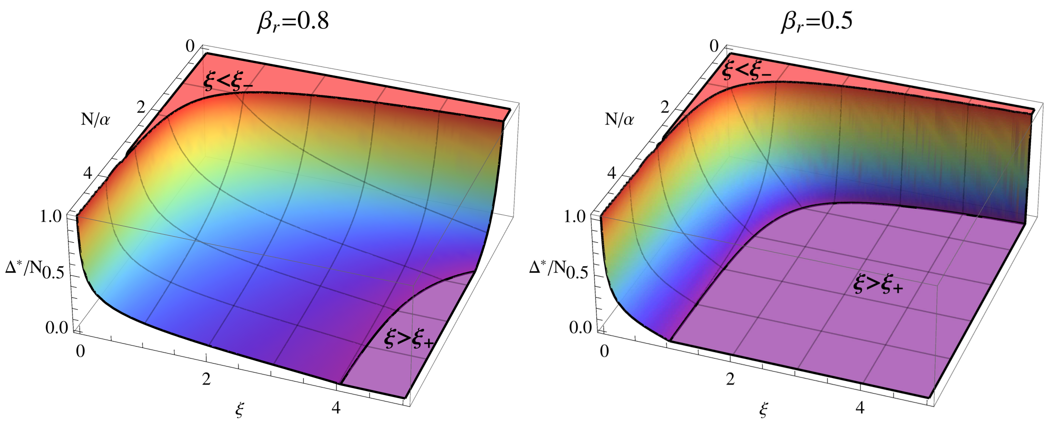
$$\text{with} \qquad \overline{w} := -\gamma e^{-\xi^{-1} - 1}, \qquad \gamma := \frac{1 - e^{-\frac{N}{\alpha}}(1 - \beta_r)}{\beta_r}, \tag{2.20}$$

$$\text{and} \qquad \xi_- := \left(\gamma e^{\frac{N}{\alpha}} - 1 - \frac{N}{\alpha}\right)^{-1}, \qquad \xi_+ := \left(\gamma - 1\right)^{-1}. \tag{2.21}$$

The trader's optimal display strategy depends on the (effective) order size N/α and the different levels of liquidity competition, specifically, the intensity of liquidity supply ξ that improves the trader's submission price level (i.e. overbidding) and the level of liquidity supply β_r that does not overbid the trader's submission price level. Figure 1 illustrates the main result.

The preceding proposition highlights the effect of (overbidding) limit order flow ξ on liquidity suppliers display decision. If traders display their orders, they trigger limit order submissions which improve (overbid) the trader's own submission price, and thus, reduce the latter's execution priority and therefore increase his transaction costs. Hence, liquidity suppliers completely hide their order ($\Delta^* = 0$) when liquidity competition exceeds a critical value ($\xi_+ \leq \xi$). On the other hand, when liquidity competition is sufficiently low ($\xi \leq \xi_-$), traders will fully reveal their intentions ($\Delta^* = N$). In the intermediate regime ($\xi_- < \xi < \xi_+$), liquidity suppliers partially reveal their trade intentions ($0 < \Delta^* < N$).

Figure 1: Optimal display rations in reduced model as a functions of the (scaled) order size N/α and the intensity of liquidity competition ξ at more aggressive price levels. The left (right) panel figure shows optimal display sizes with respect to a high (low) level of liquidity competition at the submission price level: $\beta_r = 0.8$ ($\beta_r = 0.5$).



Besides overbidding (aggressive) liquidity competition, less aggressive limit orders which submit at the liquidity suppliers price level do also affect the latter's display decision, however, in an opposite way. While aggressive limit orders force liquidity suppliers to hide their trade intentions, limit orders that are submitted at the liquidity suppliers' price level – and not ahead – force liquidity suppliers to display their trade intentions. This is because hidden orders lose time-priority against incoming limit orders at the same price level. Hence, an increased level of liquidity competition at the same price level, encourages order display. For instance, Figure 1 shows that order display is larger in the case $\beta_r = 0.8$ compared to $\beta_r = 0.5$. Thus, the order-display level will ultimately depend on the mix between liquidity competition at more aggressive price levels ξ (i.e., $\hat{\beta}_r$) and the intensity of liquidity competition at the same price level β_r.

Moreover, the proposition further shows that optimal display strategies depend on the (scaled) order size N/α. As long as traders are sufficiently small relative to the overall trade volume, the costs associated with order display due to market impact are negligible compared to the gain in execution-priority. Hence, traders fully display their trade intentions. However, if orders are large, then the costs of order display increase dis-proportionally so traders gradually decrease order display with growing order sizes. The following result is immediate.

Corollary 1. *There exists an order size N_0, such that the optimal order display Δ^* increases monotonously for $N \leq N_0$ and decreases monotonously for $N_0 > N$. More-*

over,

$$\lim_{N\to 0} \frac{\Delta^*}{N} = 1, \tag{2.22}$$

$$\lim_{N\to\infty} \Delta^* = \begin{cases} -\alpha\left(1+\xi^{-1}+\Phi_{-1}\left(-\frac{1}{\beta_r}e^{-\xi^{-1}-1}\right)\right) < \infty & \textit{if} \quad \xi < \frac{\beta_r}{1-\beta_r}, \\ 0 & \textit{else.} \end{cases} \tag{2.23}$$

An important consequence of the preceding Corollary is that there is a maximum display size beyond which traders should never display their order, when liquidity competition at aggressive price levels outweighs liquidity competition at less aggressive price levels ($\xi < \beta_r/(1-\beta_r)$).

3 Estimating Optimal Display Strategies

In this section, we use high-frequency order-message data to estimate optimal display strategies. Optimal display sizes are obtained for varying market and order settings and for a range of stocks traded on NASDAQ. We benchmark the trade performance of the optimal display sizes obtained from both, the full and reduced model against ad-hoc trading practices *full-display* ($\Delta = N$) and *zero-display* ($\Delta = 0$).

3.1 Data

Our estimates are based on NASDAQ ITCH order-message data for the period ranging from January 2011 to end of April 2011 for a random selection of 31 stocks from the S&P500 index. The dataset provides messages for every order entry, including modification, cancellation, submission and execution. The messages contain order identification numbers, time stamps, order -modifications, -submissions, -cancellations and -execution, as well as a flag marking the side of the book (buy or sell). This allows to track every order until cancellation/execution and re-construct the complete visible order book.

To estimate the model we fix the time period to be one minute and aggregate order execution, cancellation and submission volumes on a minute-by-minute basis and merge the aggregated order flow volumes with one-minute snapshots of the order book state (i.e., spread, depth, price etc.). Thus, for each minute of trading time we know the total number of market orders and limit orders that arrive at the best quotes and at more aggressive prices, respectively. We only aggregate "net" limit order flow: we did not include limit orders that were cancelled in the same minute they arrived, as this has the same effect as a zero order submission. Thus, cancellations were for standing liquidity only. The data aggregated in this way consists of 390 daily minute-by-minute information observed over 82 trading days, corresponding to 31,980 one-minute observations for each stock.

3.1.1 Descriptive Statistics

Table 1 reports group-averages of time-averaged model parameters. Group-averaging is based on one-minute dollar trade volume (TrVol): q_1 (TrVol$<$ 5000\$), q_2 (5000\$ $\leq$ TrVol $<$ 50000\$) and q_3 (50000\$ $\leq$ TrVol). The number of liquidity groups has been chosen so as to represent a proper balance between the available sample size (i.e., 31 stocks) and variation in liquidity. Time-averages for individual stocks are provided in Table 3 in the appendix.

Table 1: Time-averages of stock properties and model parameters based on liquidity groups q_1 (TrVol$<$ 5000\$), q_2 (5000\$ $\leq$ TrVol $<$ 50000\$), q_3 (50000\$ $\leq$ TrVol). Estimates report average mid-quote price (MQ), spread (S), one-minute dollar trade volume ($TrVol$), buy-side depth at the first level (D_{bid}), one-minute mid-quote return variance (Var), one-minute order size trade volume (α), one-minute arrival probability of trades (p), proportion of liquidity competition inside the best quotes ($\hat{\beta}_r$), proportion of liquidity competition at the best quote (β_r), ratio of cancellations relative to the existing depth at the first level (c), realized effective spread (μ). The spread (S) and the effective spread (μ) are time-averaged after normalizing by the prevailing mid-quote price and are given in basis points. Time averages are based on one-minute aggregated data for the time period January 2011 to April 2014 for 31 random stocks from NASDAQ.

	Average Stock Properties				Average Model Parameters						
	MQ (\$)	S (bps)	$TrVol$ (1000\$)	Var	D_{bid} ($shares$)	α ($shares$)	p	$\hat{\beta}_r$	β_r	c	μ (bps)
q_1	32	37.27	2.55	25.07	1,705	647	0.23	0.14	0.11	0.28	8.83
q_2	23	9.29	19.14	13.01	6,101	1,940	0.62	0.23	0.22	0.20	1.90
q_3	64	3.96	425.03	8.15	18,391	11,488	0.90	0.27	0.22	0.16	2.24
All	41	15.35	171.44	14.63	9,582	5,261	0.61	0.22	0.19	0.21	4.04

Table 1 shows that liquid stocks exhibit a stronger degree of liquidity competition in terms of $\hat{\beta}_r$ and β_r. In view of Proposition 2 we expect that order display must be lower in liquid stocks than in illiquid stocks. In fact, trading activity and liquidity competition are linked. Harris (1997) shows that small tick sizes provide incentives for traders to overbid existing quotes as the costs are relatively small compared to stocks with large tick sizes. Thus, markets with small tick sizes exert higher liquidity competition. At the same time, liquid stocks are known to trade at high prices that, therefore, have low (relative) tick sizes. This explains why liquidity competition is more intense for liquid stocks than for illiquid stocks.

3.1.2 Model Estimation

To estimate both, the impact of order imbalances on market dynamics and optimal display strategies we employ two different settings. The first corresponds to the general model where we allow imbalances to affect almost all model parameters. For such setting we propose a simple linear regression approach for $\hat{\beta}$, β and μ, i.e.,

$$\hat{\beta}_r(I) = \hat{b}_0 + \hat{b}_1 I + \epsilon_{\hat{b}}, \qquad \beta_r(I) = b_0 + b_1 I + \epsilon_b, \qquad \mu(I) = m_0 + m_1 I + \epsilon_m. \tag{3.1}$$

For p and α, we suggest logarithmic and sigmoidal transformations as follows:

$$\log[\alpha](I) = a_0 + a_1 I + \epsilon_a, \quad \bar{p}(I) = \kappa_0 + \kappa_1 I + \epsilon_\kappa, \quad \text{with} \quad p(I) = \frac{1}{1 + e^{-\bar{p}(I)}}. \tag{3.2}$$

The second setting corresponds to the reduced model. In this case only $\hat{\beta}_r$ varies with imbalances. The assumption on the arrival probability of market orders ($q = 1$) can be weakened. In fact, it can be seen from the proof that we only need to assume that $\hat{\beta}_r$ is inversely linear, i.e.,

$$1 - \hat{\beta}_r = \frac{1}{\alpha + \hat{\beta}_0 + \hat{\beta}_1 \Delta}.$$

This can be re-written such that the regression model reads

$$\hat{\beta}_r = 1 - \frac{1}{\zeta}, \qquad\qquad \zeta = \zeta_0 + \zeta_1 I + \epsilon_\zeta. \tag{3.3}$$

The new estimation parameters ζ_0 and ζ_1 capture the effects of α, $\hat{\beta}_0$ and $\hat{\beta}_1$ and are estimated indirectly via $\hat{\beta}_r$ in (3.3). The ϵ-variables denote iid normal error terms.

All estimations are based on weighted ordinary least squares on binned and discretized grids for the imbalance. For each stock, we discretize order-imbalance into equidistant data points and construct conditional means for $\alpha, p, \hat{\beta}_r, \beta_r, \mu$ based on the realized imbalances in the respective intervals. Using weighted ordinary least squares allows to account for heteroscedasticity as large order-imbalances occur less often than small order-imbalances and thus fewer events have to be weighed accordingly. We chose the weights to be the square-root of the sample size associated with each imbalance bin.

Table 2 reports t-statistics of coefficient estimates for each stock. The coefficient estimates and r^2-goodness-of-fit are reported in Table 4 and 5 in the appendix. For brevity of exposition, tables only show the results for the slopes of the linear regression models, not for the respective intercepts. Average r^2 goodness-of-fit ranges from 40% to 63% and is stable across the 31 stocks, suggesting that the Δ-dependencies in the model parameters are well captured. Second, in line with our central assumption of the reduced model, t-statistics broadly confirm that order display mainly affects the amount of liquidity competition. The impact on price and liquidity demand is ambiguous.

3.2 Optimal Display Estimates

Using the market impact estimates of the previous section, we calculate optimal displays for a broad range of initial market and order settings. To this end, we fix the initial sell-side depth to be the average trade volume over the same period, i.e., $D_{sell} = \alpha * p$. Then, we calculate the optimal display size for various realizations of the initial bid-side depth D_{bid} and order size N. For each such realization, we calculate the transaction cost functional for the general (W) and the reduced (W_{red}) model. Optimal display sizes

Table 2: Reported t-statistics for the model parameters of the *general model* as of (3.1), (3.2) and the *reduced model* as of (3.3). Estimates are provided for each stock separately based on one-minute aggregated NASDAQ-ITCH data ranging from January to April 2011. Stocks are a random selection from the S&P 500 during that period. Only the slope coefficients of the linear regressions are shown. For ease of exposition coefficients have been multiplied by a factor of 1000.

	General Model (3.1) and (3.2)					Reduced Model (3.3)
Stock	a_1	κ_1	$\hat{b}_1$	b_1	m_1	ζ_1
AAPL	7.36	−0.91	0.73	1.66	0.65	0.86
ADM	11.07	−5.39	10.99	15.85	−2.74	10.65
BAC	−0.82	−5.14	18.24	21.01	3.82	17.81
CALL	4.90	2.00	0.74	−2.39	−5.96	1.20
CCJ	6.70	−7.07	0.01	9.81	−3.45	0.35
COCO	3.85	−2.92	11.23	3.40	3.90	10.87
CSCO	2.28	−5.31	11.10	14.84	4.45	10.34
DBD	3.31	−4.15	−0.79	3.97	−1.81	−1.07
DNR	8.72	−7.45	9.45	22.78	−5.34	8.88
EBAY	5.05	−5.97	17.08	23.06	−1.83	16.21
ERIC	4.08	−6.51	13.95	11.82	13.79	11.20
EWA	2.39	−14.08	22.25	24.20	12.77	15.80
GE	−0.81	−5.99	14.00	21.81	3.16	13.85
HBAN	0.84	−9.40	11.63	10.21	7.40	10.97
HPQ	3.03	−4.48	12.69	18.75	−3.15	13.11
IBM	10.51	−2.48	−0.34	8.34	−1.18	−0.41
ING	5.52	−2.56	9.55	15.89	17.13	9.36
INTC	−3.69	−12.03	18.84	12.27	6.63	17.83
ITC	2.74	−1.89	1.08	2.39	−0.63	1.51
MS	3.81	−6.05	10.97	13.46	−3.84	11.43
MSFT	2.55	−8.95	19.48	21.59	3.58	18.99
ORCL	4.83	−7.99	11.86	20.07	−1.61	11.94
PFE	−0.05	−4.14	29.90	15.94	6.32	29.01
PLOW	1.26	0.47	0.71	−0.85	−3.01	0.71
RSH	6.98	−10.94	9.10	22.17	−2.81	8.99
SNP	2.82	−1.94	1.35	1.99	2.66	1.47
SWC	7.62	−2.44	0.27	5.76	1.23	−0.18
TECD	5.87	0.91	3.45	7.27	−0.76	2.01
USMO	6.33	−4.58	3.73	1.13	−2.22	3.71
VIP	7.50	−11.63	3.58	5.50	−2.17	3.53
WEN	1.78	−14.19	10.16	13.96	1.09	10.08
Average	4.14	−5.59	9.26	11.86	1.49	8.74

for the general (Δ^*), respectively, reduced (Δ^*_{red}) model are obtained by numerically minimizing W, respectively using the explicit representation of Proposition 2.

Another widely used metric for the quality of trade execution is the so called fill-rate (or execution ratio), that is the ratio of successfully executed orders at the initial submission price level. Bessembinder et al. (2009) shows that both measures are sensibly linked to the willingness to display or hide trading intentions. To further compare the performance of the general and reduced model relative to the benchmark *full-display* and *zero-display* strategy we also estimate the fill-rates for each strategy. Results are shown in Figure 2 and 3. For brevity of exposition we show the group-averaged results for liquid and il-liquid stocks. Figure 2 reports cross-sectional averages for high-liquid stocks with TrVol$>$ 50000\$, while Figure 3 reports cross-sectional averages for low-liquid stocks with TrVol$<$ 50000\$. Results for individually selected stocks are provided in the appendix.

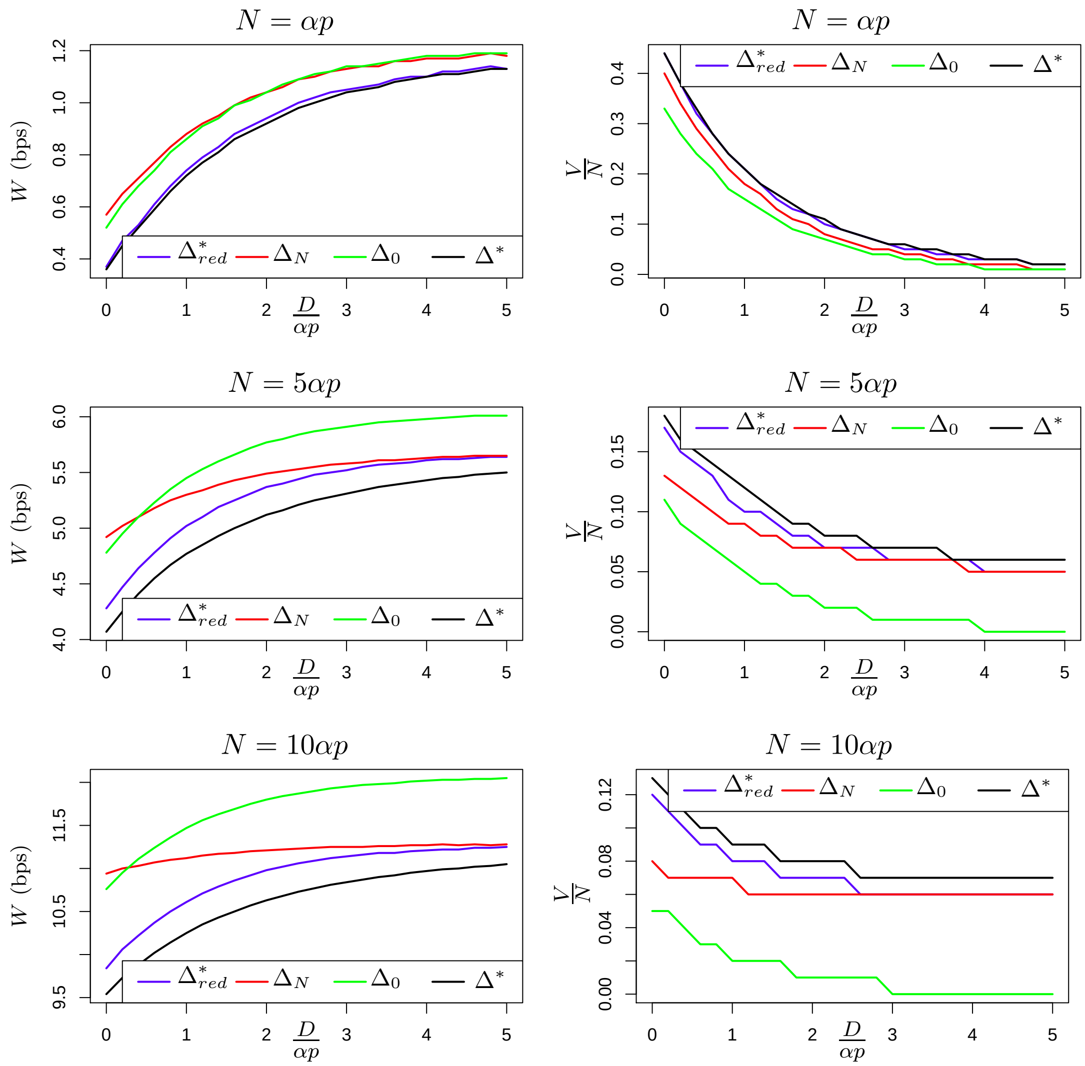
Our key results can be summarized as follows. First, we observe two distinct regimes with respect to the performance of the reduced display size. It turns out that the comparative advantage of using the reduced display strategy Δ_{red} over ad-hoc trading strategies is particularly strong in the liquid regime of stocks, for instance Apple or Oracle. For less liquid stocks, however, the full-display strategy is generally the best strategy. Generally, performance gains of strategic display strategies Δ^* and Δ^*_{red} over non-strategic strategies Δ_0 and Δ_N are significant in liquid stocks. The cross-sectional averages suggest that performance gains up to one basis points in absolute terms and up to 40% in relative terms are attainable.

We also find that the relative advantage of optimal display strategies does critically depend on the the prevailing order book depth D_{bid} relative to incoming market order flow αp, or the *exhaustion speed* $\frac{\alpha}{D_{bid}}$. The exhaustion speed reflects the speed with which depth (at the first level) is exhausted by incoming market orders. If depth is executed quickly (i.e., $\frac{\alpha}{D_{bid}} \gg 1$), then (partially) hidden orders allows to significantly reduce transaction costs. However, if the order-book is thick (i.e., $\frac{\alpha}{D_{bid}} \ll 1$), hidden strategies will not yield significant benefits. The findings suggests that stocks with a low average exhaustion speed attract more hidden liquidity on average, while stocks with high $\frac{\alpha}{D_{bid}}$ are overall more transparent. A prominent example of a stock with high exhaustion speed is Apple (AAPL). According to Table 3, this stock has an exhaustion speed of $\frac{\alpha}{D_{bid}} = 23.23$, by far the strongest exhaustion speed in our sample. Thus, it is conceivable that the use of hidden orders for this stock is more prevalent (see Figure 4).

4 Conclusion

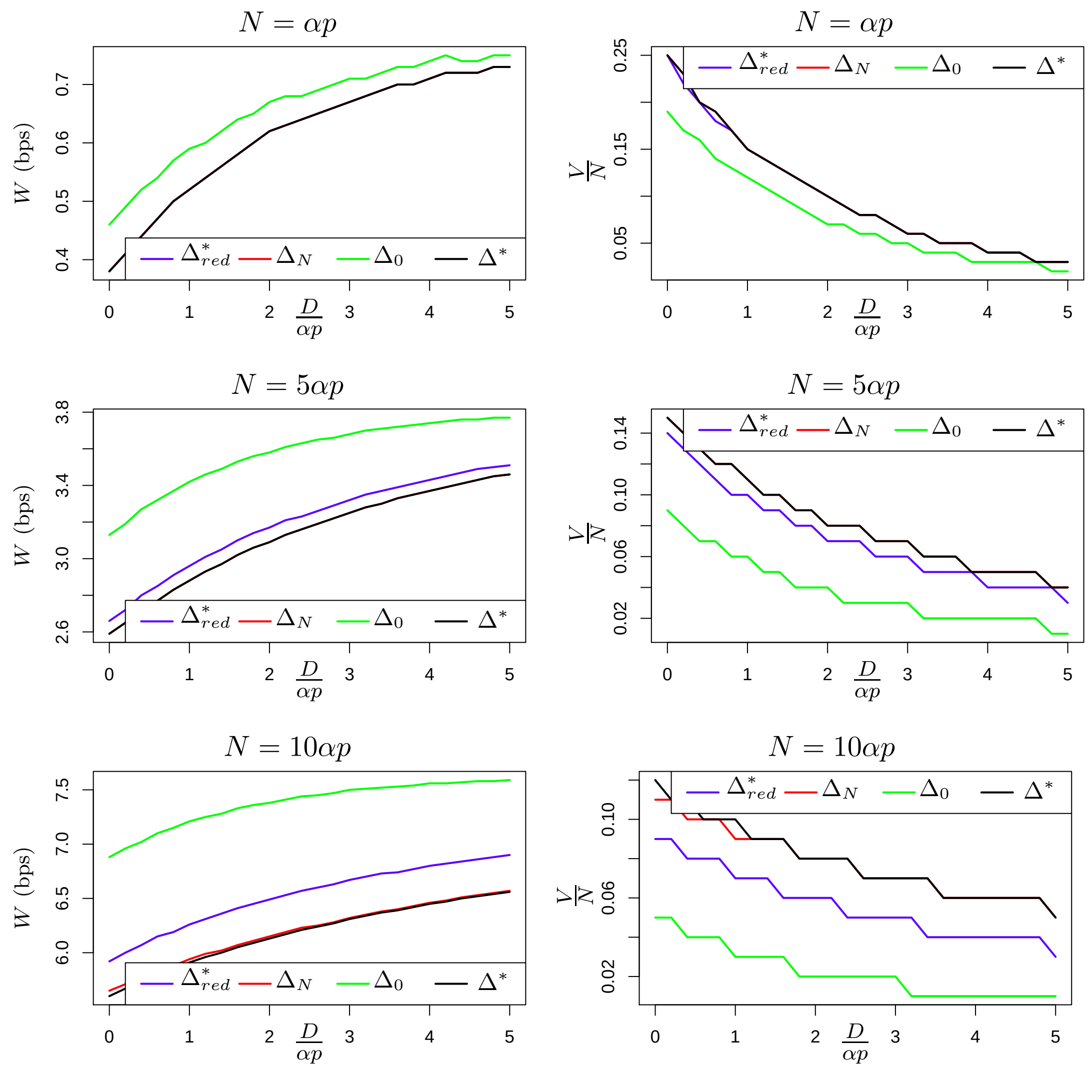
We proposed a simple model of optimal order display in which key market parameters depend on order imbalances. In a benchmark model of pure liquidity competition optimal display strategies could be given in closed form. We find that underlying liquidity

Figure 2: Figures show estimated the cross-sectional average of transaction costs W and execution ratio $\frac{V}{N}$ for **liquid** stocks with $TrVol > 50000\$$. Estimates are shown for different display strategies: zero-display ($\Delta = 0$ in green), full-display ($\Delta = N$), the reduced model optimal display ($\Delta = \Delta^*_{red}$ in blue) and the full model optimal display strategy ($\Delta = \Delta^*$ in black). Transaction cost performance W is shown in the left panel. Expected execution ratio $\frac{V}{N}$ is shown in the right panel. For each, we show three different realizations of the order size N relative to expected transaction volume αp: small ($N = \alpha p$), medium ($N = 5\alpha p$) and large ($N = 10\alpha p$).



characteristics, especially liquidity competition strongly influence display strategies. This has important implications for trading as the liquid class of stocks generate the bulk of trading activity at today's markets. The decision to hide or display also depends on the prevailing market depth at time of order submission. We further find that hidden orders are beneficial when the speed with which the order book depth is executed against market orders is high. This is specifically so for actively traded stocks at times of low market depth. Thus, we predict that hidden liquidity cumulates in times of low depth and high liquidity demand. Our analysis also generates implica-

Figure 3: Figures show estimated the cross-sectional average of transaction costs W and execution ratio $\frac{V}{N}$ for **il-liquid** stocks with $TrVol < 50000\$$. Estimates are shown for different display strategies: zero-display ($\Delta = 0$ in green), full-display ($\Delta = N$), the reduced model optimal display ($\Delta = \Delta^*_{red}$ in blue) and the full model optimal display strategy ($\Delta = \Delta^*$ in black). Transaction cost performance W is shown in the left panel. Expected execution ratio $\frac{V}{N}$ is shown in the right panel. For each, we show three different realizations of the order size N relative to expected transaction volume αp: small ($N = \alpha p$), medium ($N = 5\alpha p$) and large ($N = 10\alpha p$).



tions for the cross-sectional variation of hidden liquidity with respect to tick size rules. For instance, stocks that trade at higher prices, have lower (relative) tick sizes. Since lower tick sizes incentivize liquidity competition as bypassing orders is cheaper, such stocks encourage the use of hidden orders more than in low-priced stocks.

A Proof of Proposition 2

We need to compute the minimizer of the function

$$\Delta \mapsto W(\Delta) = \left(1 - \frac{V(\Delta)}{N}\right) \mu(\Delta).$$

Under the assumptions of Proposition 2 this reduces to computing the critical point of a function $f : [0, N] \to \mathbb{R}$ of the form

$$f(x) = -\frac{d - ce^{-x/\alpha}}{a + bx}, \quad a, b, c, d > 0.$$

Its first derivative is given by

$$f'(x) = -\frac{\frac{c}{\alpha} e^{-x/\alpha}(a + bx) - b(d - ce^{-x/\alpha})}{(a + bx)^2}.$$

Hence any *interior* critical point x^* solves an equation of the form

$$e^{x/\alpha} + B_1 x + B_0 = 0 \tag{A.1}$$

for suitable constants B_0, B_1 and can thus be expressed in terms of the Lambert function Φ as

$$x^* = -\frac{1}{B_1}\left(B_0 + B_1 \alpha \Phi\left(\frac{e^{-\frac{B_0}{B_1 \alpha}}}{B_1 \alpha}\right)\right).$$

The second derivative evaluated at the interior critical point x^* is given by

$$f''(x^*) \quad = \quad \frac{\frac{c}{\alpha^2} e^{-x^*/\alpha}(a + bx^*)}{(a + bx^*)^2}.$$

Hence, x^* is a local minimizer if $x^* \geq 0$. This is the case in our model. Indeed, in our model,

$$B_1 = \frac{\eta - 1}{\alpha(1 - \eta e^{-\frac{N}{\alpha}})}, \quad \text{and} \quad B_0 = \frac{(\eta-1)(\alpha(1+\hat{\beta}_1) + \hat{\beta}_0)}{\hat{\beta}_1 \alpha (1 - \eta e^{-\frac{N}{\alpha}})},$$

where $\eta := 1 - \beta_r$. Hence the *interior* optimal display sizes Δ^* satisfy

$$\Delta^* = -\frac{\alpha + \hat{\beta}_0}{\hat{\beta}_1} - \alpha\Big(1 + \Phi(w)\Big)$$

where

$$\begin{aligned} w \quad &= \quad -e^{-1-\frac{\alpha+\hat{\beta}_0}{\hat{\beta}_1 \alpha}} \frac{(1 - e^{-\frac{N}{\alpha}}\eta)}{1 - \eta} \\ &= \quad -e^{-1-\xi^{-1}} \gamma \\ &\in \quad (-1/e, 0). \end{aligned}$$

The Lambert function Φ is not injective on $(-\frac{1}{e}, 0)$. The additional constraints $\Phi \geq -1$, respectively $\Phi \leq -1$, define single-valued functions, the upper branch Φ_0 and lower branch Φ_{-1}. The upper branch satisfies $\Phi_0(0) = 0$ and $\Phi_0(-1/e) = -1$. It would give negative display sizes. The lower branch decreases from $\Phi_{-1}(-1/e) = -1$ to $\Phi_{-1}(0-) =$

$-\infty$; the associated optimal display size $\Delta^*_{-1} = \Delta^*_{-1}(\xi)$ satisfies $\Delta^*_{-1}(0) = +\infty$ and $\Delta^*_{-1}(\infty) = -1$. Hence it remains to identify the thresholds $\xi_\pm$ for which

$$\Delta^*_{-1}(\xi_+) = 0 \quad \text{and} \quad \Delta^*_{-1}(\xi_-) = N.$$

To this end, we use the fact that $\Phi_{-1}(x) = y$ if and only if $x = ye^y$ to find

$$\xi_+ = \Big(\gamma - 1\Big)^{-1} \geq 0 \quad \text{and} \quad \xi_- = \Big(\gamma e^{\frac{N}{\alpha}} - 1 - \frac{N}{\alpha}\Big)^{-1} \geq 0.$$

B Further empirical results

Table 3: Time averages of stock properties for individual stocks; cf Table 1.

	Average Stock Properties				Average Model Parameters						
	MQ (\$)	S (bps)	$TrVol$ (1000\$)	V	D_{bid} ($shares$)	α ($shares$)	p	$\hat{\beta}_r$	β_r	c	μ (bps)
AAPL	344	1.73	2,353.65	7.72	295	6,852	1.00	0.29	0.03	0.46	9.80
ADM	35	3.54	38.29	9.12	918	1,478	0.74	0.27	0.23	0.16	1.55
BAC	14	7.08	350.64	9.33	64,295	28,638	0.88	0.23	0.24	0.13	1.10
CALL	21	183.14	0.42	98.30	279	313	0.06	0.07	0.04	0.18	37.19
CCJ	36	5.98	30.46	18	403	1,094	0.77	0.34	0.17	0.26	2.66
COCO	5	21.36	5.79	19.92	5,438	2,330	0.50	0.10	0.20	0.08	1.07
CSCO	19	5.31	336.98	9.37	44,105	19,826	0.90	0.17	0.25	0.09	1.07
DBD	35	13.48	2.69	12.48	175	269	0.28	0.21	0.07	0.41	4.74
DNR	22	6.06	19.47	12.09	951	1,188	0.74	0.31	0.27	0.18	1.63
EBAY	31	3.41	122.51	9.73	2,075	4,332	0.91	0.31	0.23	0.12	1.57
ERIC	12	8.11	18.61	11.60	10,521	3,081	0.50	0.13	0.28	0.13	1.04
EWA	26	4	18.36	7.10	6,417	1,462	0.48	0.26	0.29	0.26	1.11
GE	20	5.07	193.32	9.52	17,181	11,399	0.85	0.26	0.26	0.14	1.17
HBAN	7	14.55	28.55	9.89	34,536	6,636	0.62	0.10	0.29	0.10	1.02
HPQ	44	2.50	155.45	9.20	1,993	4,017	0.88	0.30	0.24	0.11	1.53
IBM	161	2.04	216.54	5.17	263	1,430	0.94	0.35	0.11	0.28	4.36
ING	12	8.51	4.36	11.96	3,223	931	0.39	0.17	0.30	0.20	1.07
INTC	21	4.38	367.88	7.32	38,267	19,394	0.90	0.23	0.26	0.10	1.13
ITC	69	14.08	2.28	11.91	139	176	0.19	0.21	0.05	0.45	9.53
MS	29	3.77	80.01	8.98	2,481	3,288	0.84	0.32	0.25	0.16	1.43
MSFT	27	3.79	483.73	6.43	23,196	19,196	0.93	0.25	0.22	0.11	1.18
ORCL	33	3.14	276.90	8.13	6,176	9,309	0.90	0.30	0.24	0.10	1.38
PFE	19	5.31	162.77	6.87	20,360	10,179	0.84	0.22	0.27	0.11	1.11
PLOW	15	52.74	0.26	33.54	187	204	0.08	0.10	0.05	0.24	7.47
RSH	16	7.47	9.49	12.14	1,266	1,044	0.57	0.19	0.23	0.16	1.28
SNP	101	14.48	4.14	11.20	201	198	0.20	0.25	0.04	0.57	14.42
SWC	22	13.15	11.57	17.39	329	714	0.74	0.32	0.15	0.28	3.20
TECD	51	8.72	10.76	12.87	228	374	0.56	0.29	0.09	0.37	4.48
USMO	14	19.26	1.05	22.77	324	293	0.26	0.13	0.11	0.23	2.66
VIP	14	9.11	4.70	12.85	795	1,010	0.33	0.10	0.13	0.15	1.35
WEN	5	20.67	3.05	10.61	10,021	2,428	0.25	0.05	0.23	0.09	1.02
Average	41.29	15.35	171.44	14.63	9,582	5,261	0.61	0.22	0.19	0.21	4.04

Table 4: Reported **Coefficient Estimates** for the model parameters of the *general model* as of (3.1), (3.2) and the *reduced model* as of (3.3). Estimates are provided for each stock separately based on one-minute aggregated NASDAQ Itch data ranging from January to April 2011. Stocks are a random selection from the S&P 500 during that period. Only the slope coefficients of the linear regressions are shown. For ease of exposition coefficients have been multiplied by a factor of 1000.

			General Model (3.1) and (3.2)			Reduced Model (3.3)
Stock	a_1	κ_1	$\hat{b}_1$	b_1	m_1	ζ_1
AAPL	0.326	−0.531	0.004	0.002	19.573	0.009
ADM	0.259	−0.098	0.026	0.038	−8.894	0.047
BAC	−0.001	−0.003	0.001	0.001	0.151	0.002
CALL	1.076	0.590	0.017	−0.030	−1,829.747	0.046
CCJ	0.377	−0.186	0.0001	0.064	−38.051	0.008
COCO	0.032	−0.018	0.004	0.003	0.640	0.005
CSCO	0.004	−0.007	0.001	0.001	0.231	0.002
DBD	1.029	−0.504	−0.036	0.080	−86.001	−0.112
DNR	0.182	−0.176	0.024	0.049	−26.178	0.047
EBAY	0.057	−0.085	0.018	0.020	−3.062	0.038
ERIC	0.022	−0.024	0.006	0.004	1.471	0.010
EWA	0.031	−0.045	0.017	0.016	4.219	0.037
GE	−0.002	−0.012	0.004	0.005	0.478	0.007
HBAN	0.002	−0.008	0.001	0.002	0.219	0.001
HPQ	0.035	−0.056	0.015	0.020	−6.125	0.030
IBM	0.607	−0.271	−0.003	0.037	−31.970	−0.010
ING	0.056	−0.038	0.015	0.011	3.996	0.028
INTC	−0.005	−0.010	0.002	0.002	0.356	0.003
ITC	1.218	−0.500	0.067	0.043	−131.406	0.151
MS	0.028	−0.069	0.013	0.022	−4.798	0.026
MSFT	0.004	−0.013	0.003	0.003	0.328	0.006
ORCL	0.014	−0.044	0.007	0.009	−0.722	0.014
PFE	−0.0001	−0.007	0.004	0.004	0.644	0.007
PLOW	0.442	0.081	0.012	−0.014	−171.740	0.014
RSH	0.162	−0.182	0.019	0.040	−7.354	0.028
SNP	0.745	−0.202	0.028	0.014	332.745	0.069
SWC	0.446	−0.152	0.004	0.046	19.068	−0.006
TECD	1.033	0.121	0.126	0.134	−40.564	0.732
USMO	0.811	−0.377	0.037	0.010	−29.563	0.054
VIP	0.319	−0.153	0.007	0.016	−5.934	0.008
WEN	0.029	−0.047	0.002	0.008	0.102	0.003
Average	0.301	−0.098	0.014	0.021	−65.738	0.042

Table 5: Reported r^2 statistics for (3.1), (3.2) and (3.3) are reported. Statistics are provided for each stock separately based on one-minute aggregated NASDAQ Itch data ranging from January to April 2011. Stocks are a random selection from the S&P 500 during that period.

	General Model (3.1) and (3.2)					Reduced Model (3.3)
Stock	a_1	κ_1	$\hat{b}_1$	b_1	m_1	ζ_1
AAPL	0.77	0.07	0.02	0.08	0.03	0.02
ADM	0.82	0.34	0.68	0.82	0.22	0.67
BAC	0.04	0.41	0.90	0.92	0.45	0.89
CALL	0.65	0.13	0.03	0.21	0.72	0.07
CCJ	0.63	0.48	0.00	0.64	0.31	0.00
COCO	0.50	0.21	0.80	0.27	0.50	0.79
CSCO	0.26	0.47	0.79	0.87	0.57	0.77
DBD	0.32	0.27	0.01	0.25	0.12	0.02
DNR	0.73	0.49	0.61	0.90	0.50	0.58
EBAY	0.48	0.38	0.83	0.90	0.11	0.82
ERIC	0.41	0.46	0.80	0.74	0.89	0.72
EWA	0.21	0.81	0.91	0.93	0.88	0.84
GE	0.04	0.54	0.87	0.94	0.42	0.86
HBAN	0.04	0.73	0.81	0.77	0.78	0.79
HPQ	0.25	0.26	0.74	0.86	0.26	0.75
IBM	0.85	0.14	0.003	0.63	0.07	0.00
ING	0.59	0.13	0.67	0.85	0.93	0.67
INTC	0.55	0.86	0.94	0.86	0.80	0.93
ITC	0.31	0.09	0.03	0.15	0.02	0.06
MS	0.57	0.60	0.83	0.88	0.57	0.84
MSFT	0.26	0.67	0.90	0.92	0.40	0.90
ORCL	0.64	0.70	0.83	0.94	0.17	0.84
PFE	0.00	0.35	0.97	0.89	0.73	0.96
PLOW	0.13	0.01	0.02	0.03	0.41	0.02
RSH	0.73	0.76	0.69	0.93	0.31	0.68
SNP	0.27	0.08	0.04	0.11	0.23	0.05
SWC	0.71	0.11	0.00	0.40	0.06	0.00
TECD	0.56	0.01	0.17	0.49	0.02	0.07
USMO	0.63	0.30	0.22	0.02	0.17	0.22
VIP	0.79	0.81	0.29	0.49	0.24	0.28
WEN	0.17	0.86	0.77	0.86	0.07	0.77
Average	0.45	0.40	0.52	0.63	0.39	0.51

Figure 4: Figure shows **expected implementation shortfall** of the optimal display strategy Δ^*, the strategy of the reduced model Δ^*_{red}, the full-display strategy $\Delta_N = N$ and the zero-display strategy $\Delta_0 = 0$ for a selection of four **liquid** stocks from the S&P 500 index: *Apple* (AAPL), *Ebay* (EBAY), *Microsoft* (MSFT) and *Oracle* (ORCL). Performances are plotted against the initial (same-side) depth. Depth is denoted in multiples of average trade volume over the considered period, αp. The figure shows results for two order sizes: on the left panel, a small order that equals the average trade volume ($N = \alpha p$); on the right panel, a large order equaling ten times the average trade volume ($N = 10\alpha p$).

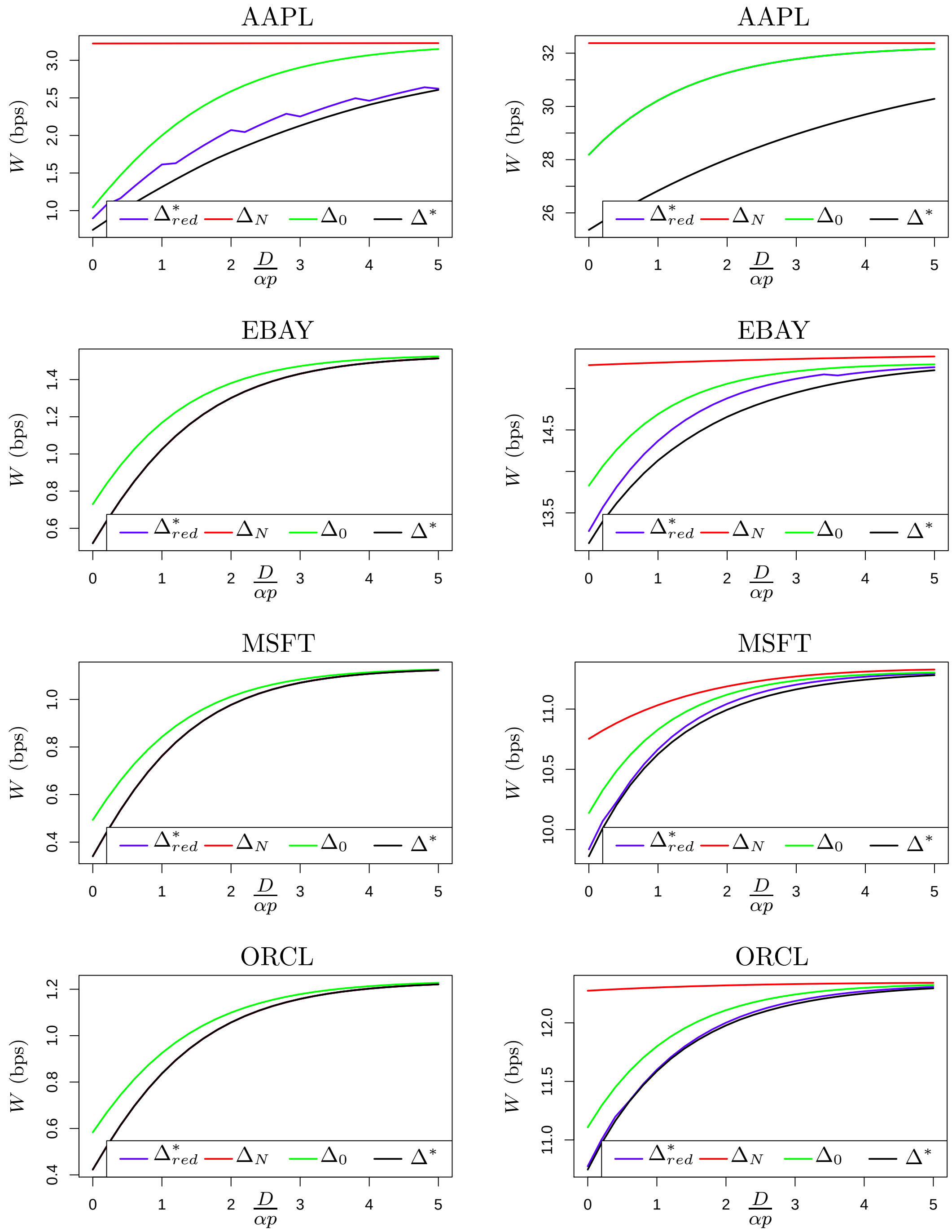


Figure 5: Figure shows **expected execution ratios** of the optimal display strategy Δ^*, the strategy of the reduced model Δ^*_{red}, the full-display strategy $\Delta_N = N$ and the zero-display strategy $\Delta_0 = 0$ for a selection of four **liquid** stocks from the S&P 500 index: *Apple* (AAPL), *Ebay* (EBAY), *Microsoft* (MSFT) and *Oracle* (ORCL). Expected execution ratios are plotted against the initial (same-side) depth. Depth is denoted in multiples of average trade volume over the considered period, αp. The expected execution ratio is defined as the expected execution volume V at the submission price level divided by the initial order size N. The figure shows results for two order sizes: on the left panel, a small order that equals the average trade volume ($N = \alpha p$); on the right panel, a large order equaling ten times the average trade volume ($N = 10\alpha p$).

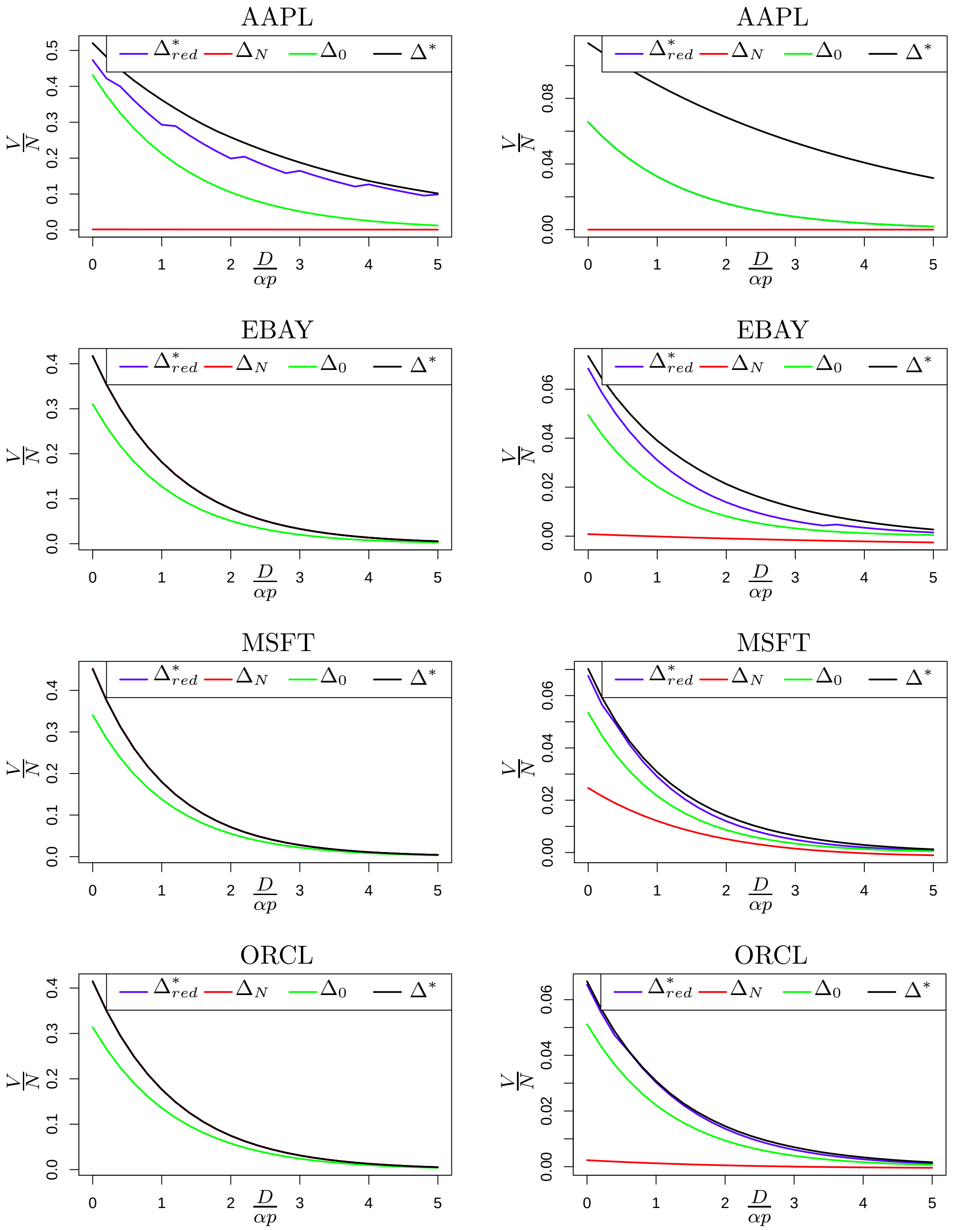


Figure 6: Figure shows **expected implementation shortfall** of the optimal display strategy Δ^*, the strategy of the reduced model Δ^*_{red}, the full-display strategy $\Delta_N = N$ and the zero-display strategy $\Delta_0 = 0$ for a selection of four **il-liquid** stocks from the S&P 500 index: *Corinthian Colleges* (COCO), *Huntington Bancshares* (HBAN), *Douglas Dynamics* (PLOW) and *The Wendy's Company* (WEN). Performances are plotted against the initial (same-side) depth. Depth is denoted in multiples of average trade volume over the considered period, αp. The figure shows results for two order sizes: on the left panel, a small order that equals the average trade volume ($N = \alpha p$); on the right panel, a large order equaling ten times the average trade volume ($N = 10\alpha p$).

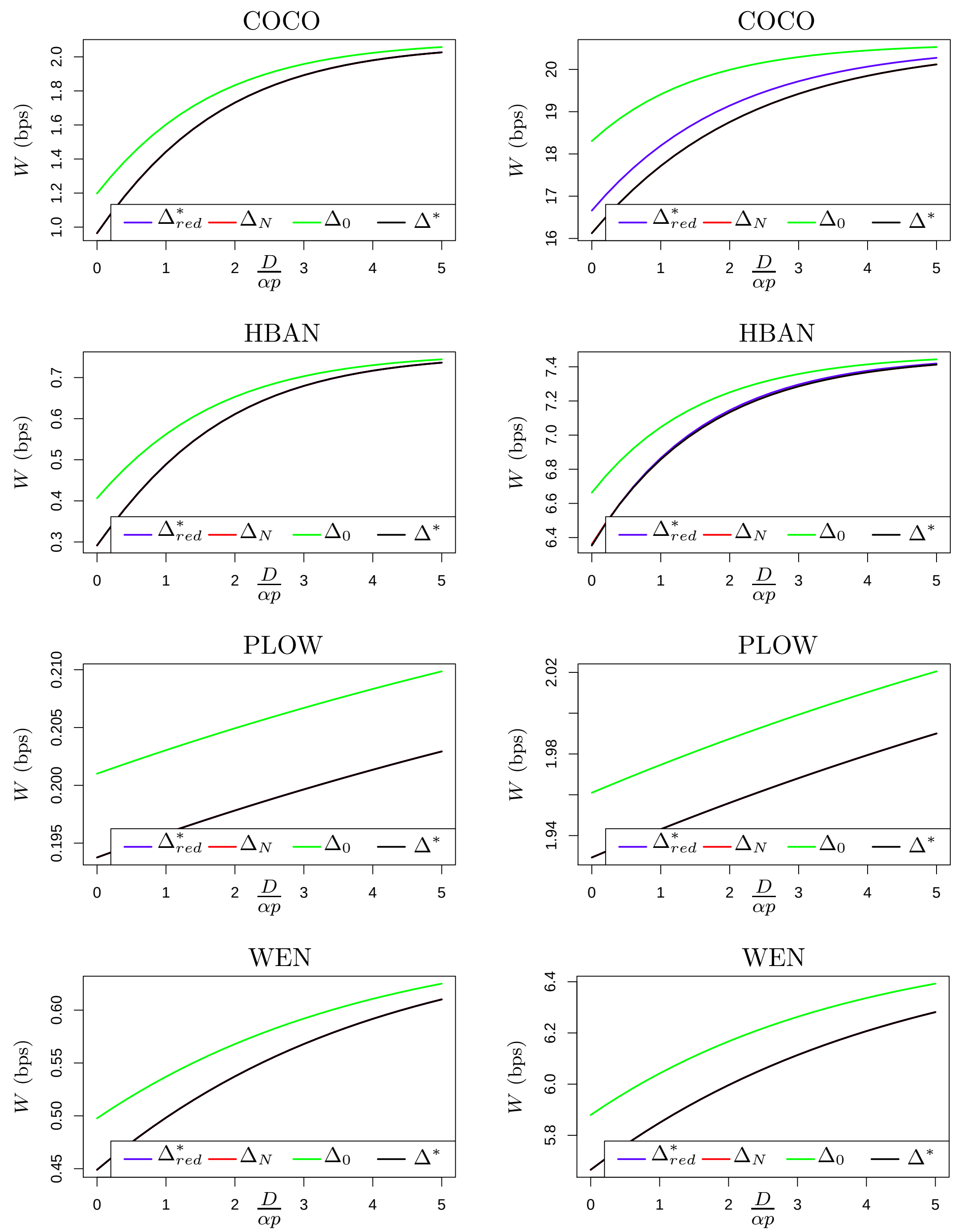
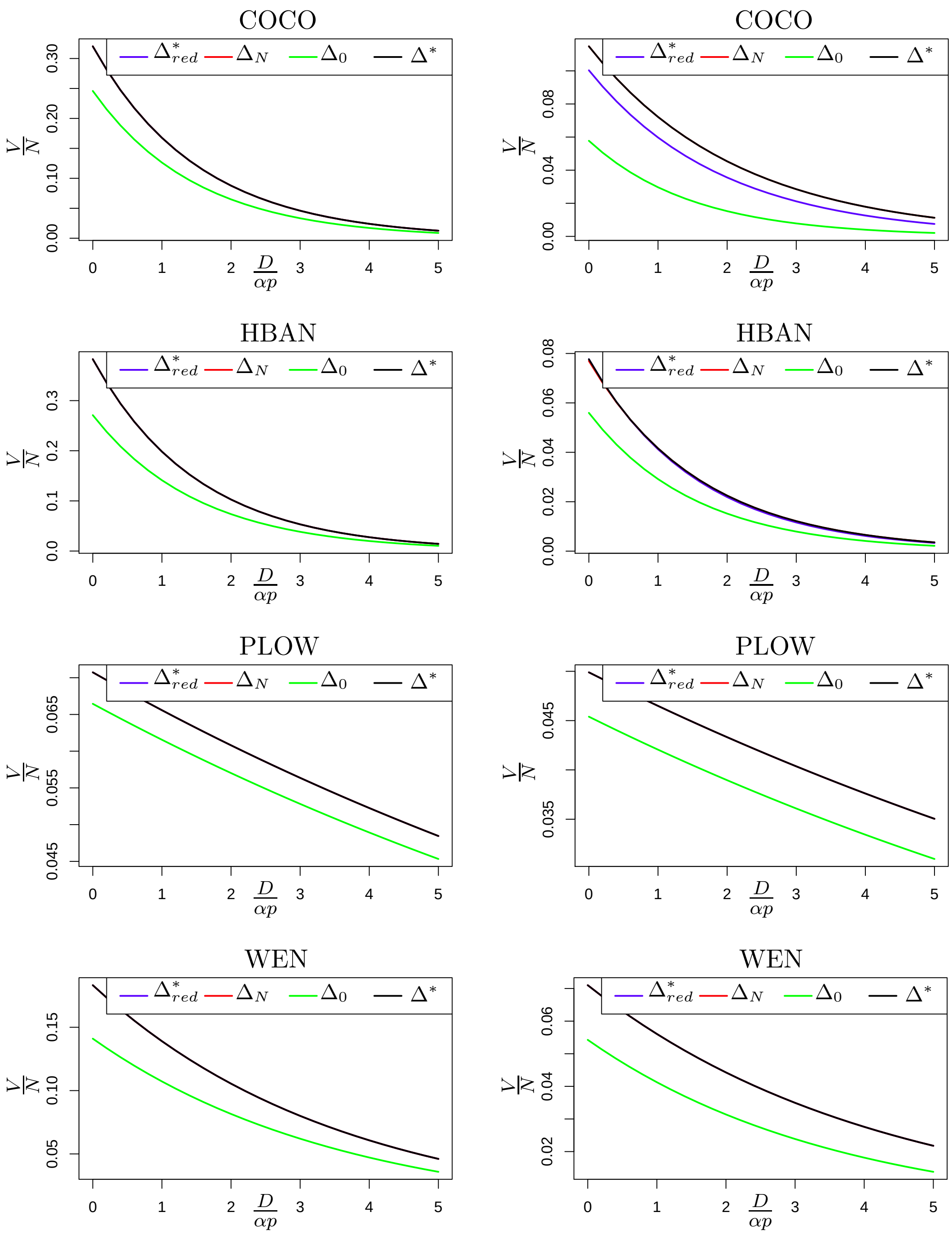


Figure 7: Figure shows **expected execution ratios** of the optimal display strategy Δ^*, the strategy of the reduced model Δ^*_{red}, the full-display strategy $\Delta_N = N$ and the zero-display strategy $\Delta_0 = 0$ for a selection of four **il-liquid** stocks from the S&P 500 index: *Corinthian Colleges* (COCO), *Huntington Bancshares* (HBAN), *Douglas Dynamics* (PLOW) and *The Wendy's Company* (WEN). Expected execution ratios are plotted against the initial (same-side) depth. Depth is denoted in multiples of average trade volume over the considered period, αp. The expected execution ratio is defined as the expected execution volume V at the submission price level divided by the initial order size N. The figure shows results for two order sizes: on the left panel, a small order that equals the average trade volume ($N = \alpha p$); on the right panel, a large order equaling ten times the average trade volume ($N = 10\alpha p$).



References

ALFONSI, A., A. FRUTH, AND A. SCHIED (2010): "Optimal Execution Strategies in Limit Order Books with General Shape Functions," *Quantitative Finance*, 2, 143–157.

ALMGREN, R. (2001): "Optimal Execution of Portfolio Transactions," *Journal of Risk*, 3, 5–39.

——— (2003): "Optimal Execution with Nonlinear Impact Functions and Trading-enhanced risk," *Applied Mathematical Finance*, 10, 1–18.

ALMGREN, R. AND N. CHRISS (1999): "Optimal Execution of Portfolio Transactions," *Journal of Risk*, 3, 5–39.

BERTSIMAS, D. AND A. W. LO (1998): "Optimal Control of Execution Costs," *Journal of Financial Markets*, 1, 1–50.

BESSEMBINDER, H., M. PANAYIDES, AND K. VENKATAMARAN (2009): "Hidden Liquidity: An Analysis of Order Exposure Strategies in Electronic Stock Markets," *Journal of Financial Economics*, 94, 361–383.

BIAIS, B., P. HILLION, AND C. SPATT (1995): "An Empirical Analysis of the Limit Order Book and the Order Flow in the Paris Bourse," *The Journal of Finance*, 50, 1655–1689.

BUTI, S. AND B. RINDI (2013): "Undisclosed Orders and Optimal Submission Strategies in a Limit Order Market," *Journal of Financial Economics*, 109, 797 – 812.

CAO, C., O. HANSCH, AND X. WANG (2009): "The Information Content of an Open Limit-order Book," *Journal of Futures Markets*, 29, 16–41.

CORLESS, R. M., G. H. GONNET, D. E. HARE, D. J. JEFFREY, AND D. E. KNUTH (1996): "On the Lambert-W function," *Advances in Computational mathematics*, 5, 329–359.

DE WINNE, R. AND C. D'HONDT (2007): "Hide-and-Seek in the Market: Placing and Detecting Hidden Orders," *Review of Finance*, 11, 663–692.

——— (2009): "Hidden Liquidity in a Pure Order-Driven Market," *Finance Letters, forthcoming.*

D'HONDT, C., R. DE WINNE, AND A. FRANCOIS-HEUDE (2004): "Hidden Orders on Euronext: Nothing is Quote as it Seems," Working paper series, FUCaM-Catholic University of Mons, Belgium.

ESSER, A. AND B. MÖNCH (2007): "The Navigation of an Iceberg: The Optimal Ese of Hidden Orders," *Finance Research Letters*, 4, 68 – 81.

FREY, S. AND P. SANDAS (2009): "The Impact of Iceberg Orders in Limit Order Books," Cfr working paper, Centre for Financial Research.

GRAEWE, P., U. HORST, AND J. QIU (2014): "A Non-Markovian Liquidation Problem and Backward SPDEs with Singular Terminal Conditions," *SIAM J. Control and Optimization, to appear.*

GRIFFITHS, M. D., B. F. SMITH, D. ALASDAIR, S. TURNBULL, AND R. W. WHITE (2000): "The Costs and Determinants of Order Aggressiveness," *Journal of Financial Economics*, 6, 65 – 88.

HARRIS, L. (1997): "Order Exposure and Parasitic Traders," Working paper series, University of Southern California.

HASBROUCK, J. AND G. SAAR (2009): "Technology and Liquidity Provision: The Blurring of Traditional Definitions," *Journal of Financial Markets*, 12, 143 – 172.

HAUTSCH, N. AND R. HUANG (2012): "The Market Impact of a Limit Order," *Journal of Economic Dynamics and Control*, 36, 501–522.

HAUTSCH, N., P. MALEC, AND M. SCHIENLE (2013): "Capturing the Zero: A New Class of Zero-Augmented Distributions and Multiplicative Error Processes," *Journal of Financial Econometrics.*

HOLLIFIELD, B., R. A. MILLER, P. SANDÅS, AND J. SLIVE (2006): "Estimating the Gains from Trade in Limit-Order Markets," *The Journal of Finance*, 61, 2753–2804.

HORST, U. AND F. NAUJOKAT (2014): "When to Cross the Spread: Trading in a Two-sided Limit Order Book," *SIAM J. Financial Mathematics*, 5, 278–315.

HUANG, R. D. AND H. R. STOLL (1997): "The Components of the Bid-ask Spread: A General Approach," *Review of Financial Studies*, 10, 995–1034.

KRATZ, P. AND T. SCHÖNEBORN (2014): "Optimal Liquidation in Dark Pools," *Mathematical Finance, to appear.*

MADHAVAN, A., M. RICHARDSON, AND M. ROOMANS (1997): "Why Do Security Prices Change? A Transaction-level Analysis of NYSE Stocks," *Review of Financial Studies*, 10, 1035–1064.

MOINAS, S. (2010): "Hidden Limit Orders and Liquidity in Order Driven Markets," IDEI Working Papers 600, Institut d'Economie Industrielle (IDEI), Toulouse.

OBIZHAEVA, A. AND J. WANG (2013): ""Optimal Trading Strategy and Supply/Demand Dynamics"," *Journal of Financial Markets*, 16, 1–32.

PASCUAL GASCO, R. AND D. VEREDAS (2008): "What Pieces of Limit Order Book Information Matter in Explaining Order Choice by Patient and Impatient Traders?" Working Paper Series.

Perold, A. F. (1988): "The Implementation Shortfall: Paper versus Reality," *Journal of Portfolio Management*, 14, 4–9.

Ranaldo, A. (2004): "Order Aggressiveness in Limit Order Book Markets," *Journal of Financial Markets*, 7, 53 – 74.

Tuttle, L. A. (2003): "Hidden Orders, Trading Costs and Information," Working Paper Series.